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Some Theorems on Polygons with One-line Spectral Proofs

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Abstract. We use discrete Fourier transforms and convolution products to give one-line proofs of some theorems about planar polygons. We illustrate the method by computing the perspectors of a pair of concentric equilateral triangles constructed from a hexagon and leave the proofs of Napoleon's theorem, the Barlotti theorem, the Petr–Douglas–Neumann theorem, and other theorems as an exercise.

1. Introduction

The Fourier decomposition of a planar (or nonplanar [4]) polygon and circulant matrices have been used for a long time in the study of polygon transformations with a circulant structure (see [6] for a list of references). The replacement of circulant matrices with convolution products simplifies the approach [6, 7] and allows one-line proofs of many theorems about polygons: Napoleon's theorem, the Barlotti theorem, and the Petr–Douglas–Neumann theorem are such examples (Section 7). Sections 3–5 provide a short but self-contained overview of the necessary theory (see [6, 7] for more details). As an application we determine in Section 6 the perspectors of the pair of triangular Fourier components of a planar hexagon and find so an elegant and enlightening solution to a problem treated in [3]. In preparation for the hexagon problem we begin our exposition by expressing the perspectors of two concentric equilateral triangles.

2. Perspectors of two concentric equilateral triangles

By a theorem attributed to D. Barbilian (1930), but which is older, two concentric equilateral triangles are triply perspective [9] (with a short proof in trilinears), [5], [2, p. 71], [8, pp. 91–92]. We prove this result by giving an explicit formula for the perspectors (Figure 1).

Theorem 1. (1) *Two equilateral triangles centered at the origin of the complex plane with vertices* 1 *and* v*,* $|v| \neq 1$*, respectively, have the perspectors*

$$p_k = \frac{v^2 - \overline{v}}{1 - |v|^2}\, \omega^k = p_0\, \omega^k, \quad k = 0,\, 1,\, 2, \quad \textit{where} \quad \omega = e^{i2\pi/3}. \tag{1}$$

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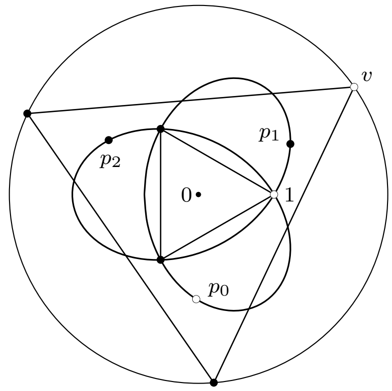

Figure 1. Common locus of the three perspectors for $|v| = 2.5$

The position of p_k *on the line of the corresponding vertices* ω^ℓ *and* $v\omega^{-\ell-k}$ *is given by the real quotient*

$$\frac{p_k - \omega^\ell}{v\omega^{-\ell-k} - \omega^\ell} = \frac{1 + 2\,\mathrm{Re}\left(v\omega^{\ell-k}\right)}{1 - |v|^2}, \quad \ell = 0,\, 1,\, 2. \tag{2}$$

When one triangle has its vertices on the sidelines of the other, the perspectors p_k *are the vertices of the second triangle.*

(2) *If* $v \notin \{1,\, \omega,\, \overline{\omega}\}$ *lies on the unit circle, the successive perspectors* p_k *are the points at infinity of the lines through* 1 *and* $v\omega^{-k}$ *obtained from one another by a rotation of* $2\pi/3$ *about* 1.

(3) *The origin is a further perspector when* $\arg v$ *is an integer multiple of* $\pi/3$.

