Forum Geometricorum
Volume 16 (2016) 29–35.

FORUM GEOM
ISSN 1534-1178

Geogebra Construction of the Roots of Quadratic, Cubic and Quartic Equations

Nikolaos Dergiades

Abstract. We construct geometrically the roots of a quadratic equation, which construction is ruler and compass possible, where the roots are signed segments and the roots of a cubic or quartic equation, that are not ruler and compass constructible, for which we give conic (with intersections of parabolas) dynamic constructions, easily performed via computer applications such as Geogebra etc and apply this construction to the trisection of an acute angle.

1. Introduction

Our attention is for pure geometric constructions of the roots of equations that are the result of real geometric problems, where the coefficients are not simple values [1].

1.1. *Quadratic equation, a ruler and compass construction.* A geometric quadratic equation with unknown a segment x can take the form

$$x^2 + \varepsilon ax + \delta b^2 = 0,$$

where a and b are known segments and $\varepsilon, \delta = \pm 1$. It is easy to see that the above equation has roots the abscissae of the intersections of the circle

$$\left(x + \frac{\varepsilon a}{2}\right)^2 + \left(y + \frac{(\delta+1)b}{2}\right)^2 = \frac{a^2}{4} + \left(\frac{(\delta-1)b}{2}\right)^2 = R^2$$

with the x-axis $y = 0$. It is also easy to see that the above circle passes through the points $A(0, -b)$ and $B(-\varepsilon a, -\delta b)$, and that the distance of these points is $2R$, the diameter of the above circle. Hence the construction of the roots of the quadratic equation consists of constructing the circle with diameter AB and finding its intersections with the x-axis.

1.2. *Not ruler and compass cases.* A geometric construction, that is not possible with ruler and compass, of a segment y sometimes leads to the solution of a cubic or quartic equation. The general form of such a cubic equation is

$$y^3 \pm py^2 \pm q^2 y \pm r^2 s = 0,$$

Publication Date: March 6, 2016. Communicating Editor: Paul Yiu.

where p, q, r, s are lengths of known segments. By the substitution $y = x \mp \frac{p}{3}$ we get the simpler equation

$$x^3 \pm a^2x \pm d^2e = 0 \tag{1}$$

where a, d, e are constructible segments. If $a \neq 0$, then we can construct the segment b such that $d^2e = a^2b$ with the following construction (Figure 1). On a line take the consecutive segments $AB = a$, $BE = e$, Draw $AD \perp AB$, $AD = d$, $AC \perp BD$. If DF is parallel to CE, then $EF = b$.

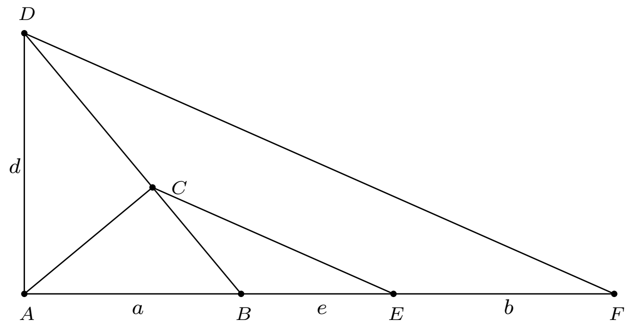

Figure 1

Hence, if $a \neq 0$, we have the equation

$$x^3 + \varepsilon a^2x + \delta a^2b = 0, \qquad \varepsilon, \delta = \pm 1. \tag{2}$$

If $a = 0$, we have the equation

$$x^3 \pm d^2e = 0. \tag{3}$$

Similarly the general form of a quartic equation

$$y^4 \pm py^3 \pm q^2y^2 \pm r^2sy \pm t^2u^2 = 0$$

by the substitution $y = x \mp \frac{p}{4}$ gives the simpler equation

$$x^4 \pm a^2x^2 \pm d^2ex \pm v^2w^2 = 0 \tag{4}$$

where $vw \neq 0$ and a, d, e, v, w are constructible segments.

