Forum Geometricorum
Volume 15 (2015) 211–228.

FORUM GEOM
ISSN 1534-1178

Circle Incidence Theorems

J. Chris Fisher, Eberhard M. Schröder, and Jan Stevens

Abstract. Larry Hoehn discovered a remarkable concurrence theorem about pentagrams. Draw circles through two consecutive vertices and the intersection points of the sides in between. Then the radical axes of each pair of consecutive circles are concurrent or parallel. In this note we prove a generalization to n-gons.

1. Introduction

Given a triangle, there are unexpected triples of lines that pass through one point; e.g, the three medians, altitudes, and angle bisectors are all concurrent. Larry Hoehn discovered a remarkable concurrence theorem about pentagons, illustrated in Figure 1, see [2]. In this note we prove a generalization to n-gons.

Let $A_1, \ldots, A_n$ be n points in the plane, no three on a line, and such that the lines $l_{i+1} = \langle A_i, A_{i+2} \rangle$ and $l_i = \langle A_{i-1}, A_{i+1} \rangle$ are not parallel, where we consider the indices modulo n. Let $B_{i,i+1}$ be the intersection point of l_i and l_{i+1}. Through the three points A_i, $B_{i,i+1}$ and A_{i+1} passes a unique circle $c_{i,i+1}$. Let g_i be the radical axis of the two consecutive circles $c_{i-1,i}$ and $c_{i,i+1}$.

Theorem 1 ([2])**.** *Given five points $A_1, \ldots, A_5$ in the plane the five radical axes $g_1, \ldots, g_5$, constructed as above are concurrent or parallel* (see Figure 1).

We use the terminology that lines *lie in a pencil* if they are concurrent or parallel. For $n \geq 6$ the radical axes in general do not lie in a pencil. For $n = 6$ we show that it is necessary and sufficient that the six points $B_{i,i+1}$ lie on a conic. This is equivalent to the condition that the three lines $\langle A_i, A_{i+3} \rangle$ lie in a pencil. In fact, the initial six points have to be in a special position for just three consecutive axes to lie in a pencil: Fisher, Hoehn and Schröder showed that this condition implies that than the remaining three axes lie in the same pencil [3]. Our main result generalizes this to $n > 6$.

Theorem 2. *Let $A_1, \ldots, A_n$ be n points in the plane, no three on a line, and such that the lines $l_{i-1} = \langle A_{i-1}, A_{i+1} \rangle$ and $l_{i+1} = \langle A_i, A_{i+2} \rangle$ intersect in a point $B_{i,i+1}$ (indices considered modulo n). Let $c_{i,i+1}$ be the circle through A_i, $B_{i,i+1}$ and A_{i+1}, and let g_i be the radical axis of the circles $c_{i-1,i}$ and $c_{i,i+1}$.*

If the lines $g_1, g_2, \ldots, g_{n-3}$ lie in a pencil, then the remaining three radical axes g_{n-2}, g_{n-1} and g_n lie in the same pencil.

We prove the theorem under weaker assumptions and in a more general setting. As shown in [3], the theorem is a result in affine geometry: a radical axis g_i can be constructed by drawing parallel lines.

Publication Date: October 23, 2015. Communicating Editor: Paul Yiu.

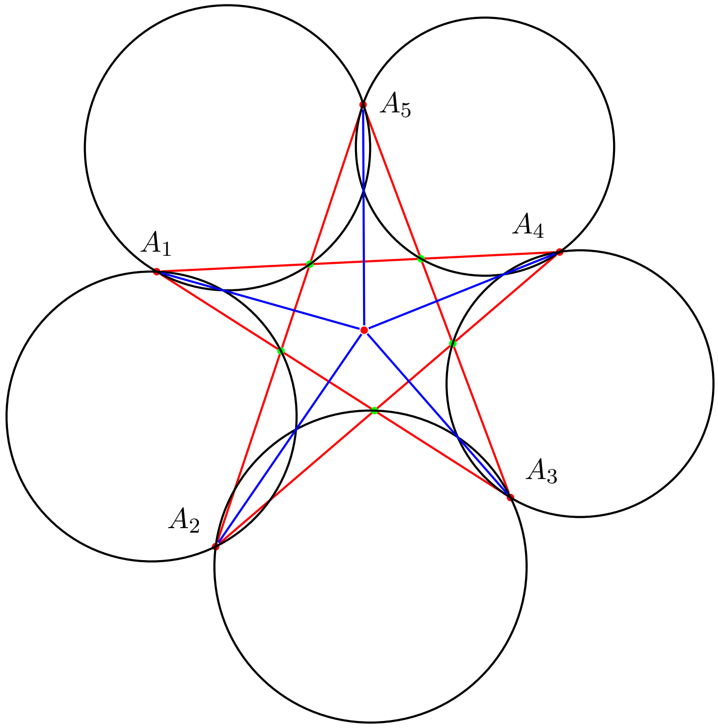

Figure 1. The 5-circle theorem

We can relax the condition that no three points lie on a line. In fact, the theorem continues to hold in certain limiting cases, if the elements of the construction are suitably reinterpreted. We make one case for $n = 5$ explicit for later use.

2. Preliminaries

We work in the affine plane $\mathbb{A}^2(k)$ over an arbitrary field k, which we view as embedded in $\mathbb{P}^2(k)$. All lines considered are projective lines. Two lines (different from the line at infinity) are parallel if their intersection point is a point at infinity. A general reference for this section is the book [1].

Definition. Let (P, Q) and (R, S) be two pairs of finite points on a line l; it is allowed that $P = Q$ or $R = S$, but neither R nor S may coincide with P or Q. Let $A \notin l$ be a finite point. Denote by l_P be the line through P that is parallel to the line $\langle A, R\rangle$ and take $l_S \parallel \langle A, Q\rangle$ through S. Set $B = l_P \cap l_S$. The line $g = \langle A, B\rangle$ is the *axis* of the configuration, see Figures 2 and 3.

The difference $P - Q$ of two points in the affine plane is a well defined vector in the associated vector space. For points P, Q, R, S on a line with $R \neq S$, the vector $P - Q$ is a scalar multiple of the vector $R - S$, so the ratio $\frac{P-Q}{R-S}$ is an element of the ground field k. We use the convention that $\frac{P-Q}{P-R} = 1$ if P lies at infinity and Q and R are distinct finite points.

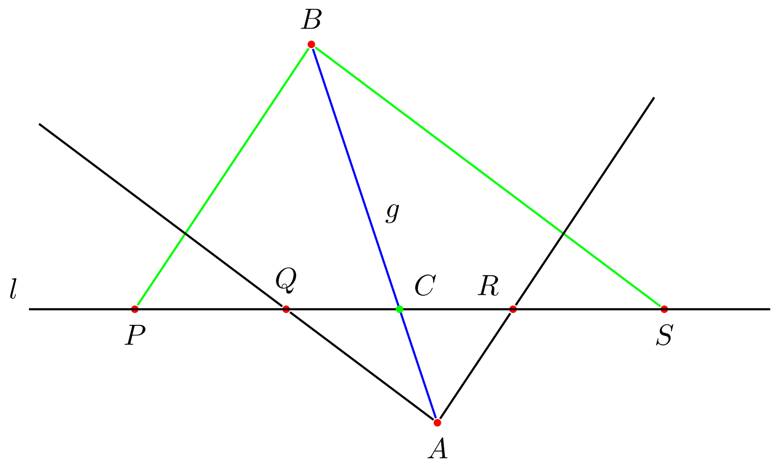

Figure 2. Construction of the axis

Lemma 3. *The intersection point $C = g \cap l$ is determined by the equivalent conditions*

$$\frac{C-Q}{C-R} = \frac{Q-S}{R-P},$$

which in case $P \neq Q$ is equivalent to

$$\frac{C-Q}{C-P} = \frac{R-Q}{R-P}\frac{S-Q}{S-P}$$

and to

$$\frac{C-S}{C-R} = \frac{Q-S}{Q-R}\frac{P-S}{P-R}$$

in case $R \neq S$.

Notation. We denote the point so determined by $C = [P, Q \mid R, S]$.