Proof. Plug formula (1) into formula (2) and verify directly. □

If v lies neither on the unit circle nor on a sideline of the triangle $(1,\, \omega,\, \overline{\omega})$, the map $v \mapsto p_0 = \left(v^2 - \overline{v}\right)/\left(1 - |v|^2\right)$ is an involution whose fixed points z form the circle $|z + 1| = 1$ without ω and $\overline{\omega}$. If in addition the triangle $(v,\, v\omega,\, v\overline{\omega})$ has no vertex on this circle, *i.e.*, if 1 is not on its sidelines, the (different) triangles $(1,\, \omega,\, \overline{\omega})$, $(v,\, v\omega,\, v\overline{\omega})$, and $(p_0,\, p_1,\, p_2)$ form a triad: each of them is perspector triangle of the others.

3. Spectral decomposition of a planar polygon

For an integer $n \geq 2$, an n-gon P in the complex plane is the sequence $P = (z_k)_{k=0}^{n-1}$ of its vertices in order representing the closed polygonal line

$$z_0 \to z_1 \to \cdots \to z_{n-1} \to z_0$$

starting at z_0. The vertices are indexed modulo n. We set $\zeta = e^{i2\pi/n}$ and use the Fourier basis of $\mathbf{C}^n$ (Figure 2) constituted by the standard regular $\{n/k\}$-gons

$$F_k = \left(\zeta^{k\ell}\right)_{\ell=0}^{n-1}, \quad k = 0,\, 1, \ldots,\, n-1.$$

After the starting vertex 1, each vertex of F_k is the kth next nth root of unity. $F_0 = (1,\, 1, \ldots,\, 1)$ is a trivial polygon and the other basis polygons are centered

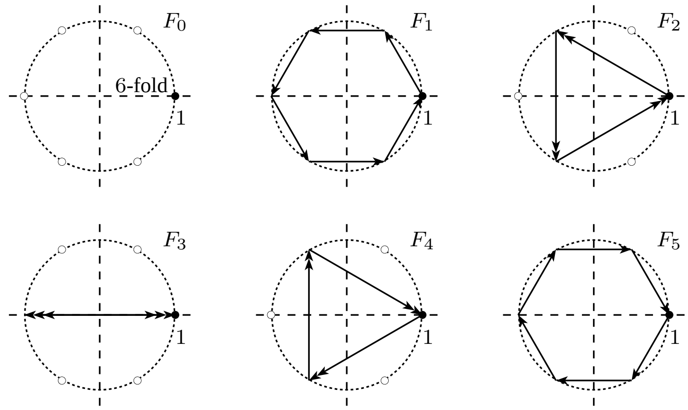

Figure 2. Fourier basis of $\mathbf{C}^6$

at the origin with $\overline{F_k} = F_{n-k}$. The Fourier basis is orthonormal with respect to the inner product of $\mathbf{C}^n$ given by

$$\langle P, Q \rangle = \left\langle (z_k)_{k=0}^{n-1}, (w_k)_{k=0}^{n-1} \right\rangle = \frac{1}{n} \sum_{k=0}^{n-1} z_k \overline{w_k}.$$

The *discrete Fourier transform* or *spectrum* of P is the polygon $\widehat{P} = (\hat{z}_k)_{k=0}^{n-1}$ given by the *spectral decomposition* of P in the Fourier basis:

$$P = \sum_{k=0}^{n-1} \hat{z}_k F_k \quad \text{with} \quad \hat{z}_k = \langle P, F_k \rangle\,, \quad k = 0, 1, \ldots, n-1,$$

where each nonzero $\hat{z}_k$ rotates and scales up or down the basis polygon about the origin. The trivial polygon $\hat{z}_0 F_0$ corresponds to the (vertex) centroid $\hat{z}_0$ of P.

