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Transforming Tripolar into Barycentric Coordinates

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Abstract. A simple construction is presented to find a point with given tripolar coordinates, i.e. the ratios of its distances to the points A, B, C of a reference triangle. This construction leads to a very nice transformation formula for tripolar into barycentric coordinates, that simplifies considerably an already existing transformation formula in Kimberling's *Encyclopedia of Triangle Centers*. The necessary and sufficient conditions for the constructibility are encoded in a triangle whose side lengths are products of the side lengths of ABC with the tripolar coordinates. Formulas for the area of this triangle are presented showing the role of inversion in this construction. As applications, one-line proofs for the formula of the pedal triangle area and the factorization of the dual of the circumcircle are given as well as simplifications of some formulas from ETC.

1. Introduction

When thinking about tripolar coordinates and looking for an idea how to transform results of [7] into the barycentric calculus, I was taken aback by the complicated transformation formulas from tripolar into barycentric coordinates I detected in the entry $X(5002)$ in [9]. Combining the techniques of inversion and of gluing similar figures, that played separate roles in my former FG articles, I found a simple construction leading to nice formulas.

2. Visual proof of Ptolemy's inequality

Gluing similar copies of ABD to CD and of BCD to AD leads to Ptolemy's inequality $\frac{bd}{f} \leq e + \frac{ac}{f}$ or

$$BC \cdot DA \leq AC \cdot DB + AB \cdot DC. \tag{1}$$

See Figure 1 and [3; p. 241, formula (49)], [5, pp. 42-43], [12, pp. 29ff].

This method is very likely behind the impressive calculations of Bretschneider [3]. In Figure 1 in [15], it leads to a short and visual proof of the maximum area property for cyclic quadrilaterals.

3. Existence of a point with given ratios of tripolar coordinates

Ptolemy's inequality (1) encodes the necessary condition for the existence of points D with given tripolar coordinates (see [1], [2], [6, pp. 6–10], [11]), i.e., the distances to the vertices A, B, C of a reference triangle have given ratios. Bottema

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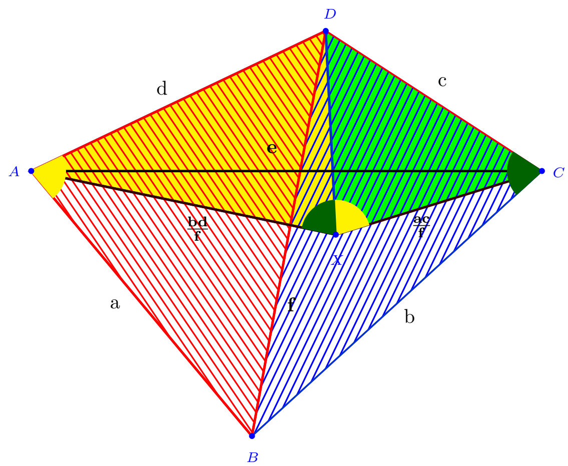

Figure 1

poses this problem in [1, Section 8.2] and proves analytically the following theorem by considering the intersections of Apollonian circles.

Theorem 1. *Given a triangle ABC and three positive numbers p, q, r. There exists (one or two) points D for which the ratios of its distances to A, B, C, are given by $DA : DB : DC = p : q : r$ if and only if we can draw a triangle, possibly degenerated, with sides $BC \cdot p$, $CA \cdot q$, $AB \cdot r$.*

A pure constructive proof, that the triangle inequalities $BC \cdot p \leq AC \cdot q + AB \cdot r$ etc. are equivalent to the existence of a point D for which $DA : DB : DC = p : q : r$, is based on the above mentioned gluing method.