As previously, if $a \neq 0$, it is easy to construct segments b, c such that (4) has the form

$$x^4 + \varepsilon a^2x^2 + \delta a^2bx + \eta a^2c^2 = 0, \qquad \varepsilon, \delta, \eta = \pm 1. \tag{5}$$

It is easy to see that (4) can have the following other forms

$$x^4 + \varepsilon a^2bx + \eta a^2c^2 = 0, \tag{6}$$

$$x^4 + \varepsilon a^2x^2 + \delta a^2b^2 = 0 \qquad \text{(biquadratic)} \tag{7}$$

and $x^4 - v^2w^2 = 0$ with a trivial construction of the mean proportional of v and w.

2. Cubic equations

2.1. The roots of the cubic equation $x^3+\varepsilon a^2 x+\delta a^2 b=0$ are the abscissae of the intersections of parabola $x^2=a\left(y-\frac{\varepsilon a}{2}\right)$ with the parabola $y^2=-\delta b\left(x-\frac{a^2}{4\delta b}\right)$ provided that $x\neq 0$, as it is easily seen after elimination of y. Hence we exclude as common point the vertex $A\left(0,\,\frac{\varepsilon a}{2}\right)$ of the first parabola. These parabolae are easily constructible with Geogebra. The first focus is $F_1\left(0,\,\frac{a}{4}+\frac{\varepsilon a}{2}\right)$ and the directrix is parallel to x-axis at $D_1\left(0,\,-\frac{a}{4}+\frac{\varepsilon a}{2}\right)$. The vertex of the second is $B\left(\frac{a^2}{4\delta b},\,0\right)$, the focus $F_2\left(-\frac{\delta b}{4}+\frac{a^2}{4\delta b},\,0\right)$ and the directrix is parallel to y-axis at $D_2\left(\frac{\delta b}{4}+\frac{a^2}{4\delta b},\,0\right)$. The points A, F_1, D_1 are easily constructible. In order to construct the point B, we construct the point $B'(-\delta b,0)$, the point A' symmetric of A relative to the x-axis. The circumcircle of triangle $AB'A'$ meets again the x-axis at B. The points F_2, D_2 are easily constructible since $F_2B=BD_2=\frac{\delta b}{4}$ (see Figure 2a).

Case 2.1.1. $\varepsilon=\delta=1$ (Figure 2a).

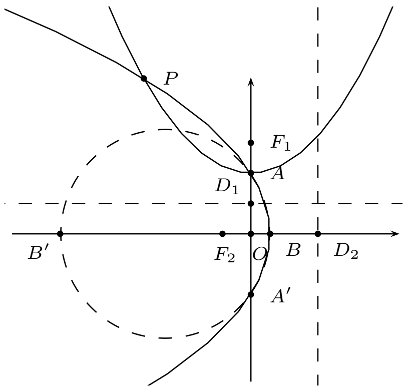

Figure 2a

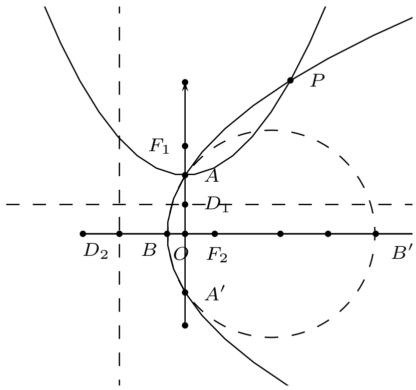

Figure 2b

The parabolae have two common points A, P. Hence we have only one negative root, the abscissa of the point P.

Case 2.1.2. $\varepsilon=1$, $\delta=-1$ (Figure 2b). For $x=-x$ we get case 2.1.1. Hence we have only one positive root.

Case 2.1.3. $\varepsilon=-1$, $\delta=1$ (Figure 2c).

Case 2.1.4. $\varepsilon=\delta=-1$ (Figure 2d). For $x=-x$ we get Case 2.1.3.