The lemma can be proved by direct computation. It also follows (if the four points P, Q, R and S are all distinct) from [3, Lemma 1] and its corollary, which

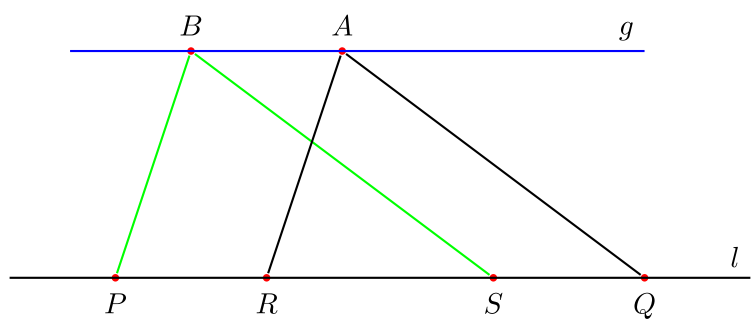

Figure 3. Axis parallel to the line

moreover establish that the above affine definition of the axis gives the radical axis of circles as in Figure 4, in the context of general affine metric planes.

Remark 1. For the euclidean plane these properties can easily be established with geometric arguments. To prove the lemma we use similarity of triangles in Figure 2, in case C is a finite point. We have that $\Delta BCP \sim \Delta ACR$ and $\Delta BCS \sim \Delta ACQ$. Therefore

$$\frac{C-P}{C-R} = \frac{C-B}{C-A} = \frac{C-S}{C-Q}\ .$$

It follows that

$$\frac{R-P}{C-R} = \frac{C-P}{C-R} - 1 = \frac{C-S}{C-Q} - 1 = \frac{Q-S}{C-Q}\ .$$

In the case that C lies at infinity (Figure 3) we have $R-P = B-A = Q-S$.

To find the axis as radical axis we add circles to the figure (see Figure 4).

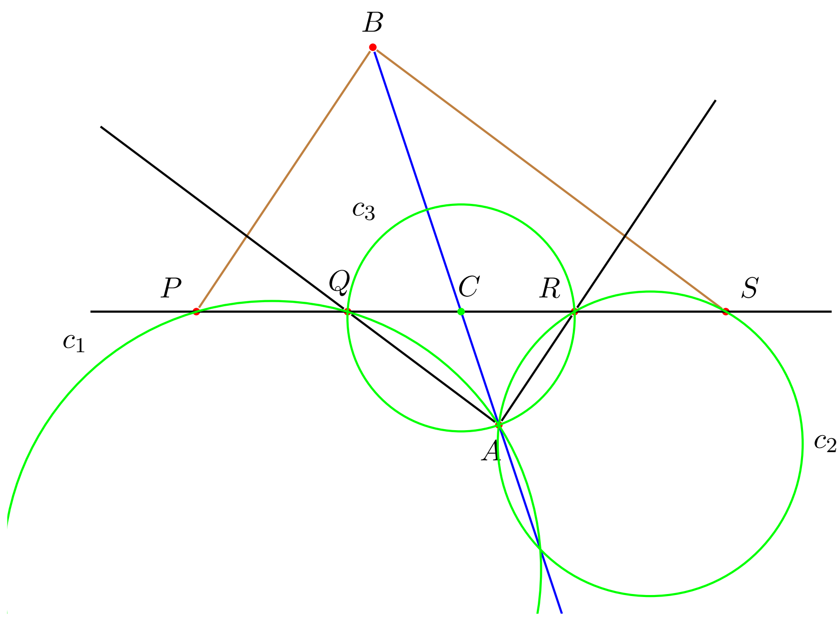

Figure 4

Let c_1 be the circle through A, P, Q and c_2 the circle through A, R, S. If $P = Q$, then c_1 is the circle through A which is tangent to the line l in the point $P = Q$; if $R = S$, the circle c_2 is tangent to l. Consider also the circle c_3 through A, Q and R. Then c_1 and c_3 intersect in A and Q, so the line $\langle A, Q\rangle$ is the radical axis of c_1 and c_3. The parallel line l_S is the locus of points for which the power with respect to c_1 has constant difference with the power with respect to c_3, the difference being $(S-P)\cdot(S-Q)-(S-Q)\cdot(S-R) = (S-Q)\cdot(R-P)$. The line l_P is the locus where the power with respect to c_2 differs from the power with respect to c_3 by the same quantity, as $(P-S)\cdot(P-R)-(P-R)\cdot(P-Q) = (P-R)\cdot(Q-S)$.

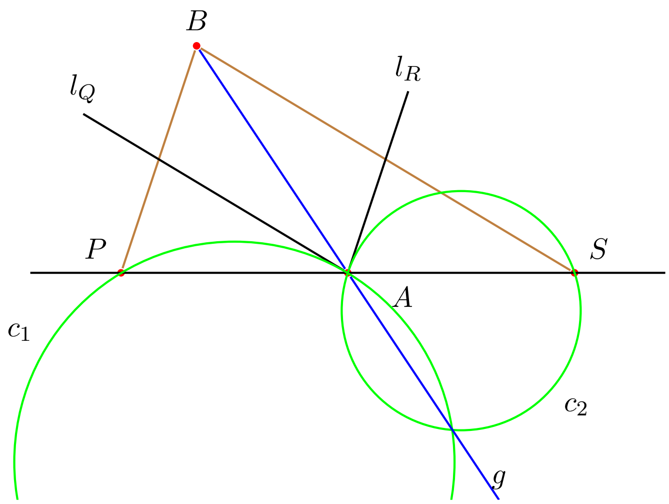

Figure 5

Therefore the intersection point $B = l_S \cap l_P$ lies on the radical axis of c_1 and c_2, so this radical axis is the axis $g = \langle A, B\rangle$.

In the situation of Figure 3 the center of the circle c_1 lies on the perpendicular bisector of PQ, which is also the perpendicular bisector of RS, on which the center of c_2 lies. Therefore the radical axis is parallel to l and B lies on it.

Lemma 4. *Given C and (R, S) on l, the map $\gamma\colon l \to l$, sending $X \in l$ to the point $\gamma(X)$, determined by $C = [X, \gamma(X) \mid R, S]$ is an involutive projectivity.*

Proof. To find $\gamma(X)$ we choose a point $A \notin l$ and draw the line l_X through X, parallel to $\langle R, A\rangle$ (see Figures 2 and 3, reading X and $\gamma(X)$ for P and Q). It intersects the line g in a point Y. Through Y we draw the line $l_S = \langle Y, S\rangle$. Then we draw a line m through A parallel to l_S and define $\gamma(X) = l \cap m$. This construction can be described as first projecting the line l from the point at infinity on the line $\langle A, R\rangle$ onto the line g, then projecting G from S onto the line l_∞ at infinity and finally projecting l_∞ onto l from A. This shows that the map γ is a projectivity.

That $\gamma^2 = \mathrm{id}$ can be seen from the formulas in Lemma 3 or by observing that γ interchanges R with S, and C with the point at infinity on the line l. □

Remark 2. Given the involution $\gamma\colon l \to l$ the point C is determined as the image of the point at infinity on the line l.

Remark 3. The point C on l is determined by the unordered pairs (P, Q) and (R, S), independent of the point A outside the line. We have emphasized the construction using a particular choice of points (Q and R) connected to A, as the construction with the points $A_1, \ldots, A_n$ naturally leads to this situation: the line $l = l_i$ is determined by the points points $P = A_{i-1}$ and $S = A_{i+1}$, while $Q = B_{i-1,i}$

and $R = B_{i,i+1}$ arise as intersection points of l with the lines $l_{i-1} = \langle A_{i-2}, A_i \rangle$ and $l_{i+1} = \langle A_i, A_{i+2} \rangle$. This extra structure makes it possible to define the axis if $A_i \in l_i$; in such a case there would be no involution on the line l_i.

Let A be a point on the line $l = \langle P, S \rangle$, different from P and S and let l_Q and l_R be two lines through A. Denote by B the intersection point of the line l_P through P, parallel to l_R and l_S through S, parallel to l_Q. We define the axis of this configuration as the line $\langle A, B \rangle$. In the case of the Euclidean plane it is the radical axis of the circle though P, tangent to l_Q in A, and the circle through S, tangent to l_R in A. The proof of Remark 1 extends to this situation, with the circle c_3 reduced to the point $A = Q = R$ (compare Figure 5 with Figure 4).

