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Bounds for Elements of a Triangle Expressed by R, r, and s

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Abstract. Assume that a triangle is defined by the triple (R, r, s) fulfilling the conditions (1) and (2) (R - the circumradius, r - the inradius, s - the semiperimeter). We find some bounds for the trigonometric functions of the angles and for the sides of the triangle expressed by R and r (see the formulas (3) and (7) - (13)).

It is well-known that the positive numbers R, r, s may be the circumradius, the inradius, and, respectively, the semi-perimeter of a triangle if and only if these numbers satisfy Euler's inequality

$$R \geq 2r, \tag{1}$$

and the fundamental double inequality

$$\begin{aligned} &2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \\ \leq\ &s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}. \end{aligned} \tag{2}$$

This double inequality was found in 1851 and it was subsequently rediscovered in many different forms. It often appears in the literature under the name of *Blundon's inequality*. A history of this inequality can be found in [2, pp.1–5].

In the following, we consider that the triangles are given by triples (R, r, s) that verify (1) and (2). The objective of this note is to find some bounds (expressed by R, r) for the sides and trigonometric functions of angles of the triangle. We recall a well-known result on the conditions in which the inequalities in (2) become equalities. There is a rich literature on this subject. In a recent short paper [1], we have presented a simple geometrical proof of Theorem 1 below.

We say that a triangle is wide-isosceles if it is isosceles with the base greater than or equal to the congruent sides, and is tall-isosceles if it is isosceles with the congruent sides greater than or equal to the base. The equilateral triangle is both wide-isosceles and tall-isosceles (see Figure 1).

Theorem 1. *In the fundamental double inequality,*
(a) *the first inequality is an equality if and only if the triangle is wide-isosceles;*
(b) *the second inequality is an equality if and only if the triangle is tall-isosceles;*
(c) *both inequalities are equalities if and only if the triangle is equilateral.*

We now state and prove our first result.

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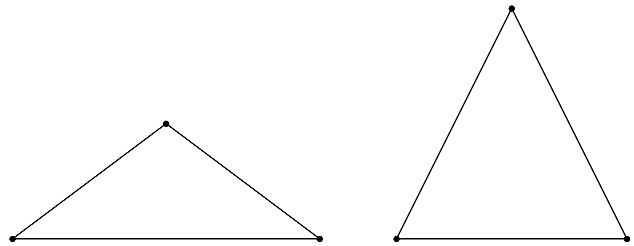

Figure 1

Proposition 2. *In any triangle ABC,*

$$\frac{1}{2}\left(1-\sqrt{1-\frac{2r}{R}}\right) \le \sin\frac{A}{2} \le \frac{1}{2}\left(1+\sqrt{1-\frac{2r}{R}}\right). \tag{3}$$

Moreover,
(a) *the first inequality is an equality if and only if the triangle is tall-isosceles;*
(b) *the second inequality is an equality if and only if the triangle is wide-isosceles;*
(c) *both inequalities are equalities if and only if the triangle is equilateral.*

Proof. Let O and I be the circumcenter and the incenter of the triangle ABC. Applying the triangle inequality for the triangle AOI, we have

$$AO - OI \le AI \le AO + OI. \tag{4}$$

The left-side of (4) is positive because I is contained in the circumcicle of triangle ABC. Taking into account the formulas $OA = R$, $OI^2 = R^2 - 2Rr$, and $AI = \frac{r}{\sin\frac{A}{2}}$, we can write (4) in the form

$$R - \sqrt{R^2 - 2Rr} \le \frac{r}{\sin\frac{A}{2}} \le R + \sqrt{R^2 - 2Rr}$$

or

$$\frac{R - \sqrt{R^2 - 2Rr}}{2R} \le \sin\frac{A}{2} \le \frac{R + \sqrt{R^2 - 2Rr}}{2R}.$$

Therefore, the inequalities (3) are valid.

To prove the assertion (a) (resp. (b)) is equivalent to the fact that the right (respectively left) inequality of (4) becomes an equality if and only if the triangle ABC is tall-isosceles (respectively wide-isosceles).

(a) The equality $AI = AO + OI$ is equivalent with the fact that the triangle ABC is isosceles, with $AB = AC$, and O lying in the segment AI. Obviously, O and I coincide if and only if ABC is an equilateral triangle.

