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Do Dogs Play with Rulers and Compasses?

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Abstract. A dog runs at the speed of 1 and swims at the speed of $s < 1$. If the dog is at point A on the shoreline and tries to get to a ball in water at point B in the least time, what path should the dog take? In this article we discuss geometric solutions to this optimization problem and its variations.

1. Introduction

A dog runs at the speed of 1 and swims at the speed of $s < 1$. If the dog is at point A on the shoreline and tries to get to a ball in water at point B in the least time, what path should the dog take?

This problem and equivalent versions of it have been typical exercises in calculus textbooks for decades. But in [5] the author discovered that his dog Elvis seemed to follow instinctively the optimal path. Since then this problem has gone "viral" and several follow-up articles [1, 2, 4, 6, 7] have discussed variations and different perspectives for the dog.

In this article we give the dog a simple geometric perspective.

2. A ruler-compass solution

As in Figure 1(a), construct the circle Σ with diameter BD, where D is the foot of perpendicular from B onto the shoreline. Let $d = BD$ and construct point Q on Σ such that $DQ = sd$. This is useful because the dog runs the distance d in the same time as he swims the distance sd. If BQ intersects AD at a point E between A and D, then the dog should run from A to E and swim from E to B. How could the dog know for sure?

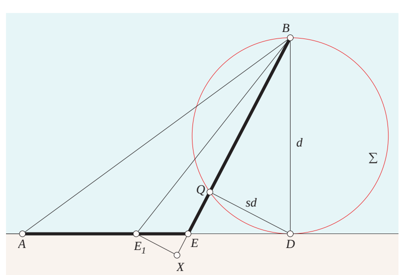

Figure 1(a). Construction of the optimal path AEB

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Take a different point E_1, as in Figure 1(a). Let X be the foot of perpendicular from E_1 onto BE. Note that $\triangle EE_1X \sim \triangle DBQ$, so the dog runs the distance E_1E in the same time as he swims the distance XE. But E_1B is a longer distance to swim than the distance XB. Therefore, the path AE_1B takes longer time than the path AEB (denoted by $AE_1B \succ AEB$ from now on). The proof remains valid if E_1 is taken on the other side of E.

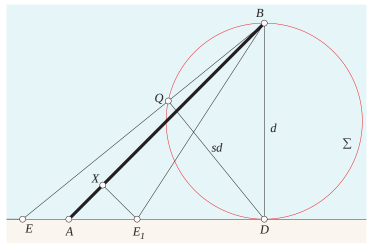

Figure 1(b). Proof that the optimal path is AB

If E falls beyond A, as in Figure 1(b), then the dog should directly swim from A to B. Indeed, take a different point E_1 between A and D. Let X be the foot of perpendicular from E_1 onto AB. Then the dog swims the distance AX in less time than he runs the distance AE_1, and XB is a shorter distance than E_1B to swim. Hence $AE_1B \succ AB$. Equivalently, $EE_1B \succ EAB$, which means that the time for the path EE_1B increases as E_1 moves away from E. Similarly, if E_1 is to the left of E, then the time for the path E_1EB decreases as E_1 moves towards E.

3. Snell's law

If A is further inland, as in Figure 2, then it is the well-known Snell's law that $s \cdot \sin\alpha = 1 \cdot \sin\beta$ at the optimal point E. The geometric proof that AEB takes the least time is similar.

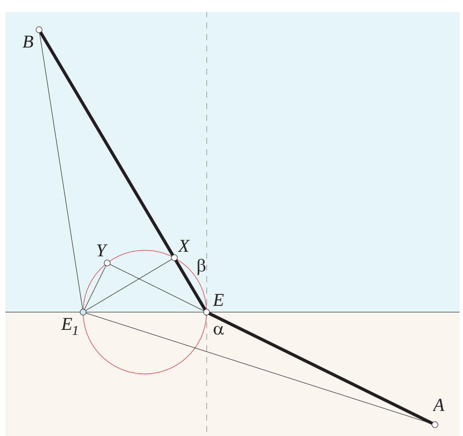

Figure 2. Proof of Snell's law

Take a different point E_1 from E. Let X and Y be the feet of perpendiculars from E_1 onto BE and AE respectively. Then

$$sEY = sE_1E\sin\alpha = E_1E\sin\beta = EX,$$

that is, the dog runs the distance EY in the same time as he swims the distance EX. But AE_1 is a longer distance than AY to run, and E_1B is a longer distance than XB to swim. Hence $AE_1B \succ AEB$. Following the references in [1] we learn that this is a rediscovery of Huygens's proof from 1678 [3]. So an old dog may not be taught new tricks, but an old dog may be motivated to rediscover old tricks. The location of the point E in this case requires the solution of a quartic equation, thus can not be constructed by ruler and compass. But [7] gives a geometric construction using the trammel of Archimedes.

