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Reflections on Poncelet's Pencil

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Abstract. We illustrate properties of the conics in Poncelet's pencil using some new insights motivated by some elementary triangle constructions of García.

1. Introduction

The conics passing through the vertices A, B, C of a triangle and its orthocenter H is the Poncelet pencil; any conic of this pencil is an equilateral hyperbola. The isogonal transform of a line through the circumcenter O gives a conic of this pencil and conversely. We described this pencil in [1] and used it to solve some triangle constructions in [2].

Here is a brief review of some properties of the conics in this pencil. For a triangle Δ and an equilateral hyperbola $\mathcal{K}$ passing through the vertices of Δ, let $\mathcal{C}$ be the circumcircle of Δ and S the fourth point of intersection of these two conics. Let S' be the antipodal of S on $\mathcal{C}$. Let L be the line through O parallel to the Wallace-Simson line of S'. Then the isogonal transform of L is $\mathcal{K}$. The center of $\mathcal{K}$ is denoted Z. The nine point circle of any triangle on the equilateral hyperbola passes through Z since the same equilateral hyperbola serves for any triangle on it.

García [4] has recently introduced some elementary triangle constructions which we will use to give some alternate constructions of some of the data of the conics in the Poncelet pencil. This provides some new insights into the properties of the conics in Poncelet's pencil.

2. Review of García's results and some extensions

Consider triangle $\Delta = \Delta ABC$; symmetries of a point P in the midpoints of Δ gives $\Delta_1 = \Delta_1(P)$ with vertices A_1, B_1, C_1. A second triangle $\Delta_2 = \Delta_2(P)$ is constructed with vertices A_2, B_2, C_2 which are the reflections of the vertices of Δ_1 in corresponding sides of triangle Δ (see Figure 1).

We review García's Theorems and develop some useful corollaries.

The triangles Δ and Δ_1 have centroids G and G_1.

Let Z be obtained by application of the similarity $\sigma = \sigma_{G,-\frac{1}{2}}$ (centered at G with scale factor $-\frac{1}{2}$) to P.

Theorem 1 (García). *Triangle Δ_1 is a symmetry of Δ about Z.*

Corollary 2. *The points P, G, Z, G_1 lie on a line.*

Proof. σ transforms P to Z, so P, Z, G lie on a line. Then also G_1 lies on this line since it is a symmetry about Z of G. ☐

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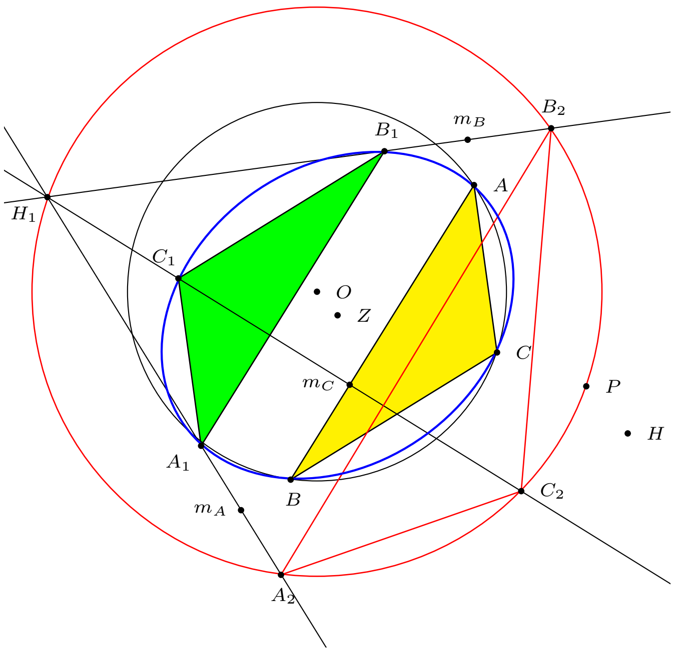

