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Another Construction of the Simson Lines Through a Given Point

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Abstract. We give a simple conic construction of the points on the circumcircle whose Simson line go through a given point.

The construction problem of the Simson lines through a given point has been solved elegantly by Jean Pierre Ehrmann in [1]. Given a point P in the plane of triangle ABC (with orthocenter H), the three points whose Simson lines pass through P are the intersections of the circumcircle and the *translation* by the vector $\mathbf{HP}$ of the rectangular circum-hyperbola through P. Ehrmann obtained this ingenious construction by applying remarkable results of Lalesco ([2]) on Simson lines. In this note we give another construction resulting from a simple-minded analysis.

We use barycentric coordinates with reference to triangle ABC. Let $P = (u : v : w)$. For an arbitrary point $M = (x : y : z)$, let B_0, C_0 be the pedals of M on the sidelines CA and AB respectively. When M lies on the circumcircle, B_0C_0 becomes the Simson line of M. Now, the line B_0C_0 contains the point P if and only if M lies on the conic Γ_a with equation

$$c^2(S_Au - S_Cw)y^2 + b^2(S_Au - S_Bv)z^2 + ((S^2 + 2S_A^2)u - S_{AB}v - S_{AC}w)yz \\ - b^2(c^2v + S_Aw)xz - c^2(b^2w + S_Av)xy = 0.$$

where S is, as usual, twice the area of triangle ABC.

Clearly, the conic Γ_a contains the vertex A. Proposition 1 exhibits five more points on the conic, which can be easily constructed; see Figure 1.

Proposition 1. *Let the perpendicular from P to AP^* intersect AC at M and AB at N.*
(a) *If the perpendicular from P to AB intersects CA at Y, then Y lies on Γ_a. In the same way, if the perpendicular from P to CA intersects AB at Z, then Z also lies on Γ_a.*
(b) *If the perpendicular to AQ at P intersects CA, AB at M, N, then the perpendiculars to CA, AB at M, N intersect at a point L on Γ_a.*

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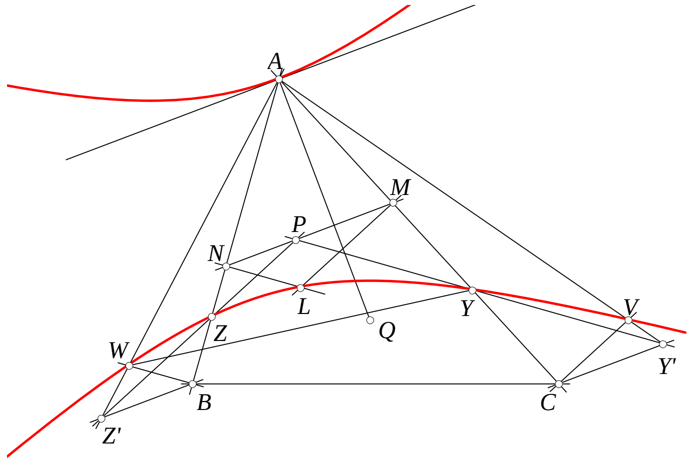

Figure 1.

(c) *If the perpendicular to AQ at C intersects PY at Y', then the perpendicular to CA at C intersects AY' at a point V on Γ_a. Likewise, if the perpendicular to AQ at B intersects PZ at Z', then the perpendicular to AB at B intersects AZ' at a point W on Γ_a.*

The conic Γ_a contains the infinite points of the altitudes through B and C. Therefore, it is a hyperbola. Proposition 2 gives a simple construction of the center of Γ_a, and hence its asymptotes (see Figure 2). Indeed, Γ_a goes through A and the normal at A is the A-cevian of the isogonal conjugate Q of P.

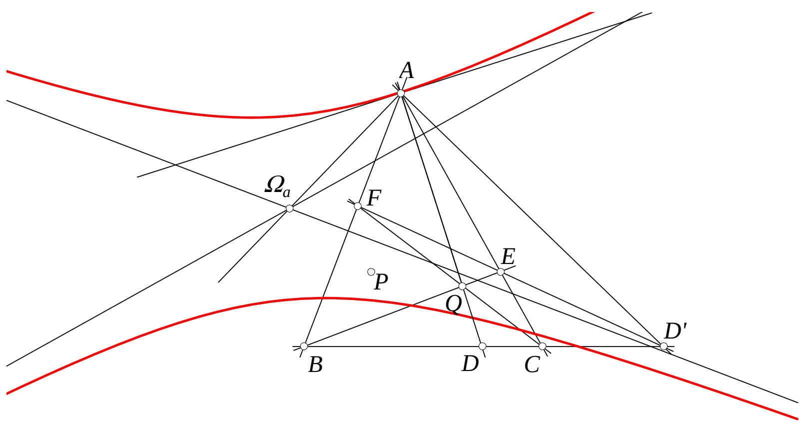

Figure 2.

Proposition 2. *If Ω_a is the center of Γ_a, the perpendicular to $A\Omega_a$ at A is the harmonic conjugate of AQ with respect to AB, AC. In other words, if DEF is the cevian triangle of Q, let $D' = DE \cap BC$ be the harmonic conjugate of D with respect to BC. Then AD' and $A\Omega_a$ are perpendicular.*

Clearly, apart from the vertex A, the common points of Γ_a and the circumcircle are the points whose Simson lines pass through the given point P. See Figure 3.