4. Convolution filters

We consider a *filter* $\Phi_\Gamma \colon \mathbf{C}^n \to \mathbf{C}^n$ given by the cyclic convolution $*$ with a fixed polygon $\Gamma = (c_0,\, c_1, \ldots,\, c_{n-1})$: the kth entry of $\Phi_\Gamma(P) = P * \Gamma = \Gamma * P$ is

$$\sum_{\ell_1+\ell_2=k \,(\mathrm{mod}\ n)} z_{\ell_1} c_{\ell_2} = \sum_{\ell=0}^{n-1} z_\ell c_{k-\ell}, \quad k = 0,\, 1, \ldots,\, n-1.$$

A circulant linear transformation of a polygon in the complex plane that is given by the coefficients $(a_k)_{k=0}^{n-1}$ of the circulant linear combination of the vertices is simply the convolution of the initial polygon with the polygon $(a_0, a_{n-1}, a_{n-2}, \ldots, a_1)$ obtained from $(a_0, a_1, \ldots, a_{n-1})$ by going the other way around. The operator $*$ is commutative, associative and bilinear.

Since $F_k * F_\ell = \begin{cases} nF_k & (k=\ell) \\ (0,0,\dots,0) & (k\neq\ell) \end{cases}$, one has

$$\Phi_\Gamma(P) = P * \Gamma = \left(\sum_{k=0}^{n-1} \hat{z}_k F_k\right) * \left(\sum_{\ell=0}^{n-1} \hat{c}_\ell F_\ell\right) = \sum_{k=0}^{n-1} n\hat{c}_k \hat{z}_k F_k,$$

i.e.,

$$\widehat{P * \Gamma} = n\widehat{P} \cdot \widehat{\Gamma},$$

where $\cdot$ is the entrywise product: the Fourier basis is a basis of eigenvectors of the convolution Φ_Γ with eigenvalues $n\hat{c}_k$ (geometrically clear!). $\Phi_\Gamma(P)$ and P always have the same centroid if and only if $\sum_{k=0}^{n-1} c_k = 1$, which means $\hat{c}_0 = 1/n$; the centroid is always translated to the origin if and only if $\hat{c}_0 = 0$.

5. Ears and diagonals

A *Kiepert n-gon* consists of the apices of similar triangular ears that are erected in order on the sides of the initial polygon $P = (z_k)_{k=0}^{n-1}$ (beginning with the side $z_0 \to z_1$) and that are directly similar to the normalized triangle $(0,1,a) \in \mathbf{C}^3$ with apex a: the apex of the ear for the side $z_0 \to z_1$ is defined as $z_1 + a(z_0 - z_1)$; it is a right-hand ear if $\operatorname{Im} a > 0$. The corresponding Kiepert polygon is thus given by the centroid-preserving convolution of P with

$$K(a) = (a, 0, \dots, 0, 1-a).$$

An *ℓ-diagonal midpoint n-gon* consists of the midpoints of the diagonals $z_k \to z_{k+\ell}$ taken in order over the initial polygon $P = (z_k)_{k=0}^{n-1}$. As its first vertex is $(z_0 + z_\ell)/2$, the ℓ-diagonal midpoint n-gon is given by the centroid-preserving convolution of P with

$$M_\ell = \frac{1}{2}(\underset{\substack{\uparrow \\ 0}}{1},\ 0, \dots,\ 0,\ \underset{\substack{\uparrow \\ n-\ell}}{1},\ 0, \dots,\ \underset{\substack{\uparrow \\ n-1}}{0}). \quad \text{(position)}$$

We will only use the fact that these transformations are convolution products since they are circulant linear maps. We need neither the explicit convolving polygon nor its spectrum.

6. Filtered hexagons

Theorem 2. *Erect right-hand equilateral triangles on the sides of a planar hexagon. The midpoints of the opposite ear centers are the vertices of an equilateral triangle T. Left-hand ears lead to an equilateral triangle T' centered, as T, at the vertex centroid of the hexagon.*

Proof. For the hexagon

$$H = (z_k)_{k=0}^{5} = \sum_{k=0}^{5} \hat{z}_k F_k$$

the triangle T corresponding to right-hand ears is simply

$$T = H * K(a_{\pi/6}) * M_3 \quad \text{with} \quad a_{\pi/6} = \tfrac{1}{\sqrt{3}} e^{i\pi/6}.$$

