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Trisecting an Angle Correctly up to Arcminute

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Abstract. We present a simple compass-and-straightedge construction method of approximately trisecting an angle. This method is applicable to both acute and obtuse angles. With an original angle α, the construction gives an angle τ with error $|\epsilon| = |\tau - \alpha/3| < .0155^\circ$.

1. Introduction

Angle trisection is one of a few infamous problems that originated all the way back in ancient times but had required modern mathematics to settle. It was not until in the late 19th centuries that it could be proved rigorously that it is impossible to divide an arbitrary angle into three equal angles using only compass and straightedge.

However, for practical purposes, there are many approximate constructions that can trisect an angle up to a small error [1, 2, 3, 4]. Here we will present a new method of approximate construction. Start with an angle α, the proposed method constructs an angle τ with the error

$$\epsilon = \tau - \frac{\alpha}{3} = \frac{\alpha}{6} - \arctan \frac{\sin \frac{\alpha}{2}}{\sqrt{9 - 4\sin^2 \frac{\alpha}{2}}}.$$

This error is very small (less than an arcminute) which is due to the fact that its Taylor series have zero coefficients in degrees 0, 1, and 2.

$$\epsilon = -\frac{1}{1296}\alpha^3 + \frac{7}{20736}\alpha^5 + \frac{54903553}{3085588961280}\alpha^7 + \dots$$

It is quite simple to calculate this error and it would be a perfect trigonometry problem for students. It is also a nice calculus problem for students to find the Taylor series of the error function and to establish the maximum bound on the error.

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2. The construction

Given an angle $\angle xOy = \alpha$, the construction is as follows (see Figure 1)

- construct the bisector Ot,
- on Oy construct arbitrary points U, A, V such that $OU = UA = AV$,
- construct a circle centre at A with radius equal to OV which meets Ot at B,
- construct a point C on Ot such that $OB = BC$,
- through C, construct a line perpendicular to Ot which meets the line AB at D
- draw OD which makes $\angle xOD = \tau \approx \frac{\alpha}{3}$.

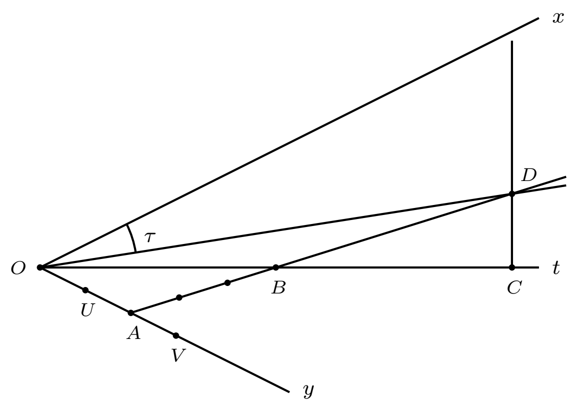

Figure 1. The proposed trisection construction

3. Calculating the error $\epsilon = \tau - \frac{\alpha}{3}$

Using the law of sines on triangle OAB, we have

$$\sin \angle OBA = \frac{2}{3} \sin \angle BOA = \frac{2}{3} \sin \frac{\alpha}{2}.$$

Comparing the two right triangles DOC and DBC we have

$$\tan \angle DOC = \frac{1}{2} \tan \angle DBC = \frac{1}{2} \tan \angle OBA.$$

Thus,

$$\tan \angle DOC = \frac{\sin \angle OBA}{2 \cos \angle OBA} = \frac{\frac{2}{3} \sin \frac{\alpha}{2}}{2\sqrt{1 - \frac{4}{9} \sin^2 \frac{\alpha}{2}}} = \frac{\sin \frac{\alpha}{2}}{\sqrt{9 - 4 \sin^2 \frac{\alpha}{2}}},$$

and

$$\tau = \angle xOD = \frac{\alpha}{2} - \angle DOC = \frac{\alpha}{2} - \arctan \frac{\sin \frac{\alpha}{2}}{\sqrt{9 - 4 \sin^2 \frac{\alpha}{2}}}.$$

