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A Simple Dynamic Localization of the Gravitational Center of a Triangle

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Abstract. We describe the gravitational center of a triangle as the common point of three ellipses and give a one-line proof of its existence and uniqueness.

We consider a nondegenerate triangle ABC of constant area mass density with sides $AB = c$, $BC = a$, and $CA = b$. A point mass of the triangle plane is at a *gravitational center* of the triangle when the force of gravity exerted by the triangle on the point mass is zero (with $1/r$ potential). As shown in [1], the gravitational centers are the inner points of the triangle whose distances r_A, r_B, and r_C from the vertices fulfill

$$\left(\frac{r_A + r_B + c}{r_A + r_B - c}\right)^{1/c} = \left(\frac{r_B + r_C + a}{r_B + r_C - a}\right)^{1/a} = \left(\frac{r_C + r_A + b}{r_C + r_A - b}\right)^{1/b} \tag{1}$$

and there is exactly one such point, labeled $X(5626)$ in [3].

The gravitational center can be localized very easily as follows. Construct with a dynamic geometry software an ellipse $\mathcal{E}_c$ of foci A and B with some ratio

$$\frac{\text{major axis}}{c} = \frac{1}{\text{eccentricity}} =: \tan\theta_c, \quad \frac{\pi}{4} < \theta_c < \frac{\pi}{2}. \tag{2}$$

Permuting vertices and sides cyclically, construct ellipses $\mathcal{E}_a$ and $\mathcal{E}_b$ similarly, but let θ_a and θ_b depend on θ_c and on the ratio a/c or b/c by setting

$$\theta_{\text{side}} = \frac{3\pi}{4} - \arctan\left(\tan^{\text{side}/c}\left(\frac{3\pi}{4} - \theta_c\right)\right), \quad \text{side} = a,\, b. \tag{3}$$

(Relation (3) between θ_{side} and θ_c is then true when "side" and c are replaced with any elements of $\{a,\, b,\, c\}$.) Vary θ_c by moving the corresponding slider (Figure 1) until the three ellipses share a common point inside the triangle: this is the gravitational center.

Proof. Consider a point P at some distance r_A from A and r_B from B (and not on the triangle's boundary). Draw the ellipse $\mathcal{E}_c$ of foci A and B through P, with major axis μ. The quotient on the left of (1) is by (2)

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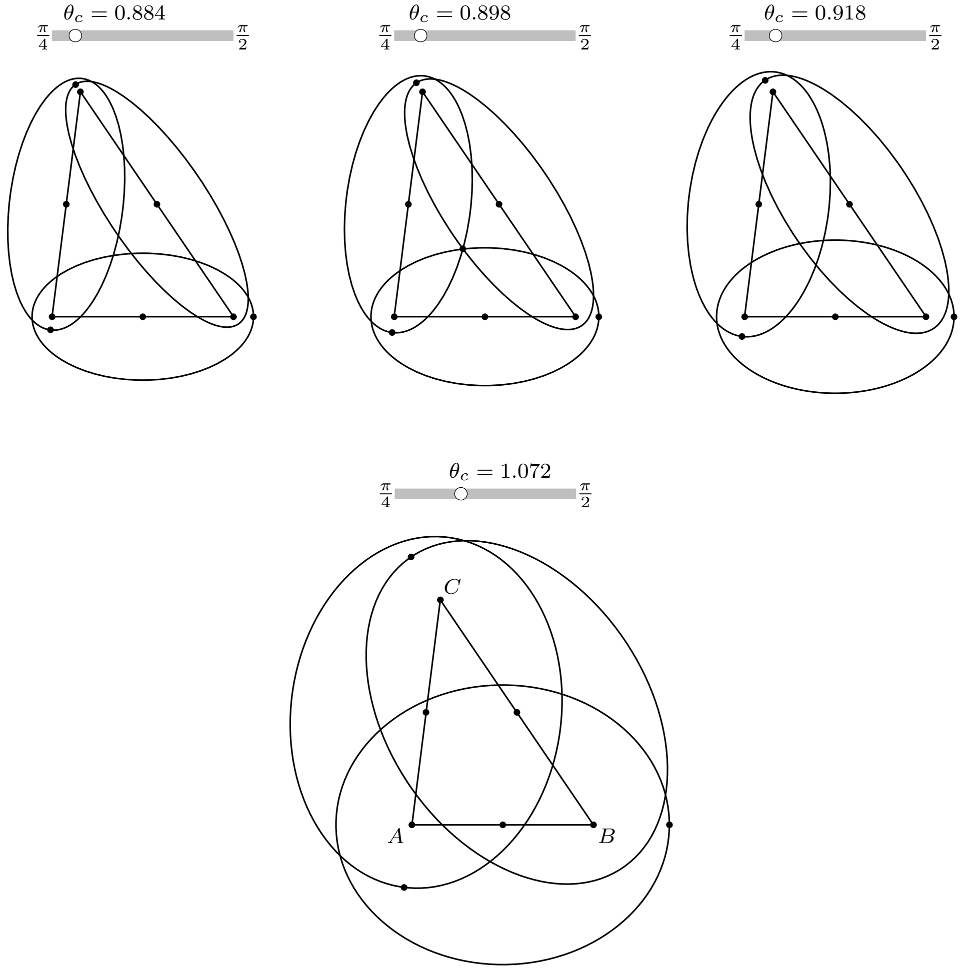

Figure 1. Triangle with sides $a : b : c = 6 : 5 : 4$

$$\frac{r_A + r_B + c}{r_A + r_B - c} = \frac{\frac{\mu}{c} + 1}{\frac{\mu}{c} - 1} = \tan\left(\frac{3\pi}{4} - \theta_c\right). \tag{4}$$

For the same point P, the other quotients of (1) without exponents are as in (4) $\tan\left(\frac{3\pi}{4} - \theta_a\right)$ and $\tan\left(\frac{3\pi}{4} - \theta_b\right)$ for some θ_a and θ_b between $\pi/4$ and $\pi/2$. The logarithmic version of (1) characterizes the gravitational center by

$$\frac{1}{a}\log\tan\left(\frac{3\pi}{4} - \theta_a\right) = \frac{1}{b}\log\tan\left(\frac{3\pi}{4} - \theta_b\right) = \frac{1}{c}\log\tan\left(\frac{3\pi}{4} - \theta_c\right). \quad \square$$

One sees at once that θ_a and θ_b given by (3) increase with θ_c and have the same range $(\pi/4, \pi/2)$ as θ_c. For values of the θs near $\pi/4$, the ellipses surround a gap inside the triangle (Figure 1); for values of the θs near $\pi/2$, the three ellipse interiors have a common overlap. It is immediate that the transition gap/overlap takes place at the solution of (1), *i.e.*, at the gravitational center, which lies in every gap and every overlap. It is also clear from Figure 1 that the three ellipses share a common point *inside* the triangle for a *unique* θ_c: this proves the existence and uniqueness of the gravitational center. (We conjecture that (1) never has a solution outside the triangle.)

Remark. As the three ellipses share pairwise a focus, the three pairwise common chords are concurrent (if existing) [4, 2], but the concurrency point varies with θ_c.

References

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