3. An n-axes theorem

We now formulate our main theorem.

Theorem 5. *Let $A_1, \ldots, A_n$ be a sequence of $n \geq 5$ distinct points in $\mathbb{A}^2(k)$, and define $l_i = \langle A_{i-1}, A_{i+1} \rangle$ (indices considered modulo n). Assume that*

(i) $A_i \notin l_{i-2}, l_i, l_{i+2}$,
(ii) $l_{i-1} \neq l_{i+1}$,
(iii) $l_i \nparallel l_{i+1}$,

and set $B_{i,i+1} = l_i \cap l_{i+1}$, $C_i = [A_{i-1}, B_{i-1,i} | B_{i,i+1}, A_{i+1}]$, and, finally, let $g_i = \langle A_i, C_i \rangle$ be the axis through A_i. If the $n-3$ axes $g_1, g_2, \ldots, g_{n-3}$ lie in a pencil, then the remaining three axes g_{n-2}, g_{n-1}, g_n lie in the same pencil.

As $A_i \in l_{i+1} = \langle A_i, A_{i+2} \rangle$ but $A_i \notin l_i$ by assumption (i), we have that $l_i \neq l_{i+1}$ and therefore assumption (iii) guarantees the existence of the point $B_{i,i+1}$ as a well-defined finite point.

By (i) and (ii) the points A_{i-1}, $B_{i-1,i}$, $B_{i,i+1}$ and A_{i+1} are four distinct points on the line l_i and A_i is a point outside, so that the axis g_i is defined. The condition $l_{i-1} \neq l_{i+1}$ means that A_{i-2}, A_i and A_{i+2} are not collinear. It is therefore equivalent to each of the conditions $A_{i-2} \notin l_{i+1}$ and $A_{i+2} \notin l_{i-1}$. Therefore the assumptions (i) and (ii) can be replaced by

(iv) $A_i \notin l_{i-3}, l_{i-2}, l_i, l_{i+2}, l_{i+3}$.

In particular this means that for $n \leq 6$, (i) and (ii) together are equivalent to the condition that no three points are collinear. Therefore the Theorem holds for $n = 5$ and $n = 6$ by the results of [3].

Definition. We call the common (finite or infinite) point of the pencil $\{g_i\}$ the *center* of the sequence $A_1, \ldots, A_n$.

4. A degenerate case of the 5-axes theorem

The 5-axes theorem states that for five points $A_1, \ldots, A_5$ in the plane, no three collinear, and $\langle A_{i-1}, A_{i+1} \rangle \nparallel \langle A_i, A_{i+2} \rangle$, the five axes $g_1, \ldots, g_5$ lie in a pencil. Motivated partly because they will be required later, but also because they are themselves of some interest, we study in this section some special and limiting cases. We first consider when the center is a point at infinity. More generally, we investigate the relationship of the center to the position of the initial five points.

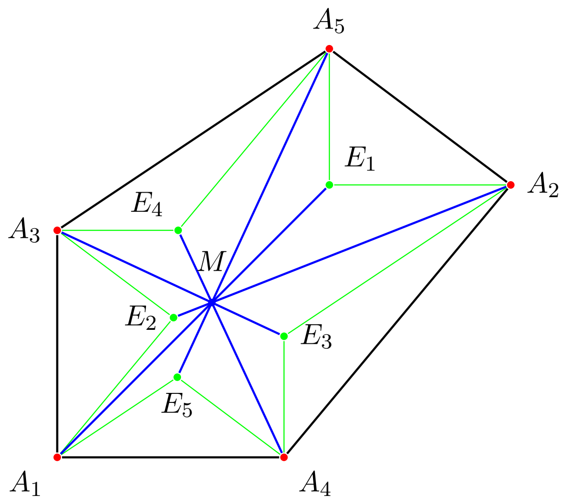

Figure 6

Theorem. *Consider four points A_1, A_2, A_3 and A_4 in an affine plane $\mathbb{A}^2(k)$, such that no three are collinear and such that $l_2 = \langle A_1, A_3\rangle$ is not parallel to $l_3 = \langle A_2, A_4\rangle$. A point A_5 in the plane, such that the assumptions of the 5-axes theorem are satisfied* (*i.e.,* A_5 *does not lie on a line* $\langle A_i, A_j\rangle$, *while* $l_i \nparallel l_{i+1}$ *for all* $i \neq 2$) *determines a center M in the extended plane $\mathbb{P}^2(k)$. The correspondence $A_5 \mapsto M$ is the restriction of a projective transformation $\mathbb{P}^2(k) \to \mathbb{P}^2(k)$. In particular, the locus of points A_5 for which M is a point at infinity* (*i.e., for which the axes are parallel*) *is a line.*

Proof. This is a computation. We construct the axis g_i from the intersection point E_i of the line through A_{i-1}, parallel to $\langle A_i, B_{i,i+1}\rangle = \langle A_i, A_{i+2}\rangle$, with the parallel to $\langle A_i, B_{i,i-1}\rangle = \langle A_i, A_{i-2}\rangle$ through A_{i+1}, see Figure 6.

We use homogeneous coordinates and take $A_1 = (0 : 0 : 1)$, $A_3 = (0 : 1 : 1)$, $A_4 = (1 : 0 : 1)$, $A_2 = (a : b : c)$ and $A_5 = (x : y : z)$.

The point E_1 is easily seen to be $(cx : bz : cz)$. We compute $E_4 = (bx + (c - a)y + (a - c)z : bz : bz)$ and find M as the intersection of the axes $g_1 = \langle A_1, E_1\rangle$ and $g_4 = \langle A_4, E_4\rangle$. The result is

$$M = (cx : bz : (c - b)x + (a - c)y + (c - a + b)z) \ .$$

In particular, M is at infinity if and only if $(c-b)x + (a-c)y + (c-a+b)z = 0$, which is the equation of a line whose slope is $\frac{c-b}{c-a}$. □

Remark 4. With a little more effort one can compute all points E_i and check that M lies on all axes $g_i = \langle A_i, E_i\rangle$. This gives a computational proof of the five-axes theorem.

Our stipulation that the conditions of the five-axes theorem be satisfied was sufficient for defining the five axes. But the resulting formula for M makes sense under more general circumstances, indicating that the theorem also holds in degenerate

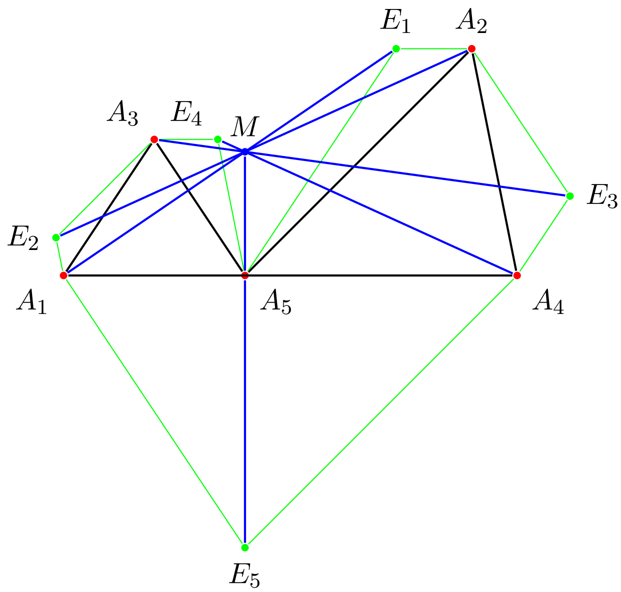

Figure 7. $A_5 \in l_5$

cases with a suitable definition of the axes. The point M fails to be determined only if $cx = bz = -bx + (a-c)(y-z) = 0$. When A_2 and A_5 are finite points ($c \neq 0$ and $z \neq 0$), this happens if either $A_2 = A_4$ and $A_5 \in \langle A_1, A_3\rangle$ or $A_5 = A_3$ and $A_5 \in \langle A_1, A_4\rangle$. If, say, A_5 lies at infinity ($z = 0$), then $A_5 = \langle A_1, A_3\rangle \cap \langle A_2, A_4\rangle$. Note that our coordinates are based on the assumption that A_1, A_3, A_4 form a triangle. In general we can say that the center is undefined when for some i, A_{i-1} coincides with A_{i+1} and the remaining three points are collinear, or when $\langle A_{i-1}, A_{i-3}\rangle \parallel \langle A_{i+1}, A_{i+3}\rangle$ with A_i being their intersection point at infinity, or when all five points are collinear. Moreover, if M is defined, but coincides with the point A_i, then the axis g_i is not defined.