We derive the error of the construction:

Theorem 1.

$$\epsilon = \tau - \frac{\alpha}{3} = \frac{\alpha}{6} - \arctan \frac{\sin \frac{\alpha}{2}}{\sqrt{9 - 4\sin^2 \frac{\alpha}{2}}}.$$

Comparing with the errors ϵ_S and ϵ_G of Steinhaus' [3, 4] and Gauld's constructions [2].

$$\epsilon_S = \arctan \frac{2\sin \frac{\alpha}{2}}{1 + 2\cos \frac{\alpha}{2}} - \frac{\alpha}{3},\ \epsilon_G = \arctan \frac{\sin \frac{\alpha}{2} + 2\sin \frac{\alpha}{4}}{\cos \frac{\alpha}{2} + 2\cos \frac{\alpha}{4}} - \frac{\alpha}{3}$$

our method is more accurate than Steinhaus' but a bit weaker than Gauld's. The following table shows the errors in degree.

α	$\lvert\epsilon\rvert$	$\lvert\epsilon_G\rvert$	$\lvert\epsilon_S\rvert$	α	$\lvert\epsilon\rvert$	$\lvert\epsilon_G\rvert$	$\lvert\epsilon_S\rvert$
5	.0000293	.0000073	.0000588	50	.0191908	.0073744	.0597073
10	.0002319	.0000588	.0004704	55	.0225624	.0098235	.0797398
15	.0007694	.0001984	.0015888	60	.0249879	.0127653	.1039094
20	.0017795	.0004704	.0037703	65	.0257239	.0162461	.1326482
25	.0033635	.0009191	.0073744	70	.0238342	.0203129	.1664038
30	.0055737	.0015888	.0127653	75	.0181658	.0250129	.2056412
35	.0083990	.0025243	.0203129	80	.0073259	.0303941	.2508445
40	.0117506	.0037703	.0303941	85	.0103425	.0365048	.3025190
45	.0154458	.0053719	.0433940	90	.0367826	.0433940	.3611934

4. Estimating the error

The error function $\epsilon(\alpha)$ is an odd function, calculting the derivatives,

$$\epsilon'(0) = 0,\ \epsilon^{(3)}(0) = -\frac{1}{216},\ \epsilon^{(5)}(0) = \frac{35}{864},\ \epsilon^{(7)}(0) = \frac{54903553}{612220032},$$

we have the Taylor series

$$\epsilon = -\frac{1}{1296}\alpha^3 + \frac{7}{20736}\alpha^5 + \frac{54903553}{3085588961280}\alpha^7 + \dots$$

We can manipulate the derivative of ϵ as follows

$$\epsilon'(\alpha) = \frac{1}{6} - \frac{3\cos \frac{\alpha}{2}}{2(2 + \cos^2 \frac{\alpha}{2})\sqrt{5 + 4\cos^2 \frac{\alpha}{2}}}$$

$$= \frac{2\sin^2 \frac{\alpha}{2}(\cos^2 \frac{\alpha}{2} + \frac{\sqrt{945}+25}{8})(\frac{\sqrt{945}-25}{8} - \cos^2 \frac{\alpha}{2})}{3(2 + \cos^2 \frac{\alpha}{2})\sqrt{5 + 4\cos^2 \frac{\alpha}{2}}((2 + \cos^2 \frac{\alpha}{2})\sqrt{5 + 4\cos^2 \frac{\alpha}{2}} + 9\cos \frac{\alpha}{2})}$$

It implies that $\epsilon'(\alpha) < 0$ for $\alpha \in [0, \frac{\pi}{4}]$, and so the function $\epsilon(\alpha)$ is decreasing on $[0, \frac{\pi}{4}]$. We have

$$\epsilon(0) = 0 \geq \epsilon(\alpha) \geq \epsilon(\frac{\pi}{4}) = \frac{\pi}{24} - \arctan \frac{1}{\sqrt{32 + 18\sqrt{2}}}.$$

Thus, we derive the following bound:

Theorem 2.

$$\max_{0\le\alpha\le\frac{\pi}{4}} |\epsilon(\alpha)| = \arctan\frac{1}{\sqrt{32+18\sqrt{2}}} - \frac{\pi}{24} < .0155^\circ$$

The above bound is established only for $\alpha \in [0, \frac{\pi}{4}]$. However, we can obtain the same bound for $\alpha \in (\frac{\pi}{4}, 2\pi)$ as follows. If $\alpha \in (\frac{\pi}{4}, 2\pi)$ then we can reduce the trisection of the angle α into the problem of trisection of another angle $\alpha' \in (0, \frac{\pi}{4})$ which is specified in the following table.

$\alpha \in [\frac{\pi}{4}, \frac{2\pi}{4}]$	$\alpha \in [\frac{2\pi}{4}, \frac{3\pi}{4}]$	$\alpha \in [\frac{3\pi}{4}, \frac{4\pi}{4}]$
$\alpha' = \frac{\pi}{2} - \alpha$	$\alpha' = \alpha - \frac{\pi}{2}$	$\alpha' = \pi - \alpha$
$\tau' = \frac{\pi}{6} - \tau$	$\tau' = \tau - \frac{\pi}{6}$	$\tau' = \frac{\pi}{3} - \tau$

$\alpha \in [\frac{4\pi}{4}, \frac{5\pi}{4}]$	$\alpha \in [\frac{5\pi}{4}, \frac{6\pi}{4}]$	$\alpha \in [\frac{6\pi}{4}, \frac{7\pi}{4}]$	$\alpha \in [\frac{7\pi}{4}, 2\pi]$
$\alpha' = \alpha - \pi$	$\alpha' = \frac{3\pi}{2} - \alpha$	$\alpha' = \alpha - \frac{3\pi}{2}$	$\alpha' = 2\pi - \alpha$
$\tau' = \tau - \frac{\pi}{3}$	$\tau' = \frac{\pi}{2} - \tau$	$\tau' = \tau - \frac{\pi}{2}$	$\tau' = \frac{2\pi}{3} - \tau$

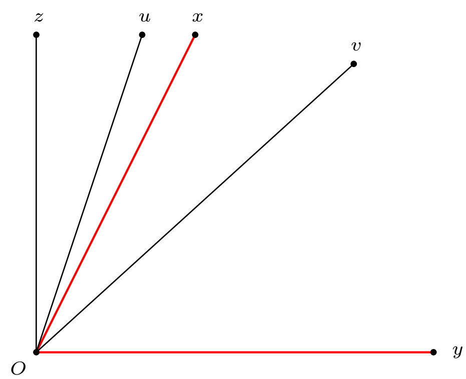

Figure 2. Trisection of α based on the trisection of $\alpha' = \frac{\pi}{2} - \alpha$ when $\alpha \in [\frac{\pi}{4}, \frac{\pi}{2}]$

For example, if $\alpha \in [\frac{\pi}{4}, \frac{\pi}{2}]$ as in Figure 2, then $\alpha' = \frac{\pi}{2} - \alpha$. First, we construct Oz perpendicular to Oy and by our above method, approximately trisect the angle $\alpha' = \angle xOz = \frac{\pi}{2} - \alpha$ by Ou, so we have $\tau' = \angle xOu \approx \frac{\alpha'}{3}$. Construct $\angle uOv = \frac{\pi}{6}$, then $\tau = \angle xOv = \frac{\pi}{6} - \tau'$ is an approximation of $\frac{\alpha}{3}$.

Since

$$|\epsilon'| = |\tau' - \frac{\alpha'}{3}| = |(\frac{\pi}{6} - \tau) - \frac{1}{3}(\frac{\pi}{2} - \alpha)| = |\tau - \frac{\alpha}{3}| = |\epsilon|,$$

the reduction from α to α' gives us the same error. Therefore, for any $\alpha \in (0, 2\pi)$, we can make a construction with an error $|\epsilon| < .0155^\circ$.

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