The convolution with $K(a_{\pi/6})$ erects right-hand isosceles ears with base angles $\pi/6$. The following facts are geometrically immediate (Figure 2): F_1, F_3, and F_5 are filtered out by the diagonal midpoint construction, whereas F_0 and F_2 are left unchanged. F_4 is deleted by the ear erection, F_0 is left unchanged, and F_2 is rotated by $\pi/3$. By linearity, associativity, and commutativity of the convolution product, T is thus the (doubly covered) equilateral triangle

$$T = \hat{z}_0 F_0 + \eta \hat{z}_2 F_2 \quad \text{for} \quad \eta = e^{i\pi/3}$$

with the same centroid as the hexagon (T collapses to the centroid if H is F_2-free). Left-hand ears lead to

$$T' = \hat{z}_0 F_0 + \overline{\eta} \hat{z}_4 F_4. \qquad \square$$

Notice that the components $T_1 = \hat{z}_0 F_0 + \hat{z}_2 F_2$ and $T_1' = \hat{z}_0 F_0 + \hat{z}_4 F_4$ of the hexagon can be retrieved from T and T', respectively: T and T_1 form a regular hexagram, as do T' and T_1' as well as the perspector triangles of T, T' and T_1, T_1'. Since

$$\hat{z}_2 = \frac{1}{6}\left(z_0 + z_3 + \overline{\omega}(z_1 + z_4) + \omega(z_2 + z_5)\right) \quad \text{and}$$
$$\hat{z}_4 = \frac{1}{6}\left(z_0 + z_3 + \omega(z_1 + z_4) + \overline{\omega}(z_2 + z_5)\right) \quad \text{for} \quad \omega = e^{i2\pi/3},$$

$(\hat{z}_0, \hat{z}_2, \hat{z}_4)$ is the spectrum of the triangle $(w_k)_{k=0}^{2} = \frac{1}{2}(z_k + z_{k+3})_{k=0}^{2}$ formed by the first lap of $H * M_3$ and depends thus only (and bijectively) on the midpoints of the opposite vertices of H. These midpoints are collinear if and only if $\hat{z}_2$ and $\hat{z}_4$ have the same modulus [7]. Otherwise, the perspector p_0 of T and T' is by Theorem 1

$$p_0 = \hat{z}_0 + \frac{v^2 - \overline{v}}{1 - |v|^2} \overline{\eta} \hat{z}_4 \quad \text{for} \quad v = \omega \hat{z}_2 / \hat{z}_4, \tag{3}$$

$\omega \hat{z}_2 / \hat{z}_4$ being the quotient of the vertices $\eta \hat{z}_2$ of $T - \hat{z}_0 F_0$ and $\overline{\eta} \hat{z}_4$ of $T' - \hat{z}_0 F_0$. After transformation, formula (3) leads to the following result.

Theorem 3. *Consider a hexagon $(z_k)_{k=0}^{5}$ for which the midpoints*

$$w_k = \frac{z_k + z_{k+3}}{2}, \quad k = 0, 1, 2,$$

of the opposite vertices are not collinear. The equilateral triangles T and T' from Theorem 2 *have then the perspectors*

$$p_k = \hat{z}_0 + \frac{\hat{z}_2^2 \overline{\hat{z}_4} - \overline{\hat{z}_2} \hat{z}_4^2}{|\hat{z}_2|^2 - |\hat{z}_4|^2} \omega^k, \quad k = 0, 1, 2, \quad \textit{where} \quad \omega = e^{i2\pi/3},$$

and p_0 can be written as

$$p_0 = \frac{\sum_{\text{cyclic}} |w_0|^2 (w_1 - w_2)}{\sum_{\text{cyclic}} \overline{w_0}(w_1 - w_2)}. \tag{4}$$

(Formula (4) corrects the corresponding formula of [3].)