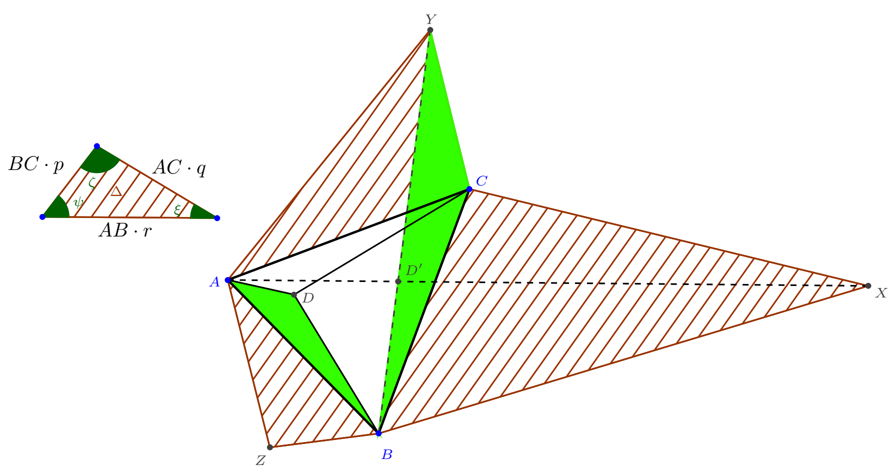

Figure 2

Proof. The necessity of the triangle inequalities, i.e. Ptolemy's inequalities, for the existence of D is obvious. To prove the sufficiency, assume first that Ptolemy's inequalities are satisfied strictly. Glue a similar copy CAY of the triangle Δ with side lengths $BC \cdot p$, $AC \cdot q$, $AB \cdot r$ to AC as in Figure 2 above. A rotation around B followed by a dilation moves triangle YBC onto a triangle ABD, transforming YB onto AB. A rotation around B followed by a dilation moves triangle YAB onto a triangle BCD, transforming YB onto BC. The images of C, respectively A, both called D, though a priori different, are indeed identical, because the angles of the rotated triangles at B sum up to $\angle ABC$, and the distances of the images of C, respectively A, to B are $a \cdot \frac{c}{YB} = c \cdot \frac{a}{YB}$. Hence, these images must coincide at a point D, which satisfies $DA : DB : DC = p : q : r$ since $DA : DB = \frac{ap}{q} : a = p : q$, and $DC : DB = \frac{cr}{q} : c = r : q$. $\square$

That there are in general two solutions $D_\pm$ to this problem becomes clear by the construction based on the Apollonius circles $XA : XB = p : q$, $XB : XC = q : r$, $XC : XA = r : p$. Two of these circles have in general two points of intersection that lie automatically on the third circle. This second solution can be obtained by gluing a similar copy of the nondegenerate triangle Δ onto the other side of AC and proceeding as above.

If Δ degenerates, suppose, for example, that $AB \cdot r + BC \cdot p = AC \cdot q$, the points A, B, C and D are concyclic by Ptolemy's theorem. The points X, Y, Z are on the sides of the triangle. In this example, Y is located in the interior of the segment AC, X and Z are on the extensions of BC and AB. AX, BY and CZ are parallel and meet at infinity, illustrating the fact that the isogonal conjugacy transforms points at infinity into points on the circumcircle, see [8; p.154, Theorem 234]. Figure 3 depicts this situation and serves also as a visual proof of Ptolemy's equation: $\frac{AB \cdot CD}{BD} + \frac{BC \cdot AD}{BD} = AC$.

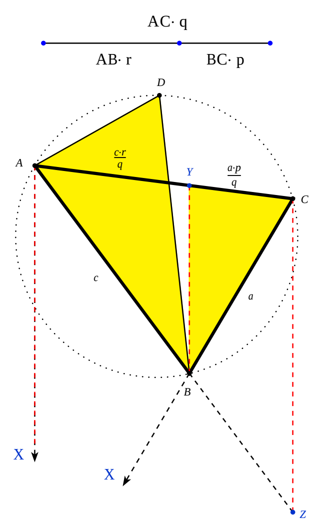

Figure 3

At the end of the next section we will have a look at the relationship between this construction and the inversion in the circumcircle k of ABC. In particular, we will see that $D_\pm$ are inverted into each other. This explains the number of solutions: two in the case of nondegenerate Δ, except for $p = q = r = 1$, with the circumcenter as the unique solution, and one in the case of degenerate Δ.