In the remaining case, we have $AO < AI$, i.e., $R < \frac{r}{\sin\frac{A}{2}}$. Let a and l denote the lengths of the base and congruent sides of the isosceles triangle. Then, using the formulas $4R\Delta = abc$ and $\Delta = rs$, we easily derive $R = \frac{l^2}{\sqrt{4l^2-a^2}}$, $r = \frac{a\sqrt{4l^2-a^2}}{2(2l+a)}$, and $\sin\frac{A}{2} = \frac{a}{2l}$. Consequently, we find that $R < \frac{r}{\sin\frac{A}{2}}$ is equivalent to $a < l$. Thus, in the second case the triangle ABC is tall-isosceles.

(b) If $AI = AO - OI$, then A, O, I are collinear and $I \in (AO]$. We proceed similarly. Again, we have to consider two cases. If O and I coincide, the triangle ABC is equilateral. If $I \in (AO)$,then $AO > AI$, and $R > \frac{r}{\sin \frac{A}{2}}$, i.e., $a > l$. In this case, ABC is wide-isosceles.

(c) follows from (a) and (b). □

We restate Proposition 2 in a symmetrical form.

Corollary 3. *In triangle ABC,*

$$\max(\sin \frac{A}{2}, \sin \frac{B}{2}, \sin \frac{C}{2}) \leq \frac{1}{2}\left(1 + \sqrt{1 - \frac{2r}{R}}\right), \tag{5}$$

$$\min(\sin \frac{A}{2}, \sin \frac{B}{2}, \sin \frac{C}{2}) \geq \frac{1}{2}\left(1 - \sqrt{1 - \frac{2r}{R}}\right). \tag{6}$$

Moreover,
(a) *equality holds in* (5) *if and only if the triangle is wide-isosceles;*
(b) *equality holds in* (6) *if and only if the triangle is tall-isosceles.*

Starting from (3), we shall obtain new inequalities by using appropriate formulas in trigonometry. Thus, we obtain from (3), after squaring and simplifying, the following inequalities:

$$\frac{1}{2}\left(1 - \frac{r}{R} - \sqrt{1 - \frac{2r}{R}}\right) \leq \sin^2 \frac{A}{2} \leq \frac{1}{2}\left(1 - \frac{r}{R} + \sqrt{1 - \frac{2r}{R}}\right). \tag{7}$$

Also, taking into account the identity $\cos^2 t + \sin^2 t = 1$, we deduce that

$$\frac{1}{2}\left(1 + \frac{r}{R} - \sqrt{1 - \frac{2r}{R}}\right) \leq \cos^2 \frac{A}{2} \leq \frac{1}{2}\left(1 + \frac{r}{R} + \sqrt{1 - \frac{2r}{R}}\right). \tag{8}$$

Because the left-side term in (8) is positive, it follows that

$$\frac{\sqrt{2}}{2}\sqrt{1 + \frac{r}{R} - \sqrt{1 - \frac{2r}{R}}} \leq \cos \frac{A}{2} \leq \frac{\sqrt{2}}{2}\sqrt{1 + \frac{r}{R} + \sqrt{1 - \frac{2r}{R}}}. \tag{9}$$

From (7) or (8), using the identities $2\sin^2 \frac{A}{2} = 1 - \cos A$ or $2\cos^2 \frac{A}{2} = 1 + \cos A$, we obtain the following inequalities:

$$\frac{r}{R} - \sqrt{1 - \frac{2r}{R}} \leq \cos A \leq \frac{r}{R} + \sqrt{1 - \frac{2r}{R}}. \tag{10}$$

Remark. As it is natural, the left-side term of (10) is not always positive. We have $\frac{r}{R} - \sqrt{1 - \frac{2r}{R}} \geq 0$ if and only if $r - \sqrt{R^2 - 2Rr} \geq 0$, that is $r \geq OI$. (Geometricaly, O is in the interior or on the incircle of the triangle ABC).