We focus now on one degenerate case, which we need later, in which three consecutive points are collinear: $A_i \in l_i = \langle A_{i-1}, A_{i+1}\rangle$. We have that $A_i = l_{i-1} \cap l_{i+1} = l_{i-1} \cap l_i \cap l_{i+1}$, so $A_i = B_{i-1,i} = B_{i,i+1}$. In this case the axis g_i can be defined as in Remark 3.

Theorem 6. *Let five points A_1, A_2, A_3, A_4 and A_5 in the affine plane be given such that $A_5 \in \langle A_1, A_4\rangle$, but no other three points are collinear. Assume that $l_i = \langle A_{i-1}, A_{i+1}\rangle$ is not parallel to $l_{i+1} = \langle A_i, A_{i+2}\rangle$. Then the five axes g_1, g_2, g_3, g_4 and g_5 lie in a pencil.*

The computation, alluded to in Remark 4, also covers this degenerate case, illustrated in Figure 7. The geometric proof of the 5-circle theorem in [2, 3] can be extended to this situation to show that the four axes g_1, g_2, g_3 and g_4 lie in a pencil. If g_i is considered as radical axis of the circles $c_{i-1,i}$ and $c_{i,i+1}$, this suffices to conclude that all five radical axes lie in a pencil: if the center M is a finite point, then the fact that M lies on g_1, g_2, g_3 and g_4 implies that the power of M with

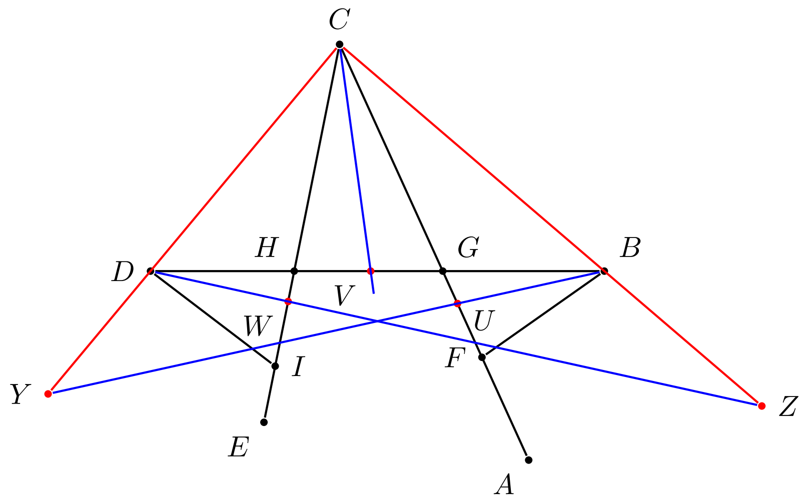

Figure 8

respect to $c_{5,1}$ is equal to the power with respect to $c_{1,2}$, equal to the power with respect to $c_{2,3}$, $c_{3,4}$ and $c_{4,5}$. As the power of M with respect to $c_{4,5}$ is equal to that that with respect to $c_{5,1}$, the point M lies on the radical axis g_5. If the center M is infinite, then the centers of all circles involved are collinear.

The main ingredient of the geometric proof is Lemma 2 of [2, 3], which we now recall.

Lemma 7. *Let A, C and E be three non collinear points in $\mathbb{A}^2(k)$, and let A, C, F, G be collinear, just as C, E, H, I and B, D, G, H (see Figure 8). Let $U = [A, F \mid C, G]$, $V = [H, D \mid B, G]$ and $W = [C, H \mid E, I]$. Then the lines $\langle B, U\rangle$, $\langle C, V\rangle$ and $\langle D, W\rangle$ lie in a pencil if and only if*

$$\frac{B-G}{B-H}\frac{E-H}{E-C}\frac{F-C}{F-G} = \frac{D-H}{D-G}\frac{A-G}{A-C}\frac{I-C}{I-H}\ . \tag{1}$$

Lemma 8. *The above lemma also holds if A and F coincide (see Figure 9).*

Proof. The proof follows [2, 3]. Let $Y = \langle B, U\rangle \cap \langle C, D\rangle$ and $Z = \langle D, W\rangle \cap \langle C, B\rangle$. By Ceva's theorem, applied to ΔBCD and its cevians $\langle B, Y\rangle$, $\langle C, V\rangle$ and $\langle D, Z\rangle$, the lines $\langle B, U\rangle$, $\langle C, V\rangle$ and $\langle D, W\rangle$ lie in a pencil if and only if

$$\frac{Y-C}{Y-D}\frac{V-D}{V-B}\frac{Z-B}{Z-C} = -1\ .$$

Menelaus' theorem first for ΔCDG and the points B, U and Y and then for ΔCBH and the points D, W and Z gives

$$\frac{Y-C}{Y-D} = \frac{U-C}{U-G}\frac{B-G}{B-D} \quad\text{and}\quad \frac{Z-B}{Z-C} = \frac{D-B}{D-H}\frac{W-H}{W-C}\ .$$

The condition $W = [C, H \mid E, I]$ gives by Lemma 3 that $\frac{W-C}{W-H} = \frac{E-C}{E-H}\frac{I-C}{I-H}$, while $V = [H, D|B, G]$ gives $\frac{V-D}{V-B} = \frac{D-G}{B-H}$ and finally $U = [C, G|A, A]$ implies

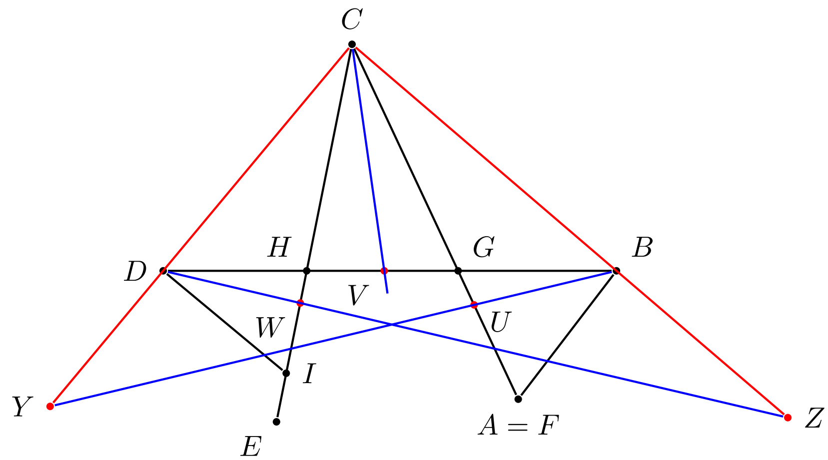

Figure 9

$\frac{U-G}{U-C} = (\frac{A-G}{A-C})^2$. Plugging these expression in in the equation and rearranging gives that $\langle B, U\rangle$, $\langle C, V\rangle$ and $\langle D, W\rangle$ lie in a pencil if and only if

$$\frac{A-C}{A-G}\frac{B-G}{B-H}\frac{E-H}{E-C} = \frac{A-G}{A-C}\frac{I-C}{I-H}\frac{D-H}{D-G}\ .$$

□

Proof that $g_1, \ldots, g_4$ lie in a pencil. In order to show that the lines g_1, g_2 and g_3 lie in a pencil, we verify the condition of Lemma 8 with $(B, C, D, E, A = F) = (A_1, A_2, A_3, A_4, A_5 = B_{5,1})$, $(G, H, I, U, V, W) = (B_{1,2}, B_{2,3}, B_{3,4}, C_1, C_2, C_3)$, where $C_i = l_i \cap g_i$. Both sides of the equation are equal to 1 by Menelaus' theorem applied to ΔCGH, on the left with the collinear points B, A and E, and on the right with D, I and A. Similarly one shows that g_2, g_3 and g_4 lie in a pencil. □

5. Six points

For six points the axes in general do not lie in a pencil.