4. Transforming tripolar into barycentric coordinates

If we glue similar copies ABZ, BCX, CAY of the triangle Δ outward, respectively inward, to all sides of the original triangle we see that the two solution $D_\pm$ to Bottema's problem are the isogonal conjugates ([8, pp. 153-157]) of the intersections $D'_\pm$ of AX, BY and CZ; see Figure 2. That these lines are concurrent can easily be seen by drawing the circumcircles of ABZ, BCX, CAY, and showing that these circles meet at one point which, by angle chasing, is the intersection of AX, BY and CZ (see [13, Theorem 4.2]). Another proof is by barycentric calculus as in [16; §3.5.2].

This construction is well known in a particular case, namely, the definition of the isodynamic points as isogonal conjugates of the Fermat points. In this case, p, q, r are the reciprocals $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ of the side lengths and triangle Δ and its copies ABZ, BCX, CAY are equilateral. See [1, §23.2], [4, p. 303], [8, p. 295ff], and, as a curiosity, [13, p. 138], where the author decrees: "There may well be an existing name for D and E [the isodynamic points], but we shall call them the Napoleon points."

Let ξ, ψ, ζ be the angles of the triangle Δ, see Figure 2. By Conway's formula ([16, §3.4.2]), with $S_\theta = S \cdot \cot\theta$, $S = 2T_{ABC}$, and $2S_A = 2bc\cos\alpha = b^2+c^2-a^2$ etc., the barycentric coordinates of X, Y, Z are $X(-a^2 : S_C \pm S_\zeta : S_B \pm S_\psi)$, $Y(S_C \pm S_\zeta : -b^2 : S_A \pm S_\xi)$, $Z(S_B \pm S_\psi : S_A \pm S_\xi : -c^2)$.

Hence, the lines AX, BY, CZ intersect at $D'_\pm((S_A \pm S_\xi)^{-1} : (S_B \pm S_\psi)^{-1} : (S_C \pm S_\zeta)^{-1})$. Since the points $D_\pm$ with tripolar coordinates $(p : q : r)$ are isogonal conjugates of $D'_\pm$, we have just proved

Theorem 2. *The barycentric coordinates of the points D_ε, $\varepsilon = \pm 1$, with tripolar coordinates $(p : q : r)$ are*

$$(a^2(S_A + \varepsilon S_\xi) : \; b^2(S_B + \varepsilon S_\psi) : \; c^2(S_C + \varepsilon S_\zeta)). \tag{2}$$

The only place where I could find an expression for the barycentric coordinates $(x : y : z)$ of the points with given ratios $(p : q : r)$ of its distances from A, B, C is in the entry $X(5002)$ in ETC [9]:

$$\begin{aligned} x =& \; a^2S_A + k^2(-a^2p^2 + S_Cq^2 + S_Br^2), \\ y =& \; b^2S_B + k^2(S_C^2p^2 - b^2q^2 + S_Ar^2), \\ z =& \; c^2S_C + k^2(S_Bp^2 + S_Aq^2 - c^2r^2) \end{aligned} \tag{3}$$

for

$$k^2 = \frac{a^2p^2S_A + b^2q^2S_B + c^2r^2S_C + 2SS_\Delta}{a^2(p^2-q^2)(p^2-r^2) + b^2(q^2-r^2)(q^2-p^2) + c^2(r^2-p^2)(r^2-q^2)}.$$

where S_Δ is twice the area of the triangle with sides ap, bq and cr. A similar formula is derived in [4, p. 304], from Stewart's theorem, for $k = 1$ and the distances p, q, r, (and not only their ratios) of the point with barycentric coordinates $(x : y : z)$ from A, B, C. Casey calls this result Lucas's Theorem, probably referring to [10, p. 133]. But beware, there are printing errors.

Likewise, formula (2) can be written as

$$D_{\pm}\ (a^2(\cot A \pm \cot \xi) :\ b^2(\cot B \pm \cot \psi) :\ c^2(\cot C \pm \cot \zeta)), \tag{4}$$

$$D_{\pm}\ \left(a^2 \left(\frac{b^2+c^2-a^2}{2S} \pm \frac{(bq)^2+(cr)^2-(ap)^2}{2S_\Delta} \right) : \ \cdots : \ \cdots \right), \tag{5}$$

with $2S = \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$ and a similar expression with ap, bq, cr replacing a, b, c for $2S_\Delta$. These formulas are by far more transparent than those in (3). They will be applied in §7 to simplify the barycentric coordinate formulas of some triangle centers from [9]. In §5, formula (11), we will see that the ratio $\frac{S_\Delta}{S}$ is, up to a factor, just the power of $D_\pm$ with respect to the circumcircle k of ABC.