Figure 1. García's triangles and a conic

Theorem 3 (García). *The point P lies on the circumcircle of Δ_2. The circumcenter of Δ_2 is O, the circumcenter of Δ.*

Corollary 4. *The orthocenter H_1 of Δ_1 is antipodal to P on the circumcircle of Δ_2.*

Proof. The two similarities $\sigma_{H,\frac{1}{2}}, \sigma_{G,-\frac{1}{2}}$ take the circumcircle of Δ to the circumcircle to the midpoint triangle Δ_m, hence $\sigma_{H,2}\sigma_{G,-\frac{1}{2}}$ preserves the circumcircle of Δ and hence its center O. Thus $\sigma_{H,2}\sigma_{G,-\frac{1}{2}} = \sigma_{O,-1}$.

Now evaluating both sides at P we get that $\sigma_{H,2}\sigma_{G,-\frac{1}{2}}(P) = \sigma_{H,2}(Z) = H_1$ is antipodal to P. □

Corollary 5. *Δ_2 and Δ_1 are in perspective from H_1.*

Proof. Since Δ_2 is obtained by reflection of the vertices of Δ_1 across the sides of Δ, which are parallel to the sides of Δ_1, then the altitudes of Δ_1 (concurrent at H_1) pass through the vertices of Δ_2. □

Corollary 6. *The midpoints of corresponding vertices of Δ_1 and Δ_2 lie on the corresponding sides of Δ.*

Proof. This follows immediately from the construction. □

2.1. *Similarity.*

Proposition 7. *Let H denote the orthocenter of Δ. Let $\mathcal{C}$ be a circle passing though H. The intersections of the altitudes of Δ with $\mathcal{C}$ give a triangle $\Delta' = \Delta_{\mathcal{C}}$ oppositely similar to Δ.*

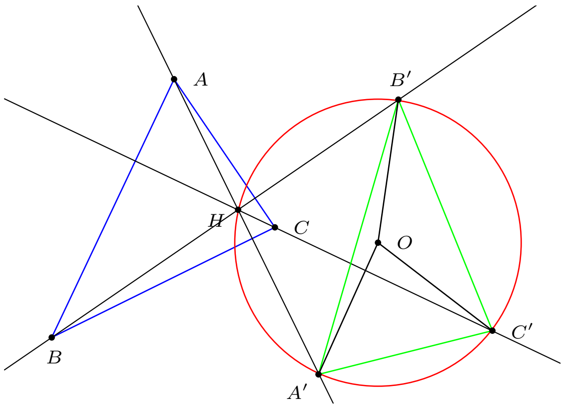

Figure 2. Similarity via H

Proof. The angles of Δ are related to the angles of Δ' via H and angles on $\mathcal{C}$ subtended at H. There are two angles at A formed by the altitude there and the adjacent sides. Consider the angle with side AC. This altitude passes through the vertex A' of Δ'. The altitude perpendicular to AC passes through B'. The angle formed by these altitudes at H is half the central angle of $A'B'$, which is the angle $OA'B'$. Similarly we can determine $OA'C'$. The sum of these two angles is $\angle A'$; using this we get the same sum as $\angle A$ since the altitudes through H are perpendicular to the adjacent sides at A. The argument is similar at the other vertices. □

Corollary 8. *Δ_2 and Δ_1 are similar with scale factor R_2/R_1.*

Proof. By Corollary 5 Δ_2 is in perspective with Δ_1 though H_1, with H_1 on the circumcircle of Δ_2; and by Proposition 7 Δ_2 is oppositely similar to Δ_1.