Theorem 9. *Let six points $A_1, \ldots, A_6$, be given, no three collinear and such that the six points $B_{i,i+1} = \langle A_{i-1}, A_{i+1}\rangle \cap \langle A_i, A_{i+2}\rangle$ are finite. Then the following are equivalent:*

(1) the six axes $g_1, \ldots, g_6$, lie in a pencil
(2) for some i the axes g_{i-1}, g_i, g_{i+1} lie in a pencil,
(3) the main diagonals of the hexagon $A_1A_2A_3A_4A_5A_6$ lie in a pencil,
(4) the six points $B_{i,i+1}$ lie on a conic.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (1):
We show that condition (3) is equivalent to g_{i-1}, g_i, g_{i+1} lying in a pencil for all i. But as the condition on the main diagonals does not single out three lines, it suffices to prove equivalence for one specific i, say $i = 3$.

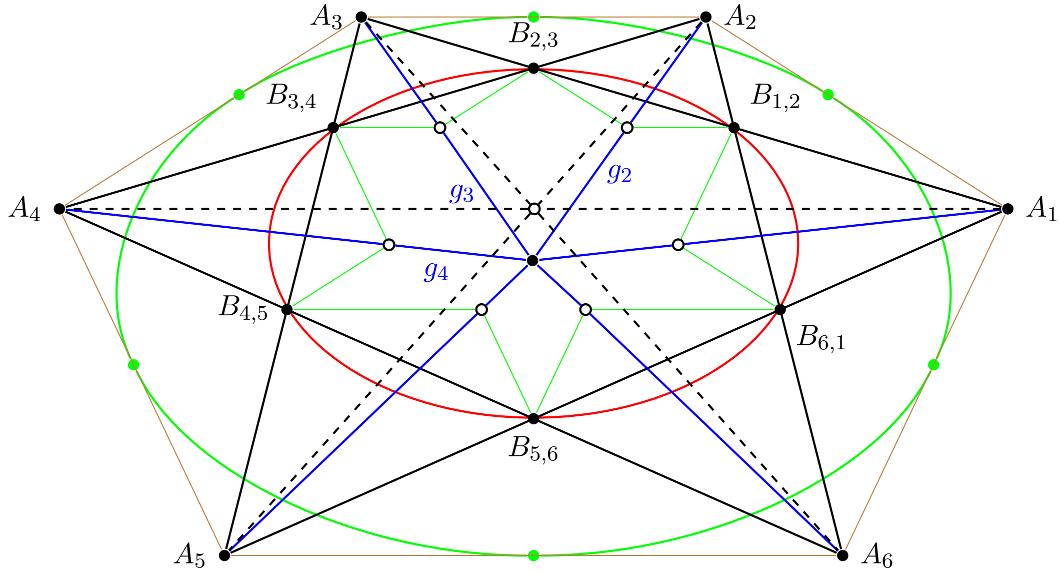

Figure 10

We take affine coordinates (x, y) with A_3 as origin, $A_2 = (0, 1)$, $A_4 = (1, 0)$, $A_1 = (a, b)$, $A_6 = (c, d)$ and $A_5 = (e, f)$. We compute $B_{2,3} = (\frac{a}{a+b}, \frac{b}{a+b})$, $B_{3,4} = (\frac{e}{e+f}, \frac{f}{e+f})$, $B_{1,2} = (\frac{ac}{bc-ad+a}, \frac{bc}{bc-ad+a})$ and $B_{4,5} = (\frac{ed}{ed-fc+f}, \frac{fd}{ed-fc+f})$. The condition (1) of Lemma 7 (with the labels $A, \dots, I$ applied, in order, to A_1, $\dots$, A_5, $B_{1,2}$, $B_{2,3}$, $B_{3,4}$, $B_{4,5}$) then becomes

$$\frac{a(e+f)}{(a+b)e} \cdot \frac{e+f-1}{e+f} \cdot \frac{c(a+b)}{(c+d-1)a} = \frac{f(a+b)}{(e+f)b} \cdot \frac{a+b-1}{a+b} \cdot \frac{d(e+f)}{f(c+d-1)} ,$$

which simplifies to

$$(e+f-1)cb = (a+b-1)de . \tag{2}$$

Here we used $a + b \neq 0$ (as $l_2 \nparallel l_3$), $e + f \neq 0$ and $a \neq 0$ (as $A_2 \notin l_2$), $f \neq 0$ and $c + d \neq 1$ (as $A_6 \notin l_3$).

The diagonal $\langle A_3, A_6\rangle$ has equation $dx - cy = 0$, the diagonal $\langle A_1, A_4\rangle$ is given by $bx + (1-a)y = b$ and $\langle A_2, A_5\rangle$ by $(1-f)x + ey = e$. The condition that these three diagonals lie in a pencil is given by the vanishing of the determinant

$$\Delta = \begin{vmatrix} b & 1-a & -b \\ 1-f & e & -e \\ d & -c & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1-a-b & -b \\ 1-e-f & 0 & -e \\ d & -c & 0 \end{vmatrix} .$$

Computing this determinant with Sarrus' rule shows that $\Delta = 0$ if and only if equation (2) holds.

(3) $\iff$ (4):
The lines $\langle A_1, A_4\rangle$, $\langle A_2, A_5\rangle$ and $\langle A_3, A_6\rangle$ lie in a pencil if and only if the triangles $\Delta A_1A_3A_5$ and $\Delta A_4A_6A_2$ are perspective from a center which, by Desargues's theorem, holds if and only if they are perspective from an axis. Note that the line

$l_i = \langle A_{i-1}, A_{i+1}\rangle$ coincides with the line $\langle B_{i-1,i}, B_{i,i+1}\rangle$. Therefore the axis of perspectivity is also the Pascal line of the points $B_{i,i+1}$, whence these points lie on a conic if and only if the original three lines lie in a pencil. □

Remark 5. If $\operatorname{char} k \neq 2$ the hexagon $A_1A_2A_3A_4A_5A_6$ circumscribes a conic by Brianchon's theorem. This is not true in characteristic 2, as then all tangents to a conic pass through one point. Figure 10 illustrates the result in the euclidean plane. To make the conics clearly visible the axes g_i are constructed by drawing parallels through $B_{i-1,i}$ and $B_{i,i+1}$.

Remark 6. The above proof shows that under weaker conditions, the equivalence between the axes g_2, g_3 and g_4 lying in a pencil and the main diagonals lying in a pencil continues to hold. The condition (1) applied to $(A, B, C, D, E) = (A_1, A_2, A_3, A_4, A_5)$ does not involve the position of the point A_6. The proof, when written in homogeneous coordinates, therefore remains valid should A_6 lie at infinity ($l_1 \parallel l_5$), or should $A_6 \in l_2, l_4, l_6$. Also the degenerations $A_1 \in l_5$, $A_5 \in l_1$, $A_2 \in l_6$, $A_4 \in l_6$ or $l_5 \parallel l_6$, $l_1 \parallel l_6$ do not affect the conclusion.

6. The proof of the main result

We have now seen that Theorem 5 holds for extended versions of the cases $n = 5$ and $n = 6$. For $n \geq 7$ we find it convenient to assume that the axes $g_2, \dots, g_{n-2}$ lie in a pencil.

The proof of Theorem 5 proceeds by induction on the number of vertices. The idea is the following. Suppose $A_1, \dots, A_n$ are given with $g_2, \dots, g_{n-2}$ in a pencil. Then we construct a sequence A_1, A_2, $A_{3,4}$, $A_5, \dots, A_n$ of $n-1$ points by replacing A_3 and A_4 by the intersection $A_{3,4}$ of l_2 and l_5. For the new configuration the axes g_2, $g_{3,4}$, $g_5, \dots, g_{n-2}$ lie in a pencil with the same center, and the induction hypothesis applies, provided the configuration satisfies the assumptions of the theorem. Sometimes this will not be the case, but we shall see that without loss of generality, one can replace the given configuration by one which does satisfy the assumptions.