Let's see what happens if the triangle Δ degenerates. Calculating barycentric coordinates according to (2) and (3) fails, but by multiplying (4) by $2S_\Delta$, we can write the remaining unique point as

$$D\ \left(a^2(b^2q^2+c^2r^2-a^2p^2) :\ \cdots :\ \cdots\right). \tag{6}$$

Supposing again $AB \cdot r + BC \cdot p = AC \cdot q$ or $ap = bq - cr$, simplifies to $D\left(\frac{a}{p} : -\frac{b}{q} : \frac{c}{r}\right)$.

Theorem 3. *A point $D\ \left(\frac{a}{p} : -\frac{b}{q} : \frac{c}{r}\right)$, $p, q, r > 0$, is on the circumcircle k of the triangle ABC if and only if $ap + cr = bq$. In this case, $(DA : DB : DC) = (p : q : r)$.*

Proof. The "if" part being dealt with just before, we are left with the "only if" part that can be shown by putting $D\left(\frac{a}{p} : -\frac{b}{q} : \frac{c}{r}\right)$ into the circumcircle equation $k(x, y, z) = a^2yz + b^2zx + c^2xy = 0$. □

The easiest way to get the tangent equation to k at D is to write it as

$$k_x\left(\frac{a}{p} : -\frac{b}{q} : \frac{c}{r}\right)x + k_y\left(\frac{a}{p} : -\frac{b}{q} : \frac{c}{r}\right)y + k_z\left(\frac{a}{p} : -\frac{b}{q} : \frac{c}{r}\right)z = 0$$

and to simplify using $ap + cr = bq$. We obtain $p^2x + q^2y + r^2z = 0$.

We summarize the last calculations in Theorem 4 that can be used to get very short and nice solutions of tangency problems, e.g. the proof of Feuerbach's theorem.

Theorem 4. *A line $p^2x + q^2y + r^2z = 0$, $p, q, r \geq 0$, in barycentric coordinates $(x : y : z)$ is tangent to the circumcircle k of a reference triangle ABC if and only if*

$$(-ap + bq + cr)(ap - bq + cr)(ap + bq - cr) = 0. \tag{7}$$

Observe that a line $ux+vy+wz=0$ with coefficients u, v, w of different signs can never be tangent to k since it contains interior points of ABC. This result in its genuinely geometric form is Theorem 117 of [8, p. 89].

Theorem 5. *Let p, q, r be the tangent lengths from A, B, C to a circle K. Then K is tangent to the circumcircle k of ABC if and only if* (7) *is satisfied.*

Just observe that for tangent circles their radical axis, i.e. the zero set of the difference of their barycentric equations, is tangent to both and that the barycentric equations for k and K are related by

$$K(x,y,z) = k(x,y,z) - (x+y+z)(p^2x + q^2y + r^2z) = 0,$$

see [16, Proposition 7.2.3].

With the appearance of $(-ap+bq+cr)(ap-bq+cr)(ap+bq-cr)$ in Theorem 4, it is tempting to put (p^2, q^2, r^2) into the equation

$$k^*(u,v,w) = a^4u^2 + b^4v^2 + c^4w^2 - 2a^2b^2uv - 2b^2c^2vw - 2c^2a^2wu = 0$$

for the coefficients of the tangent lines $ux+vy+wz=0$ to k, i. e. the equation of the dual conic of k. It is barely a surprise that S_Δ appears in this factorization of the dual equation of k:

$$k^*(p^2,q^2,r^2) = 4S_\Delta^2 = (ap+bq+cr)(-ap+bq+cr)(ap-bq+cr)(ap+bq-cr). \tag{8}$$

What is the relation of all this with the inversion? The Apollonius circle $XA : XB = p : q$ occurring in the construction of the solutions of $DA : DB : DC = p : q : r$ belongs to the pencil with limit points A and B, since it intersects the line AB harmonically or, by an analytical argument, since its equation $q^2|X-A|^2 - p^2|X-B|^2 = 0$ is a linear combination of the point-circles A and B. Therefore, this circle is orthogonal to any circle through the limit points A and B, in particular to the circumcircle k. This being so also for the circles $XB : XC = q : r$ and $XC : XA = r : p$, these Apollonius circles, left invariant by an inversion with respect to k, will therefore intersect in points $D_\pm$, which are images of each other in this inversion.