Case 2.1.5. $x^3=d^2e$ (Figure 2e). The roots of this equation are the abscissa of the intersections of the parabolas

$$x^2=dy,\qquad y^2=ex,$$

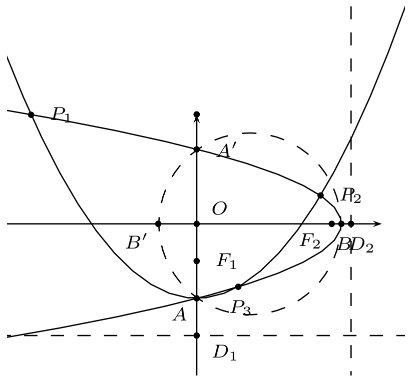

Figure 2c

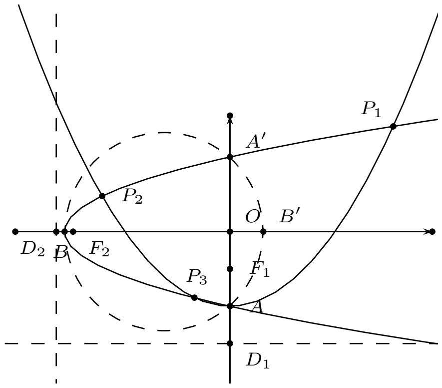

Figure 2d

where $x \neq 0$, $D_1\left(0,\ -\frac{d}{4}\right)$, $D_2\left(-\frac{e}{4},\ 0\right)$. We have only one positive root the abscissa of the point P.

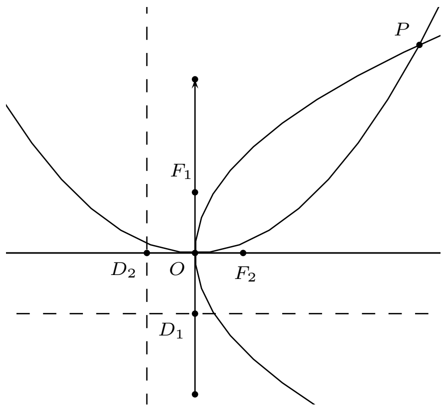

Figure 2e

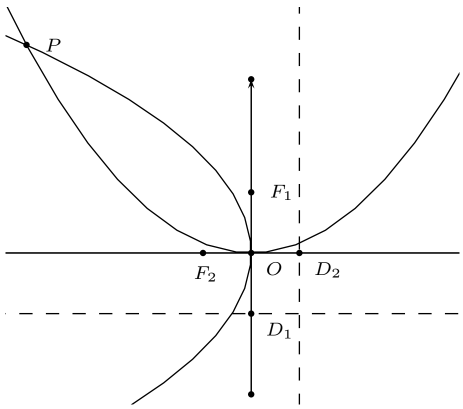

Figure 2f

Case 2.1.6. $x^3 + d^2e = 0$ (Figure 2f). For $x = -x$ we get the previous case Hence we have one negative root.

3. The doubling of a cube

This is a famous problem from antiquity with no ruler and compass solution. If we have a cube with edge a, and we want to construct a second cube with double volume of the first, then the edge x of this cube must be the root of the cubic equation $x^3 = 2a^3$. This equation has the form of Case 2.1.5 where $d = a$, $e = 2a$, and the construction of x is obvious.

4. The trisection of a given acute angle

If triangle ABC has a right angle at A (Figure 3), sidelengths a, b, c, and $\angle B = 3\varphi$, we take a point Z on CA such that $CZ = x$ and BZ trisects the angle B. Hence, $\tan 3\varphi = \frac{b}{c}$, $\tan\varphi = \frac{b-x}{c}$. Since $\tan 3\varphi = \frac{3\tan\varphi - \tan^3\varphi}{1-3\tan^2\varphi}$, we get

$$\frac{b}{c} = \frac{3c^2(b-x)-(b-x)^3}{c(c^2-3(b-x)^2)},$$

or

$$x^3 - 3a^2x + 2a^2b = 0.$$

This cubic equation is of the form of Case 2.1.3 where a and b are replaced by $\sqrt{3}a$ and $\frac{2}{3}b$ respectively. Hence we make the following