4. Bifurcation

In [4], the authors discuss the following variation: what if the dog is originally at a point A also in water? Then the optimal path could be directly swimming from A to B or a SRS path: swimming to a point F on the shoreline, then running along the shoreline to another point E, and swimming from E to B. Now we give a simple geometric solution to this problem as well.

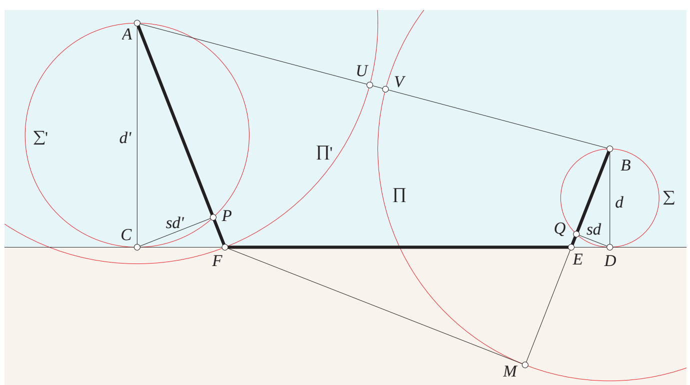

Figure 3(a). Constructing the optimal path $AFEB$

First, we construct points E and F on the shoreline as in §2. See Figure 3(a). Let M be the foot of perpendicular from F onto BE. Since $\triangle FEM \sim \triangle BDQ \sim \triangle ACP$, the dog runs the distance FE in the same time as he swims the distance ME. Now draw circles Π' and Π centered at A and B with radii AF and BM respectively. If Π and Π' are externally disjoint, as in Figure 3(a), then the SRS path $AFEB$ takes the same time as swimming from A to U and then from V to B, thus $AB \succ AFEB$. Also, it can be proved in the same way as in §2 that any other SRS paths will take longer time than $AFEB$.

On the other hand, if Π and Π' are not externally disjoint, as in Figure 3(b), then the argument above proves that the optimal path is directly swimming from A to B. The "bifurcation" mentioned in [4] occurs exactly when Π and Π' are externally tangent to each other, in which case swimming from A to B takes exactly the same time as the SRS path $AFEB$.

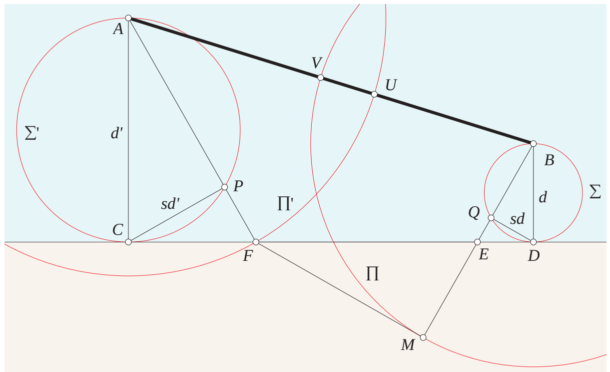

Figure 3(b). Proof that the optimal path is AB

5. Bended Shoreline

The discussions above naturally lead to the case where the shoreline is not straight. In the figures below, the shore consists of two lines l and m meeting at T. Points E and F are constructed as before, forming the optimal angle θ from B to the shorelines l and m. Note that E may not be between D and T, and F may not be between A and T, causing variations in the pictorial proofs.

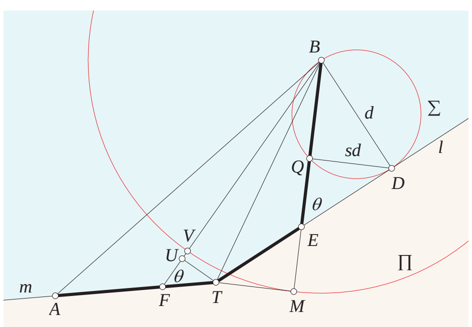

Figure 4(a). Proof that $ATEB$ is optimal, where M and U are feet of perpendiculars from T onto BE and BF