Moreover, the involvement of inversion in this problem will become clear, if we compare the area

$$T_\Delta = \frac{1}{4}\sqrt{(ap+bq+cr)(-ap+bq+cr)(ap-bq+cr)(ap+bq-cr)}$$

of the triangle Δ with sides $ap = a \cdot DA$, $bq = b \cdot DB$, $cr = c \cdot DC$, (p, q, r being now the distances from a point D to the vertices of ABC, and not only their ratios) with the area T of ABC. Bretschneider ([3, p. 241]) calls T_Δ the excentric area ("excentrische Fläche") of the quadrilateral $ABCD$. It expresses, as a numerical value for Ptolemy's theorem, the extent of the deviation of the quadrilateral from being cyclic. Using inversion, a nice formula connecting the areas of ABC and Δ will be derived in the next section.

5. Excentric area of a quadrilateral

Let us apply an inversion with respect to a circle $\kappa(D,\ \rho)$ to the triangle ABC with sides $a = BC$, $b = CA$, $c = AB$, circumcircle $k(O, R)$ and area T. The image is a triangle $A'B'C'$ with sides $a' = B'C'$, $b' = C'A'$, $c' = A'B'$, circumcircle $k'(O', R')$ and area T'.

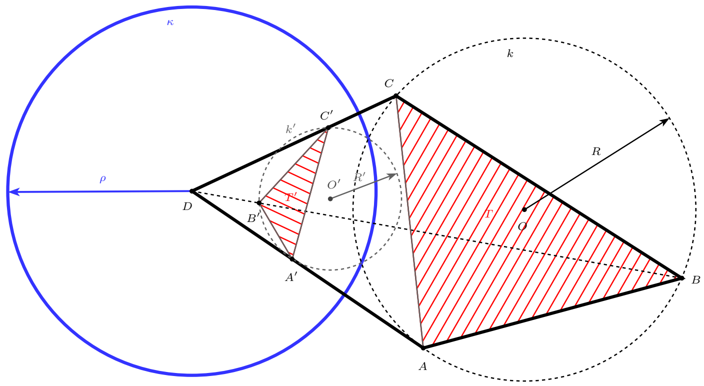

Figure 4

The side lengths transform according to

$$a' = \frac{\rho^2}{DB \cdot DC} \cdot a, \quad b' = \frac{\rho^2}{DC \cdot DA} \cdot b, \quad c' = \frac{\rho^2}{DA \cdot DB} \cdot c.$$

This leads to

$$T' = \frac{a'b'c'}{4R'} = \frac{\rho^6}{DA^2 \cdot DB^2 \cdot DC^2} \cdot \frac{abc}{4R'} \tag{9}$$

for the area T'. Inserting $R' = \frac{\rho^2}{|DO^2 - R^2|^2} \cdot R = \frac{\rho^2}{|P(D,k)|} R$ with $P(D, k) = DO^2 - R^2$, power of D with respect to k, into (9), we obtain

$$T' = \frac{\rho^4 |P(D, k)|}{DA^2 \cdot DB^2 \cdot DC^2} \cdot T_{ABC}. \tag{10}$$

Incidentally, from the triangle inequalities of $A'B'C'$ we again get Ptolemy's inequalities (1), but now by inversion.

From the similarity of Δ and $A'B'C'$ we obtain a formula for the ratio $\frac{T_\Delta}{T}$:

$$T_\Delta = \left(\frac{DA \cdot DB \cdot DC}{\rho^2}\right)^2 T' = |P(D, k)|T. \tag{11}$$

This is a nice expression of the fact that the vanishing of T_Δ is equivalent to the cyclicity of $ABCD$.