Using the formula for area $\frac{abc}{4R}$ in terms of the side lengths and circumradius then we easily deduce that the scale factor of the similarity is R_2/R_1. □

2.2. *A conic.*

Theorem 9. *The six points of Δ and Δ_1 lie on a conic $\mathcal{K} = \mathcal{K}_{\Delta,P}$ having center Z.*

Proof. Corresponding sides of the triangles meet on the line at infinity so an application of the converse of Pascal's Theorem shows that there is a conic passing through all six vertices. The point Z is the center of symmetry taking one triangle to the other; hence it must be the center of the conic. □

3. P lies on circumcircle of Δ

Corollary 10. *If P is on the circumcircle of Δ then Δ_2 is also on this circumcircle and is anti-congruent to Δ. The point H_1 is antipodal to P on the circumcircle of Δ.*

Proof. The circumcircle of Δ_2 has center O and passes through P so Δ_2 is also the circumcircle of Δ. The similarity factor is 1 by Corollary 8 so the two triangles are anti-congruent. This circumcircle also passes through H_1 using Theorem 3. □

Theorem 11. *Suppose P lies on the circumcircle of Δ then $\mathcal{K} = \mathcal{K}_{\Delta,P}$ is in Poncelet's pencil with circumcircle point H_1.*

Proof. Consider the conic passing through Δ, H and H_1. Then it is an equilateral hyperbola in Poncelet's pencil since H is on the conic. Since the conic also passes through H_1, then H_1 is the circumcircle point of this conic. Since both points H, H_1 are on the equilateral hyperbola then the midpoint Z is the center of the conic . Hence also Δ_1 is on the conic by Theorem 3. Thus the conic is $\mathcal{K}_{\Delta,P}$ by Bezout's Theorem [3]. □

Corollary 12. *As P varies on the circumcircle of Δ then the family of conics $\mathcal{K}_{\Delta,P}$ is Poncelet's pencil for Δ.*

Proof. Given a conic in Poncelet's pencil let P be the antipodal to its circumcircle point then by the Theorem above this conic is $\mathcal{K}_{\Delta,P}$. □

Corollary 13. *The conic $\mathcal{K}_{\Delta,P}$ is tangent to the circumcircle iff P is antipodal to a vertex of Δ.*

Proof. The circumcircle is tangent to $\mathcal{K}$ iff the circumcircle point H_1 is a vertex of the triangle iff (Corollary 4) P is antipodal to a vertex of Δ. □

Theorem 14. *Suppose P lies on the circumcircle of Δ. The reflections of P in the sides of Δ lie on a line M parallel to the line L, the isogonal transform of $\mathcal{K}_{\Delta,P}$. This line M is also parallel to the Wallace-Simson line of P and passes through H. Thus $L = \sigma_{G,-\frac{1}{2}}(M)$.*

Proof. This follows immediately from Corollary 7 of [2] since P is antipodal the circumcircle point H_1. The second and third statements follow from Theorems 5, 6 of [2]. Also since M passes through H, then $\sigma_{G,-\frac{1}{2}}(M)$ passes through O since $\sigma_{G,-\frac{1}{2}}(H) = O$ and thus $L = \sigma_{G,-\frac{1}{2}}(M)$. □

Theorem 15. *If P is on the circumcircle of Δ, then the midpoints of Δ_1 and Δ_2 lie on L_1, the Wallace-Simson line of H_1. The line L_1 passes through the center Z of $\mathcal{K}_{\Delta,P}$. The lines L_1 and L are perpendicular.*

Proof. As shown already in Corollary 6 and Corollary 5 these midpoints are on the sides of Δ and since the two triangles are congruent and in perspective from H_1 the midpoints are on the lines of perspectivity. But the vertices of Δ_2 are by definition the reflections of the vertices across the sides of Δ_1. Hence the midpoints are the