Construction of the point Z. On the side AB we take the point A_1 such that $AA_1 = \frac{\sqrt{3}}{2}a$, the altitude of the equilateral triangle BCD. Let A_2 be the symmetric of A_1 relative to A. We take the point B_1 on CA such that $B_1A = \frac{2}{3}b$. The circumcircle of triangle $A_1B_1A_2$ meets the line CA at B_2. Let E_1 be the midpoint of AA_2 and D_1 the symmetric of E_1 relative to A_2. We take on B_1B_2 the points E_2, D_2 such that $E_2B_2 = B_2D_2 = \frac{B_1A}{4} = \frac{1}{6}b$. We draw the parabola with focus E_1 and directrix the parallel from D_1 to CA, and the parabola with focus E_2 and directrix the parallel from D_2 to AB.

If P is the common point of the two parabolas with distance d from the line A_1A_2 that is the smallest, then $CZ = d$. Hence the point Z is constructible by this conic construction.

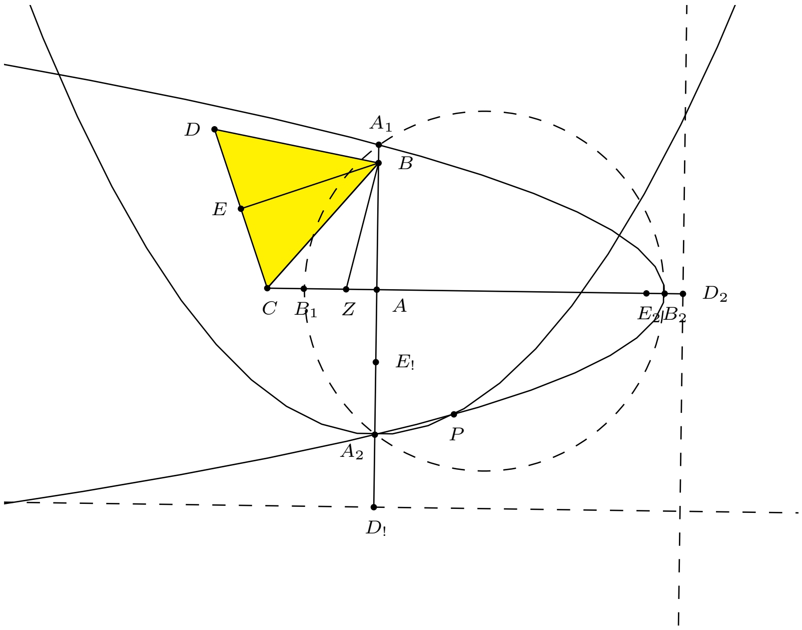

Figure 3

5. Quartic equations

5.1. *The quartic equation* $x^4+\varepsilon a^2x^2+\delta a^2bx+\eta a^2c^2=0$. The roots of the quartic equation are the abscissae of the intersections of parabola $x^2=a\left(y-\frac{\varepsilon a}{2}\right)$ with the parabola $y^2=-\delta b\left(x-\frac{a^2}{4\delta b}+\frac{c^2}{\delta\eta b}\right)$ as it is easily seen after elimination of y. We have almost the same construction as in the case of the cubic equation. The only difference is that if we construct also the points $C(0,c)$, $C'(0,-c)$, then the circumcircle of triangle $B'CC'$, that meets again the x-axis at the point B'', and we translate the second parabola (in the cubic construction) by the vector $-k\overrightarrow{OB''}$. The points F_2, D_2 again are given by $F_2B=BD_2=\frac{\delta b}{4}$.

Example. $\varepsilon=-1$, $\delta=-1$, $\eta=1$ (Figure 4) we refer to case 2.1.3 and (Figure 2d).