7. Other theorems with one-line spectral proofs

The following examples also have one-line spectral proofs, which are – with two exceptions – left to the reader as an exercise!

7.1. *Equilaterality.* A triangle (z_0, z_1, z_2) is positively oriented and equilateral (or trivial) if and only if

$$\hat{z}_2 = z_0 + \omega z_1 + \overline{\omega} z_2 = 0.$$

Negatively oriented equilateral triangles correspond to $\hat{z}_1 = 0$.

7.2. *Napoleon's theorem.* The centers of right-hand equilateral triangles erected on the sides of a triangle are the vertices of an equilateral (or trivial) triangle. The same is true for left-hand ears.

7.3. *The Barlotti theorem.* An n-gon in the complex plane is an affine image of F_k, $k \neq 0$, *i.e.*, of the form $aF_0 + bF_k + cF_{n-k}$, if and only if the centers of scaled copies of F_k erected on the sides are the vertices of a scaled copy of F_k.

7.4. *Side midpoint quadrilateral.* The side midpoints of a (planar) quadrilateral are the vertices of a parallelogram.

7.5. *The Petr–Douglas–Neumann theorem.* Start from a planar n-gon and replace it with the polygon whose vertices are the centers of scaled copies of some F_k, $k \neq 0$, erected on the sides. Repeat the operation on the actual polygon with another F_k until all integers $k \in [1, n-1]$ have been used. The result is the vertex centroid of the initial polygon.

Proof. The F_k-step erases (only) F_{n-k}. □

Remark. The F_k-step, $k \neq 0$, transforms obviously affine images of F_k into (possibly trivial) scaled copies of F_k and no other planar n-gon into an affine image of F_k: thus polygons becoming regular after more than one F_k-step do not exist – although they are explicitly described in [1] for $k = 1$!

7.6. *A theorem à la van Aubel.* The midpoints of the diagonals of a planar quadrilateral Q and the midpoints of the opposite centers of right-hand squares erected on the sides of Q form a square. The same is true for left-hand squares.

Proof. The midpoint step erases F_1 and F_3 without changing F_2. The half-square ear step turns F_2 by $\pi/2$. □

7.7. *Generalized van Aubel's theorem.* Erect right-hand squares on the sides of a planar octagon and take the quadrilateral Q whose vertices are the midpoints of the opposite square centers: Q has congruent and perpendicular diagonals and remains unchanged if one permutes the two transformations. The same is true for left-hand squares.

7.8. *Generalized Thébault's theorem.* Replace a planar octagon with the octagon of the side midpoints, erect right-hand squares on the sides of this midpoint octagon and take the quadrilateral Q whose vertices are the midpoints of the opposite square centers: Q is a square that remains unchanged for any order of the three transformations. The same is true for left-hand squares.

Remark. The transformation

$$\Phi\colon P = (z_k)_{k=0}^{n-1} \mapsto (az_k + z_{k+1} + z_{k-1})_{k=0}^{n-1}$$

multiplies the basis polygons F_ℓ and $\overline{F_\ell}$ by $a + 2\cos(2\ell\pi/n)$, and $\Phi/(a+2)$ is centroid-preserving if $a \neq -2$. The choice $a = -2\cos(2\ell_0\pi/n)$, $\ell_0 \neq 0$, erases thus exactly F_{ℓ_0} and $F_{n-\ell_0}$. To delete $F_{n-\ell_0}$ only, perform the F_{ℓ_0}-step of the Petr–Douglas–Neumann theorem.

7.9. *Filtered pentagon.* If ϕ is the golden ratio and $a = \phi$ or $1 - \phi$, the pentagon $P = (az_k + z_{k+1} + z_{k-1})_{k=0}^{4}$ obtained from $(z_k)_{k=0}^{4}$ is affinely regular and has thus a circumellipse. Unless its vertices are collinear, P is convex for $a = \phi$ and a pentagram for $a = 1 - \phi$.

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