6. The Area of the Pedal triangle

As an application of (11) we derive a formula for the area of the pedal triangle PQR of a point D (for pedal triangles see [5, pp. 22ff] or [8, pp. 135ff]). Triangle

PQR is similar to triangle Δ with side lengths $a \cdot DA$, $b \cdot DB$, $c \cdot DC$. This follows from $QR = DA \sin A = \frac{a}{2R} \cdot DA$. Hence by (11)

$$T_{PQR} = \frac{1}{4R^2} \cdot T_\Delta = \frac{|P(D,k)|}{4R^2} \cdot T_{ABC}.$$

For another proof see [8, p.139, Theorem 198].

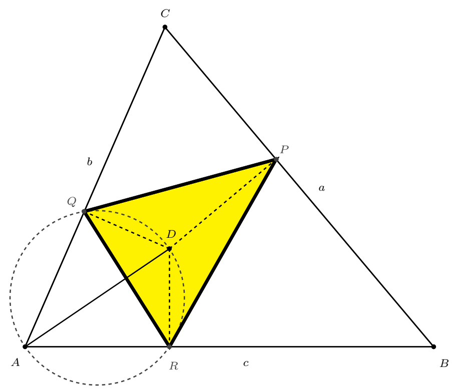

Figure 5

This formula captures the essence of the theorem about the Simson line [5, p.41, Theorem 2.51], [8, p.137, Theorem 192] that P, Q and R are collinear if and only if D is on the circumcircle of ABC.

7. Simplifications of barycentric coordinate formulas for some triangle centers

The possible simplifications, based on an application of formula (5), apply to any triangle center for which we have nice formulas for the ratios of its distances to the vertices of the triangle. As a sign change in (5) means inversion in the circumcircle, this is also a nice tool to invert triangle centers as can be seen in the following example.

For the orthocenter $D = X(4)$ we have $AD = \frac{aS_A}{S}$ etc. As tripolar coordinates of the orthocentre we take $(p : q : r) = (aS_A : bS_B : cS_C)$. From $ap + bq + cr = 2S^2$, etc. we get

$$2S_\Delta = \sqrt{(ap+bq+cr)(-ap+bq+cr)(ap-bq+cr)(ap+bq-cr)} = 4SS_AS_BS_C.$$

Similarly, $(bq)^2 + (cr)^2 - (ap)^2 = 2S_BS_C(S^2 - S_A^2)$. From (5) follows for the barycentrics of $X(4)$

$$x = a^2 \left(\frac{S_A}{S} + \frac{(bq)^2 + (cr)^2 - (ap)^2}{2S_\Delta} \right) = \frac{a^2b^2c^2}{2S} \cdot \frac{1}{S_A} = \frac{a^2b^2c^2}{2S^2} \tan A \sim \tan A.$$

The barycentrics of X(186), the inverse of X(4) in the circumcircle, are given by the minus sign:

$$x = a^2 \left(\frac{S_A}{S} - \frac{(bq)^2 + (cr)^2 - (ap)^2}{2S_\Delta} \right) = \frac{1}{2S} \cdot a^2 \cdot \frac{3S_A^2 - S^2}{S_A} \sim a^2 \cdot \frac{3S_A^2 - S^2}{S_A}.$$

We apply the formula (5) to get simpler barycentrics $(h(A,B,C) : h(B,C,A) : h(C,A,B))$ of the Walsmith point $X(5000)$ and its inverse $X(5001)$ in the circumcircle, see [9]. For this point, its distances to A, B, C have ratios $(p : q : r)$ with $p^2 = S_A$, $q^2 = S_B$, $r^2 = S_C$. Factorizing S_Δ we get the expression $h(A,B,C) = a^2(S_A\sqrt{S_AS_BS_CS_\omega} \pm SS_BS_C)$.

Applying to the first Walsmith-Moses point $X(5002)$ and its inverse $X(5003)$, the simplifications with $p = a$, $q = b$, $r = c$ lead to

$$h(A,B,C) = a^2(2S_A\sqrt{S_AS_BS_CS_\omega} \pm S(S_\omega S_A - S_BS_C)).$$

Finally, for the second Walsmith-Moses point $X(5004)$ and its inverse $X(5005)$, with $p^2 = b^2 + c^2$, $q^2 = c^2 + a^2$, $r^2 = a^2 + b^2$, we get the formula

$$h(A,B,C) = a^2S_A\sqrt{2S_\omega} \pm Sabc.$$

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