Three consecutive axes g_{i-1}, g_i, g_{i+1} are determined by seven points A_{i-3}, A_{i-2}, A_{i-1}, A_i, A_{i+1}, A_{i+2} and A_{i+3}. Let D_i be the (possibly infinite) intersection point of $l_{i-2} = \langle A_{i-3}, A_{i-1}\rangle$ and $l_{i+2} = \langle A_{i+1}, A_{i+3}\rangle$. The point D_i exists as $l_{i-2} \neq l_{i+2}$, because $A_{i+1} \notin l_{i-2}$. The axes g_{i-1}, g_i, g_{i+1} are also the axes through A_{i-1}, A_i, A_{i+1} in the hexagon $D_iA_{i-2}A_{i-1}A_iA_{i+1}A_{i+2}$. This hexagon does not necessarily satisfy all the conditions (i), (ii), (iii), but by Remark 6 less is needed to conclude that the lines g_{i-1}, g_i, g_{i+1} lie in a pencil if and only if the lines $\langle A_{i-2}, A_{i+1}\rangle$, $\langle A_{i-1}, A_{i+2}\rangle$ and $\langle A_i, D_i\rangle$ lie in a pencil (see also Figure 11). Only the three conditions $l_{i+1} \neq \langle A_{i+2}, A_{i-2}\rangle$, $l_{i-1} \neq \langle A_{i+2}, A_{i-2}\rangle$ and $l_{i-2} \neq l_{i+2}$ are not directly covered by the properties of the original configuration and the allowable degenerations from the remark. We already showed that $l_{i-2} \neq l_{i+2}$. If $l_{i-1} = \langle A_{i+2}, A_{i-2}\rangle$, then A_{i-2}, A_i and A_{i+2} are collinear, which whould imply that $l_{i-1} = l_{i+1}$, contradicting the condition (ii) for the original configuration; for the same reason $l_{i+1} \neq \langle A_{i+2}, A_{i-2}\rangle$. So the condition to test is indeed that each triple of lines $\langle A_{i-2}, A_{i+1}\rangle$, $\langle A_{i-1}, A_{i+2}\rangle$ and $\langle A_i, D_i\rangle$ lies in a pencil.

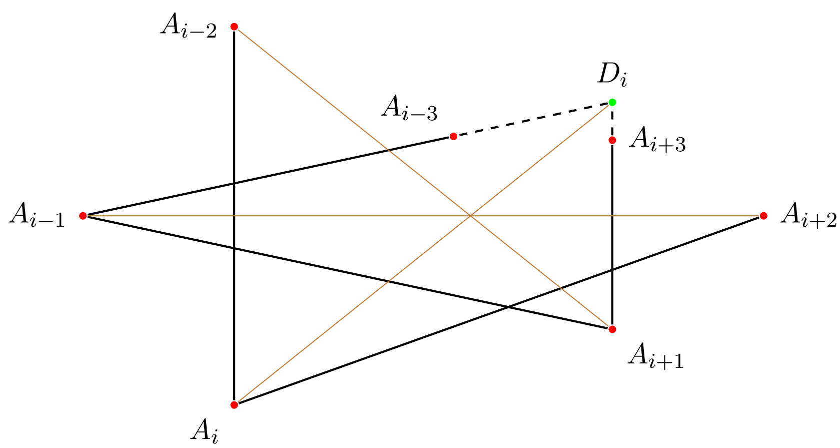

Figure 11

In the following lemma we consider a sequence of points $A_0, A_1, \ldots, A_6$, which may be part of a larger configuration. Because of the lemma's limited scope, we require only that the indices in the assumptions (i) – (iii) lie between 0 and 6.

Lemma 10. *Let A_0, A_1, …, A_5, A_6 be a sequence of distinct points satisfying the assumptions* (i) – (iii) *limited to indices between 0 and 6, such that the axes g_2, g_3 and g_4 lie in a pencil. Choose a point $A'_3 \in l_4$ with $A'_3 \neq B_{3,4}$ and $A'_3 \neq \langle A_1, A_2\rangle \cap l_4$. Let $P = \langle A_1, A_4\rangle \cap \langle A_2, A'_3\rangle$. Define the point $A'_2 \in l_1$ as $A'_2 = l_1 \cap \langle P, A_3\rangle$. Suppose that the sequence A_0, A_1, A'_2, A'_3, A_4, A_5, A_6 also satisfies the limited assumptions* (i) – (iii)*. Denote the axes of this new configuration by g'_i. Then $g_4 = g'_4$ and the the axes g'_2, g'_3 and g'_4 lie in the same pencil as g_2, g_3 and g_4. If moreover one of the axes g_1, g'_1 is defined and also lies in the same pencil, then $g'_1 = g_1$.*

Proof. The construction is illustrated in Figure 12. We want to apply the 6-axes theorem (Theorem 9) to the points A_1, A_2, A_3, A_4, A'_3, A'_2. Therefore we check that they are distinct and satisfy assumptions (i) – (iii).

By construction $A'_2 = A_2$ if and only if $A'_3 = A_3$, but then there is nothing to prove. We therefore assume $A'_3 \neq A_3$. This also gives $l'_3 \neq l_3$ and $l'_2 \neq l_2$. We have that $A'_3 \in l_4$; as $A_2 \notin l_4$ and $A_4 \notin l_4$, $A'_3 \neq A_2$ and $A'_3 \neq A_4$; similarly for A'_2. The only other requirements that do not follow from the assumptions on A_0, A_1, A_2, A_3, A_4, A_5, A_6 and A_0, A_1, A'_2, A'_3, A_4, A_5, A_6, are $A_2 \notin l'_2$, $A'_2 \notin l_2$, $A_3 \notin l'_3$ and $A'_3 \notin l_3$.

If $A'_3 \in l_3$, then $A'_3 = B_{3,4}$. If $A_3 \in l'_3 = \langle A'_2, A_4\rangle$, then $A_4 \in \langle A'_2, A_3\rangle \cap \langle A_1, A_4\rangle = \{P\}$, which again implies the excluded case $B_{3,4} = A_2A_4 \cap A_3A_5 = A_2P \cap A_3A_5 = A'_3$.

The condition $A'_3 \neq \langle A_1, A_2\rangle \cap l_4$ gives $A_2 \notin \langle A_1, A'_3\rangle = l'_2$. If $A'_2 \in l_2 = \langle A_1, A_3\rangle$, then $P \in \langle A_1, A_3\rangle$. As also $P \in \langle A_1, A_4\rangle$ this implies that $P = A_1$ and again $A_2 \in \langle A_1, A'_3\rangle = l'_2$.