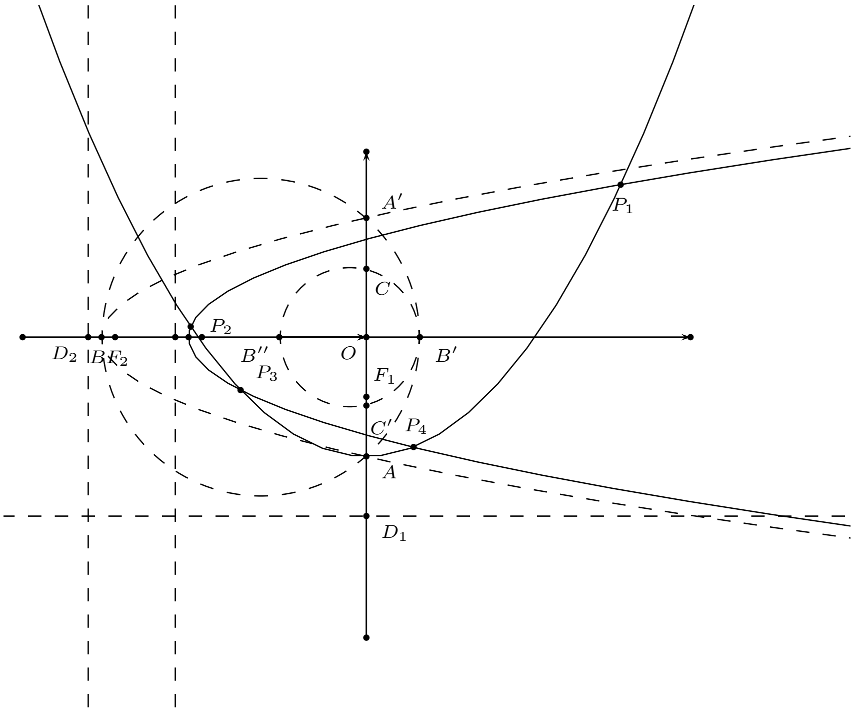

Figure 4

5.2. *The quartic equation* $x^4+\delta a^2bx+\eta a^2c^2=0$. The roots of the quartic equation, as it is easily seen after elimination of y, are the abscissae of the intersections of parabola $x^2=ay$ with the parabola $y^2=-\delta b\left(x+\frac{c^2}{\delta\eta b}\right)$. The first parabola has $A(0,0)$, $D_1\left(0,\ -\frac{a}{4}\right)$ $F_1\left(0,\ \frac{a}{4}\right)$, and the second that has $B\left(-\frac{c^2}{\delta\eta b},\ 0\right)$, $D_2\left(\frac{\delta b}{4},\ 0\right)$, $F_2\left(-\frac{\delta b}{4},\ 0\right)$ is constructed by constructing the points $C(0,c)$, $C'(0,-c)$, $B'(\delta\eta b,0)$, then the circumcircle of triangle $B'CC'$ meets the x-axis at the point B, the vertex of the second parabola.

5.3. *The quartic equation* $y^4 + \varepsilon a^2 y^2 + \delta a^2 b^2 = 0$. This quartic equation is the known biquadratic equation with roots that are ruler and compass constructible. If we put

$$y^2 = ax, \tag{8}$$

then the biquadratic equation becomes the quadratic $x^2 + \varepsilon ax + \delta b^2 = 0$, and since we know the construction of x, then from (8) we know the construction of y.

So we must construct the points $A(0, -b)$, $B(-\varepsilon a, -\delta b)$. The circle with diameter AB meets the x-axis at the points $A_1(x_1, 0)$ and $A_2(x_2, 0)$. If $\eta_1 = \text{sign}(x_1)$ and $\eta_2 = \text{sign}(x_2)$, then we construct the points $B_1(-\eta_1 a, 0)$, $B_2(-\eta_2 a, 0)$. The circles with diameters A_1B_1 and A_2B_2 meet the y-axis at four (if exist both A_1, A_2) points, the ordinates of which are the roots of the biquadratic equation.

Reference

[1] E. W. Weisstein, Carlyle Circle, *MathWorld–A Wolfram Web Resource*, `http://mathworld.wolfram.com/CarlyleCircle.html`.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: `ndergiades@yahoo.gr`