By using the double angle formula, we obtain from (3) and (9),

$$\frac{\sqrt{2}}{2}\left(1-\sqrt{1-\frac{2r}{R}}\right)\sqrt{1+\frac{r}{R}-\sqrt{1-\frac{2r}{R}}}$$
$$\leq \sin A \leq \frac{\sqrt{2}}{2}\left(1+\sqrt{1-\frac{2r}{R}}\right)\sqrt{1+\frac{r}{R}+\sqrt{1-\frac{2r}{R}}} \tag{11}$$

or, by squaring,

$$2-\frac{2r}{R}-\frac{r^2}{R^2}-2\sqrt{1-\frac{2r}{R}} \leq \sin^2 A \leq 2-\frac{2r}{R}-\frac{r^2}{R^2}+2\sqrt{1-\frac{2r}{R}}. \tag{12}$$

From (11) and (12), and taking into account the law of sines, we easily obtain upper bounds and lower bounds for the lengths of the sides. Thus, we have

$$8R^2-8Rr-4r^2-8R\sqrt{R^2-2Rr} \leq a^2 \leq 8R^2-8Rr-4r^2+8R\sqrt{R^2-2Rr}. \tag{13}$$

Because the inequality in previous section has been found by simple transformations of the inequalities (3), we can obtain the necessary and sufficient conditions for equality to occur in the inequalities (7) - (13) as immediate consequences of those specified in Proposition 2. Next we state some results along this order of ideas, leaving the details to the reader.

Proposition 4. (a) *Equality occurs in the first inequality of* (7) *if and only if triangle ABC is tall-isosceles.*
(b) *Equality occurs in the second inequality of* (7) *if and only if triangle ABC is wide-isosceles.*
(c) *Equality occurs in both cases if and only if triangle ABC is equilateral.*

Proposition 5. (a) *Equality occurs in the first inequality of* (8), (9), (10) *if and only if triangle ABC is wide-isosceles.*
(b) *Equality occurs in the second inequality of* (8), (9), (10) *if and only if triangle ABC is tall-isosceles.*
(c) *Equality occurs in both cases if and only if triangle ABC is equilateral.*

Proposition 6. *In each of the double inequalities* (11), (12) *and* (13), *equality occurs in one or both side if and only if the triangle is equilateral.*

Remarks. (1) We can formulate the inequalities (7) - (13) in a symmetrical form as we have made in Corollary 3 with the inequalities (3).

(2) Of course, one can obtain further inequalities by proceeding in the same way as above. But, it appears the risk of complicated expressions in R and r for the leftmost and rightmost sides of the derived inequalities. For example, using the formula $1+\tan^2 t = \cos^{-2} t$, it is possible to obtain some inequality for $\tan\frac{A}{2}$, $\tan A$ starting from (9), (10).

Finally, we turn our attention to the left-side of the inequalities (13), i.e., to the inequality

$$a^2 \geq 8R^2-8Rr-4r^2-8R\sqrt{R^2-2Rr}, \tag{14}$$

with equality if and only if ABC is equilateral.

If ABC is an acute triangle, this inequality can be improved. Indeed, by (10) we have

$$\cos A \leq \frac{r}{R} + \sqrt{1 - \frac{2r}{R}}.$$

In our hypothesis, $\cos A \gneqq 0$. Thus, after squaring we obtain

$$\sin^2 A \geq \frac{2r}{R} - \frac{r^2}{R^2} - 2\frac{r}{R}\sqrt{1 - \frac{2r}{R}}$$

or, equivalently,

$$a^2 \geq 8Rr - 4r^2 - 8r\sqrt{R^2 - 2Rr}. \tag{15}$$

As in (10), the equality in (15) holds if and only if the acute triangle ABC is tall-isosceles.

It is easy to see that (15) improves (14). Indeed, it is straightforward to verify that

$$8Rr - 4r^2 - 8r\sqrt{R^2 - 2Rr} \geq 8R^2 - 8Rr - 4r^2 - 8R\sqrt{R^2 - 2Rr},$$

as well as the fact that the equality sign holds only if ABC is equilateral. Consequently, apart from (14), the equality sign holds for (15) not only for equilateral triangles but also for tall-isosceles ones.

References

[1] T. Bîrsan, Blundon's double inequality revisited, *Recreaţii Matematice*, 14 (2012) 22–24 (in Romanian).

[2] D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1889.

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