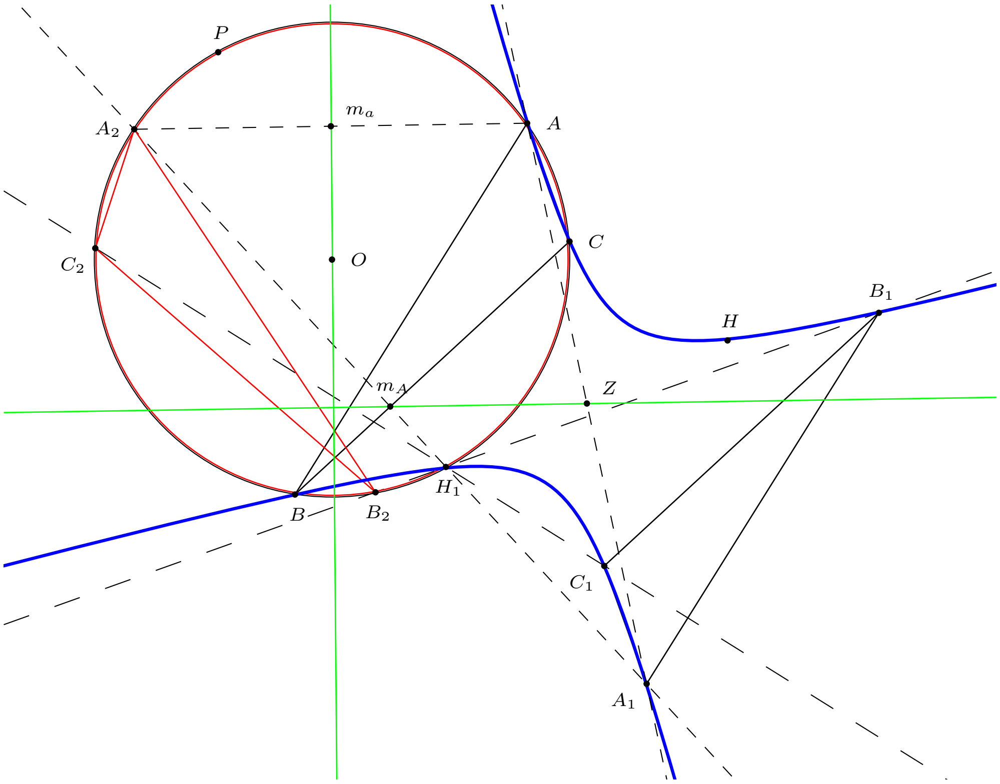

Figure 3. P on circumcircle equilateral hyperbola

feet of the altitudes from H_1 and lie on the Wallace-Simson line of H_1. Since H_1 is the circumcircle point of $\mathcal{K}_{\Delta,P}$ this line passes through Z, [2] Theorem 6. The Wallace-Simson lines of H_1 and P are perpendicular since these these points are antipodal. $\square$

Theorem 16. *If P is on the circumcircle of Δ, then the midpoints of Δ and Δ_2 lie on line L.*

Proof. From point A_1 the midpoints to A and A_2 are on the line L_1. Thus the midpoint m_A of A and A_2 lies on a line parallel to L_1. Since Δ and Δ_2 lie on the circumcircle centered at O the perpendicular bisector of A and A_2 passes through O and is perpendicular to L_1. Thus m_A lies on L. The argument is similar for the other pairs of points and hence the desired result follows.. $\square$

4. Equilateral triangles on equilateral hyperbolas

Proposition 17. *Suppose ΔABC is an equilateral triangle on the right hyperbola $\mathcal{K}$. The circumcircle meets $\mathcal{K}$ at the fourth intersection point S. The center of $\mathcal{K}$, Z, is the midpoint of OS where O is the circumcenter of Δ.*

Proof. Since Δ is equilateral $H = O$ and the result follows since Z is the midpoint of HS [2]. $\square$

Proposition 18. *Let M be a line through Z the center of the equilateral hyperbola $\mathcal{K}$ meeting at points O and S. Construct a circle $\mathcal{C}$ with center at O and passing through S. The three intersections of $\mathcal{K}$ and $\mathcal{C}$ other than S give the vertices of an equilateral triangle Δ.*

Proof. By construction O is the circumcenter of Δ and S is the circumcircle point. In general the point Z is the midpoint of HS [2]. By our construction Z is the midpoint of OS so $H = O$. Thus the triangle is equilateral. □

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