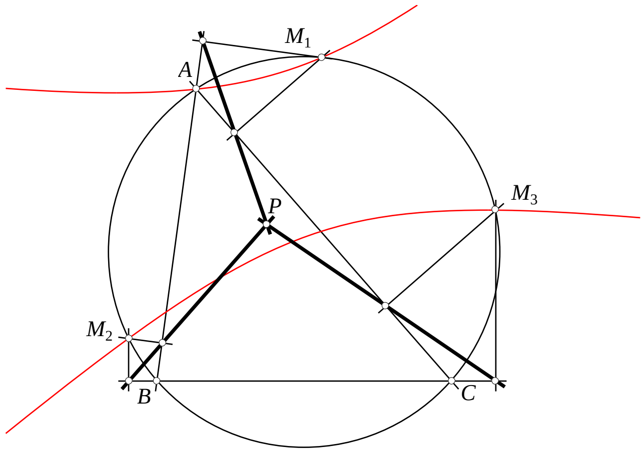

Figure 3.

Consider also the analogous hyperbolas Γ_b and Γ_c. Each of these also intersects the circumcircle at the same three points whose Simson lines pass through the given point P, as does Ehrmann's hyperbola, which has equation

$$u(S_Bv - S_Cw)\Gamma_a + v(S_Cw - S_Au)\Gamma_b + w(S_Au - S_Bv)\Gamma_c = 0.$$

Figure 4 shows the hyperbolas Γ_a, Γ_b, Γ_c, and Ehrmann's hyperbola h'.

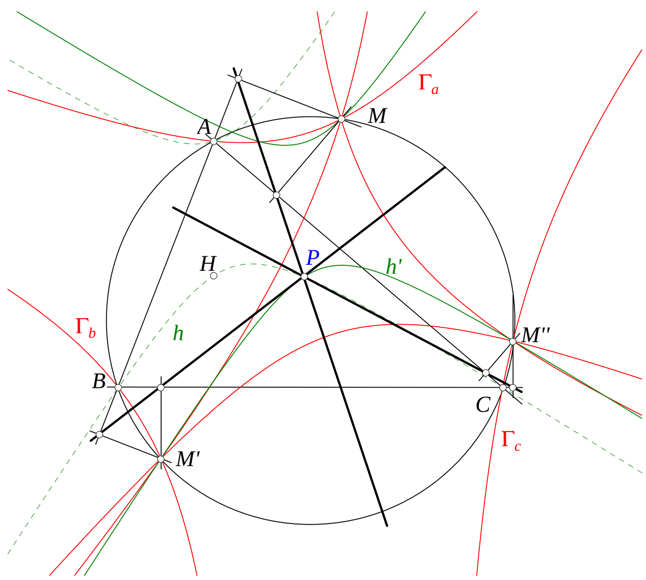

Figure 4.

We conclude this note with another construction of the center of Γ_a.

Let $A'B'C'$ be the cevian triangle of P and $A'' = B'C' \cap BC$, that is, A'' is the harmonic conjugate of A' with respect to B, C. If U is the midpoint of AP, let the parallel to AA'' intersect AB, AC at K, L. Then K and L are the orthogonal projections of Ω_a on CA and AB respectively. In other words, $\Omega_a K$ and $\Omega_a L$ are are the asymptotes of the hyperbola (see Figure 5).

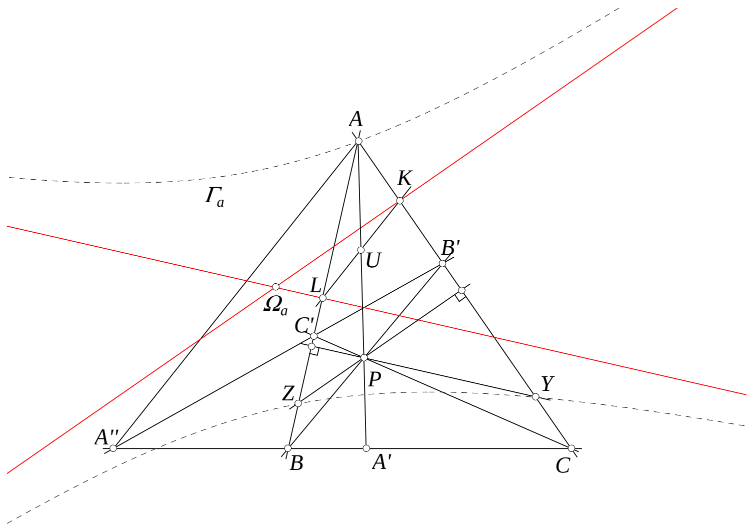

Figure 5.

References

[1] J.-P. Ehrmann, Some geometric constructions, *Forum Geom.*, 6 (2006) 327–334.
[2] T. Lalesco, *La geometrie du Triangle*, Paris Vuibert 1937; Jacques Gabay reprint 1987.

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