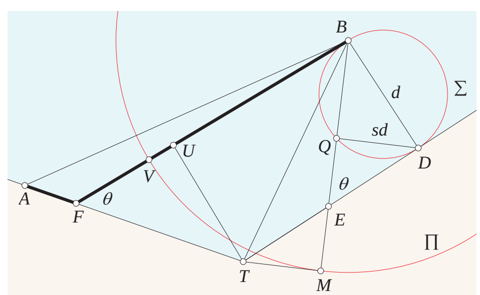

Figure 4(b). Proof that AFB is optimal

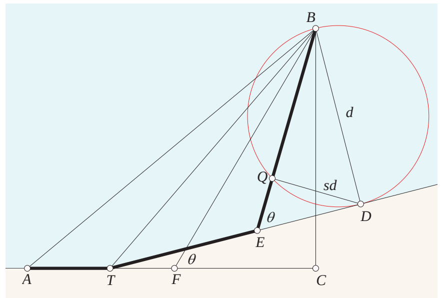

Figure 5. Proof that $ATEB$ is optimal

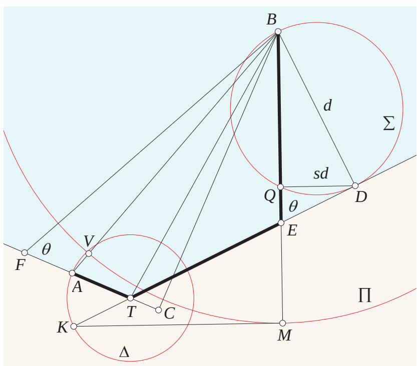

Figure 6(a). Proof that $ATEB$ is optimal, where M is the foot of perpendicular from K onto BE

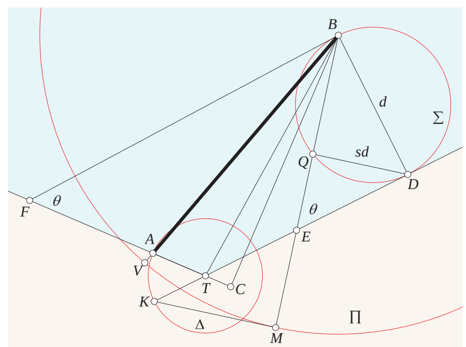

Figure 6(b). Proof that AB is optimal

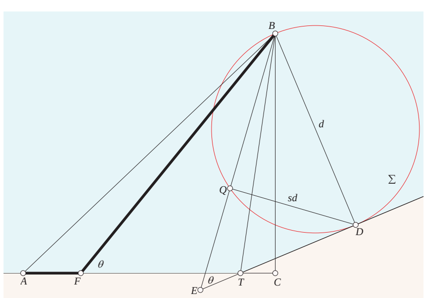

Figure 7. Proof that AFB is optimal

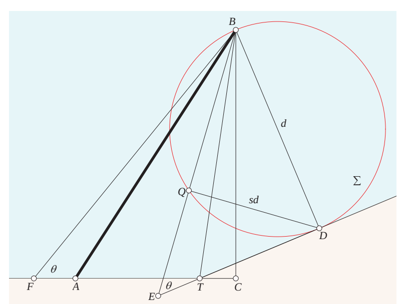

Figure 8. Proof that AB is optimal

To challenge the dog more, we can also move A into the water, as in Figure 9. Then the dog has to decide between AB, $AGFB$, $AHEB$, and $AGTEB$. The interested readers are invited to play with rulers and compasses, or with their dogs.

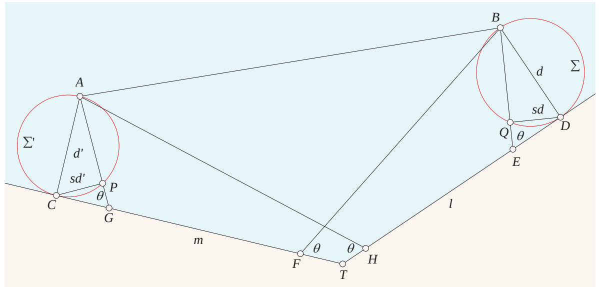

Figure 9. What is the optimal path?

Finally, we can also have more bends in the shoreline. The solutions and proofs are all the same, only with more overwhelming numbers of cases!

6. Dogs' Day-Dreams

Consider a lake Ω in the shape of a convex polygon. For any two points A and B in Ω (including the shoreline), define the dog-distance $\delta_s(A, B)$ to be the least time the dog (with running speed 1 and swimming speed $s < 1$) can get from A to B. Then all the geodesics can be constructed by ruler and compass. Fix a point A, what is the locus of points B such that there are more than one geodesics between A and B? Fix two points A and B, what is the shape of $\mathcal{N}_s(A) = \{X \in \Omega : \delta_s(A, X) < \delta_s(X, B)\}$? Perhaps dogs day-dream many more questions about the geometry of (Ω, δ_s).

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