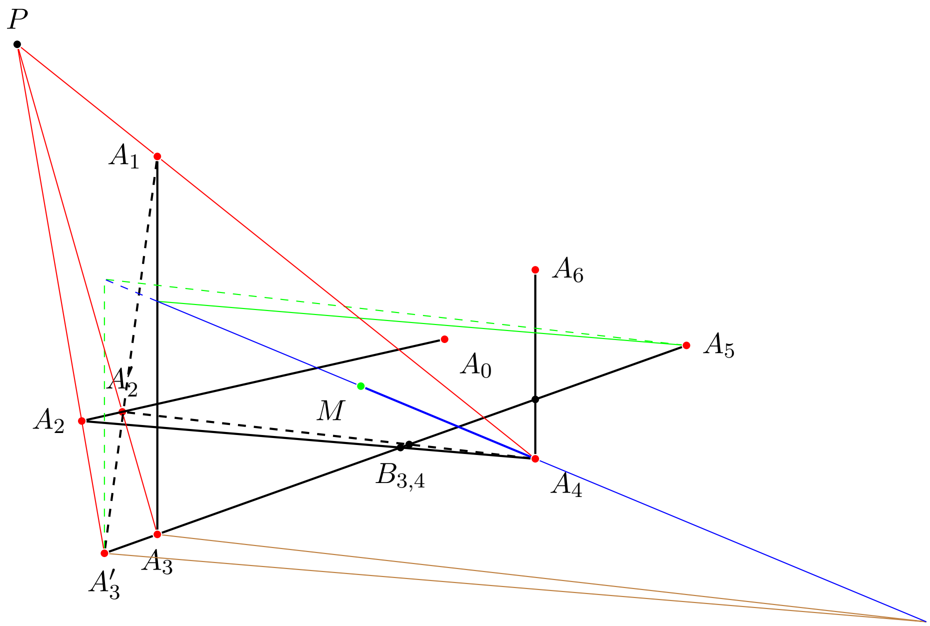

Figure 12

As $\langle A_1, A_4\rangle$, $\langle A_2, A_3'\rangle$ and $\langle A_3, A_2'\rangle$ lie in a pencil, the axes $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, $\bar{g}_4$, $\bar{g}_3'$ and $\bar{g}_2'$ of the hexagon $A_1A_2A_3A_4A_3'A_2'$ lie in a pencil. Because A_0, A_2 and A_2' are collinear and also A_5, A_3 and A_3', we have that $\bar{g}_2 = g_2$, $\bar{g}_3 = g_3$, $\bar{g}_2' = g_2'$ and $\bar{g}_3' = g_3'$.

As g_4 lies in the pencil of $g_2 = \bar{g}_2$ and $g_3 = \bar{g}_3$, the axis g_4 also coincides with $\bar{g}_4$. The axis g_4 is constructed as $\langle A_4, E_4\rangle$, with E_4 the intersection point of the parallel to $l_5 = \langle A_4, A_6\rangle$ through A_3 and the parallel to $l_3 = (A_4, A_2)$ through A_5. For $\bar{g}_4 = \langle A_4, \bar{E}_4\rangle$ one finds $\bar{E}_4$ as intersection point of the parallel to $\bar{l}_3 = l_3 = \langle A_4, A_2\rangle$ through A_3' and the parallel to $\bar{l}_3' = l_3' = \langle A_4, A_2'\rangle$ through A_3. For $g_4' = \langle A_4, E_4'\rangle$ one intersects the parallel to $l_5 = \langle A_4, A_6\rangle$ through A_3' with the parallel to $l_3' = (A_4, A_2')$ through A_5. By Pappus' theorem applied to the collinear points A_3, A_3' and A_5 and the points at infinity of the three lines l_3, l_3' and l_5 the points E_4, $\bar{E}_4$ and E_4' are collinear. As E_4 and $\bar{E}_4$ lie on $g_4 = \bar{g}_4$, the point E_4' also lies on it and therefore $g_4' = g_4$.

Furthermore, if g_1 lies in the pencil the same argument gives that $g_1' = g_1$. □

Remark 7. The requirement that the sequence A_0, A_1, A_2', A_3', A_4, A_5, A_6 also satisfy assumptions (i) – (iii) implies only finitely many forbidden positions for A_3'. Those can be made explicit. One finds that A_3' should not be equal to A_5, $\langle A_0, A_1\rangle \cap l_4$, $l_1 \cap l_4$, $l_5 \cap l_4$, $\langle A_1, A_4\rangle \cap l_4$ and also not equal to $l_4 \cap m_1$, where m_1 is the line through A_1, parallel to l_1. There will, of course, be a few more forbidden positions when the given points are part of a larger configuration. For A_2' one has corresponding forbidden positions. As one finds A_2' from A_3' by first projecting l_4

from A_2 onto the line $\langle A_1, A_4\rangle$ and then projecting from A_3 on the line l_1, those positions of A_2' yield further forbidden positions of A_3'.

This covers all assumptions except $l_2 \nparallel l_3'$, $A_2' \notin l_2'$ and $A_3' \notin l_3'$. They involve the position of the point $B_{2,3}$: in the first case it lies at infinity, in the second $B_{2,3} = A_2'$ and finally $B_{2,3} = A_3'$. The point $B_{2,3}'$ is the intersection $\langle A_1, A_3'\rangle \cap \langle A_4, B_{3,4}'\rangle$ and as $(A_3', B_{3,4}')$ is a pair of an involution on l_4 the point $B_{2,3}'$ moves on a (possibly degenerate) conic through A_1, A_4 and $B_{2,3}$, as A_3' moves on l_4. The intersection of this conic with the line at infinity, l_1 and l_4 gives at most six forbidden positions for $B_{2,3}'$ and therefore for A_3'.

On the other hand, because we could, if needed, embed the given plane in a plane over a field extension we can assume without loss of generality that there are infinitely many allowable positions for A_3' on l_4.

Proof of Theorem 5. Suppose distinct points $A_1, \dots, A_n$ ($n \geq 7$) are given, satisfying the assumptions (i), (ii), (iii) and such that the $n-3$ lines $g_2, \dots, g_{n-2}$ lie in a pencil. The lines l_2 and l_5 are not equal, as $A_4 \in l_5$, but $A_4 \notin l_2$.
Let

$$A_{3,4} = l_2 \cap l_5 \text{ (possibly at infinity)}, \quad l_{3,4} = \langle A_2, A_5\rangle,$$
$$B_3 = l_{6,4} \cap l_2 \text{ and } B_4 = l_{3,4} \cap l_5.$$

Consider the sequence of $n-1$ points A_1, A_2, $A_{3,4}$, $A_5, \dots, A_n$. Suppose first that $A_{3,4}$ is a finite point and that the sequence also satisfies the assumptions (i) – (iii), as in Figure 13.

The lines l_2 and l_5 occur both in the configuration of n points and of $n-1$ points, and also in the configuration formed by the five points A_2, A_3, A_4, A_5, $A_{3,4}$. Now $A_{3,4} \neq A_3$, as $A_{3,4} \in l_5$ but $A_3 \notin l_5$; similarly $A_{3,4} \neq A_4$.

We verify the conditions (i) – (iii) for the pentagon $A_2A_3A_4A_5A_{3,4}$. Most of them are conditions which also appear as conditions for A_1, A_2, $A_{3,4}$, $A_5, \dots, A_n$ or A_1, A_2, A_3, $A_4, \dots, A_n$. For (i) we note that $A_3 \notin l_{3,4}$, as A_2, A_3 and A_5 are not collinear, because $A_2 \notin l_4$; likewise $A_4 \notin l_{3,4}$. Also $A_{3,4} \notin l_4$, for otherwise $A_{3,4} = l_2 \cap l_4 = A_3$, similarly $A_{3,4} \notin l_3$. For (ii) we have $l_{3,4} \neq l_4$ (and similarly $l_{3,4} \neq l_3$) because A_2, A_3 and A_5 are not collinear.

Therefore the 5-axes theorem applies to the configuration A_2, A_3, A_4, A_5, $A_{3,4}$. Its axes $\bar{g}_2$, $\bar{g}_3$, $\bar{g}_4$, $\bar{g}_5$ and $\bar{g}_{3,4}$ lie in a pencil. As $\bar{g}_3$ coincides with the axes g_3 of the configuration A_1, A_2, A_3, $A_4, \dots, A_n$, and likewise $\bar{g}_4 = g_4$, and g_2 and g_5 lie in a pencil with g_3 and g_4, we find that also $\bar{g}_2 = g_2$ and $\bar{g}_5 = g_5$. By the same argument as in the previous proof we conclude that g_5 is also the axis through A_5 in the configuration A_1, A_2, $A_{3,4}$, $A_5, \dots, A_n$, and a similar statement holds for g_2. The axes $\bar{g}_{3,4}$ is also the axis through $A_{3,4}$ in the configuartion of $n-1$ points. Therefore the $n-4$ axes g_2, $g_{3,4}$, $g_5, \dots, g_{n-2}$ lie in a pencil and by the induction hypothesis the axes g_1, g_{n-1} and g_n lie in the same pencil, which is also the pencil of g_2, g_3, g_4, $g_5, \dots, g_{n-2}$.

If $A_{3,4}$ lies at infinity or coincides with one of the other points, or the configuration A_1, A_2, $A_{3,4}$, $A_5, \dots, A_n$ does not satisfy the assumptions (i) – (iii), we use the construction of Lemma 10 to replace A_1, A_2, A_3, A_4, $A_5, \dots, A_n$ by another

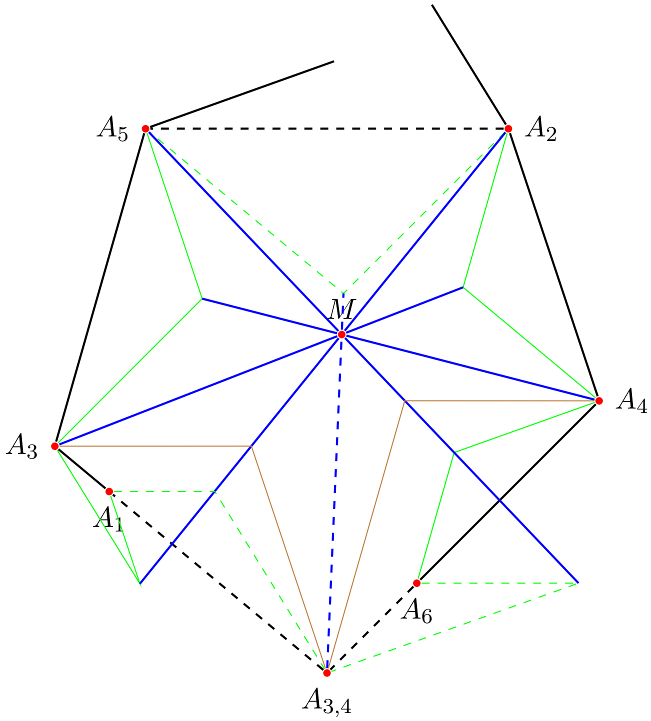

Figure 13

one $A_1', \ldots, A_n'$ with the same center, such that A_1', A_2', $A_{3,4}'$, A_5', …, A_n' does satisfy the assumptions. As mentioned earlier, the new sequence need not be defined over the field k; it suffices for the induction that it is defined over a field extension.

Some of the assumptions (i) – (iii) for the configuration A_1, A_2, $A_{3,4}$, A_5, …, A_n follow directly from the properties (i) – (iv) of the n points A_1, …, A_n, but for the others we have to modify the given configuration. We do this step by step. At each step we maintain $n-2$ points from the previous step and move the other two in a way that corrects one specific shortcoming (it is here that we might have to make use of a field extension). We then relabel the points so that the resulting configuration is free of all previous shortcomings, yet has the same center.

We now list the conditions and discuss how to satisfy them. We treat the cases which are connected by the symmetry $A_n \mapsto A_{7-n}$ together, postponing $A_{3,4} \notin l_{3,4}$ to the end.

- $l_2 \nparallel l_5$.
  This condition implies that the point $A_{3,4} = l_2 \cap l_5$ is a finite point, as desired. As $l_2 \neq l_4$, $A_1 \notin l_4$. Moving A_3 on l_4 means that the line l_2 moves in the pencil of lines through A_1, whereas l_4 does not change. Therefore, if we were given $l_2 \parallel l_5$, we could make these lines intersecting by moving A_2 and A_3.

- $A_1 \neq A_{3,4}$ and $A_6 \neq A_{3,4}$.
  If $A_{3,4} = A_6$, then $l_2 = \langle A_1, A_3 \rangle$ intersects $l_5 = \langle A_4, A_6 \rangle$ in A_6. Moving A_3 on l_4 means that l_2 moves in the pencil of lines with center A_1. As $A_6 \neq A_1$, this means that $A_{3,4}$ moves. If $A_{3,4} = A_1$, we move instead A_4 on l_3.
- $A_{3,4} \neq A_j$ for $j = 7, \ldots, n$.
  If $A_j = l_2 \cap l_5$, we move l_2 in the pencil of lines through A_1.
- $A_1 \notin l_{3,4}$ and $A_6 \notin l_{3,4}$.
  If $A_1 \in l_{3,4}$, we move $l_{3,4}$ in the pencil through A_5 by moving A_2 on l_1.
- $A_{3,4} \notin l_1$ and $A_{3,4} \notin l_6$.
  Moving A_2 and A_3 means that $A_{3,4}$ moves on $l_5 \neq l_6$, while moving A_4 and A_5 makes $A_{3,4}$ to move on $l_2 \neq l_1$.
- $l_1 \neq l_{3,4}$ and $l_6 \neq l_{3,4}$.
  This first condition means that A_2, A_5 and A_n are not collinear, and the second that A_2, A_5 and A_7 are not collinear. For $n = 7$ these conditions coincide and are satisfied because $l_6 \neq l_1$. Let $n > 7$ and suppose $A_5 \in l_1$. Then $A_5 = l_1 \cap l_4$ ($l_1 \neq l_4$ as $A_2 \notin l_4$). We can move A_5 and A_6, moving A_5 on l_4 off l_1. If $A_2 \in l_6$, then moving A_5 on l_4 moves l_6 in the pencil of lines through A_7.
- $l_2 \neq l_5$.
  This holds as $A_3 \notin l_5$.
- $l_2 \nparallel l_{3,4}$ and $l_5 \nparallel l_{3,4}$.
  If $l_5 \parallel l_{3,4}$ we move A_2 and A_3, moving A_2 on l_1. As $A_5 \notin l_1$ by a previous step, this means that $l_{3,4}$ moves, whereas l_5 does not move. If $l_2 \nparallel l_{3,4}$ we move A_4 and A_5.

The last condition to be satisfied is $A_{3,4} \notin l_{3,4}$. If $A_{3,4} \in l_{3,4}$, then $l_{3,4}$, l_2 and l_5 are concurrent and $A_{3,4} = B_3 = B_4$. Now the conditions for the degenerate case of the 5-axes theorem (Theorem 6) are satisfied. We find that $\bar{g}_2$, $\bar{g}_3$, $\bar{g}_4$ and $\bar{g}_5$ lie in a pencil. We conclude that $\bar{g}_5 = g_5$ also in this case.

We compute the image of $A_{3,4}$ under the involution on l_5 determined by A_4, l_4 and g_5, both when $A_{3,4} \in l_{3,4}$ and $A_{3,4} \notin l_{3,4}$. According to the proof of Lemma 4 we have to intersect the line through $A_{3,4}$, parallel to l_4 with g_5 and connect the intersection point with A_4. Then we draw parallel to this last line a line through A_5. The construction of the axis $\bar{g}_5 = g_5$ shows that the line through A_4 is parallel to $\langle A_2, A_5 \rangle$. Therefore the image of $A_{3,4}$ is $B_4 = l_5 \cap \langle A_2, A_5 \rangle$.

If $A_{3,4} = B_4$, then it is a fixed point of the involution and by moving A_2 on l_1 and A_3 on l_4 we move $A_{3,4}$ on l_5, so that it is no longer a fixed point of the involution, and therefore $A_{3,4} \neq B_4$, giving $A_{3,4} \notin l_{3,4}$.

This shows that we can satisfy all assumptions. For the new configuration with the same center M we can conclude by the induction hypothesis that also g_1, g_n and g_{n-1} pass through M. This then also holds for the original configuration. $\square$

References

[1] H. Eves, *A Survey of Geometry.* Boston etc: Allyn and Bacon Inc, revised ed., 1972

[2] J. C. Fisher, L. Hoehn and E. M. Schröder, A 5-circle incidence theorem, *Math. Mag.*, 87 (2014) 44–49.
[3] J. C. Fisher, L. Hoehn and E. M. Schröder, Zwei Kreistheoreme für affin-metrische Ebenen, *Mitt. Math. Ges. Hamburg*, 33 (2013) 167–180.

J. Chris Fisher: 2616 Edgehill Road, Cleveland Heights, Ohio 44106-2806, USA
E-mail address: fisher@math.uregina.ca

Eberhard M. Schröder: FB Mathematik der Universität, Bundesstrasse 55, D 20146 Hamburg, Germany
E-mail address: eberhard.schroeder@gmx.net

Jan Stevens: Mathematical Sciences, University of Gotheburg, Chalmers University of Technology, SE 412 96 Gothenburg, Sweden
E-mail address: stevens@chalmers.se