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Isogonal Conjugates in a Tetrahedron

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Abstract. The symmedian point of a tetrahedron is defined and the existence of the symmedian point of a tetrahedron is proved through a geometrical argument. It is also shown that the symmedian point and the least squares point of a tetrahedron are concurrent. We also show that the symmedian point of a tetrahedron coincides with the centroid of the corresponding pedal tetrahedron. Furthermore, the notion of isogonal conjugate to tetrahedra is introduced, with a simple formula in barycentric coordinates. In particular, the barycentric coordinates for the symmedian point of a tetrahedron are given.

1. Introduction

The *symmedian point* of a triangle is one of the 6,000 known points associated with the geometry of a triangle [4]. To define the symmedian point, we begin with the concept of isogonal lines. Two lines AR and AS through the vertex A of an angle are said to be isogonal if they are equally inclined from the sides that form $\angle A$. The lines that are isogonal to the medians of a triangle are called symmedian lines [3], pp. 75-76. Figure 1 (a) shows that the symmedian line AP of the triangle ABC is obtained by reflecting the median AM through the corresponding angle bisector AL. The symmedian lines intersect at a single point K known as the symmedian point, also called the Lemoine point. It turns out that the symmedian point of a triangle coincides with the point at which the sum of the squares of the perpendicular distances from the three sides of the triangle is minimum (the *least squares point*, LSP), [1]. Another property of the symmedian point of a triangle is described below. As shown in Figure 1 (b), let $A'B'C'$ the pedal triangle of K (i.e., the triangle obtained by projecting K onto the sides of the original triangle). Then the symmedian point of the triangle ABC and the centroid of the triangle $A'B'C'$ are concurrent.

The existence of symmedian point of a triangle was proved by the the French mathematician Emile Lemoine in 1873 ([3], Chapter 7). Later the symmedian point was defined by Marr for equiharmonic tetrahedrons in 1919 [5]. In the present work we provide the definition and prove the existence of the symmedian point of an arbitrary tetrahedron. Then we show that the symmedian point of a tetrahedron coincides with the LSP of that tetrahedron and the centroid of the corresponding

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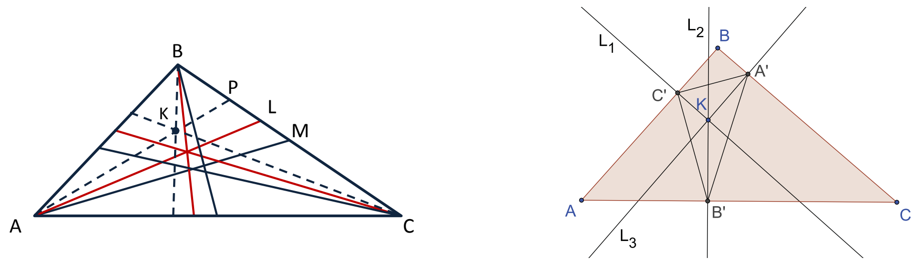
petal tetrahedron. Furthermore, we will demonstrate the utility of *least squares solution* for determining the location of the least squares points and hence the symmedian points.

The rest of this paper is organized as follows. In section 2, the existence of the symmedian point of a tetrahedron is proved. In section 3, it is shown that the symmedian point and LSP of a tetrahedron are concurrent. In section 4, the concurrency of the symmedian point and the centroid of the corresponding petal tetrahedron is proved. In section 5, a discussion of the main results is provided.



(a) (b)

Figure 1. *(a) The symmedian lines of triangle ABC intersect at the symmedian point K. (b) The symmedian point K of the ABC triangle coincides with the centroid $\widehat{C}$ of its pedal triangle, which is the $A'B'C'$ triangle formed by connecting the intersection points of the perpendicular lines L_1, L_2 and L_3 from K to the sides of the ABC triangle.*

2. Symmedian point of a tetrahedron and barycentric coordinates

Let $ABCD$ be a tetrahedron. Two planes (P) and (Q) through $\overline{AB}$, for instance, are said to be *isogonal* conjugates if they are equally inclined from the sides that form the dièdre angle between the planes of the triangles ABC and ABD. (P) is called the isogonal conjugate of (Q) and vice versa. If a point X of $ABCD$ is joined to vertex A and vertex B, the plane through $\overline{XA}$ and $\overline{XB}$ has an isogonal conjugate at A. Similarly, joining X to vertices B and D, D and C, A and C, B and C, C and D, produce five more conjugate planes. There is no immediately obvious reason why these six conjugates should be concurrent. However, that this is always the case will follow from lemma 2 below. Let M be the midpoint of $\overline{CD}$. The plane containing $\overline{AB}$ and that is isogonal to the plane of triangle ABM is called a *symmedian plane* of tetrahedron $ABCD$. Taking the midpoints of the six sides of the tetrahedron $ABCD$ and forming the associated symmedian planes, we call the intersection point of these symmedian planes the *symmedian point* of the tetrahedron. In this section we show that all six symmedian planes are indeed concurrent at a point. This definition of the symmedian point differs from the one given in [5], which was only defined for equiharmonic tetrahedrons [6]. For the existence of the symmedian point of an arbitrary tetrahedron, we first need the following two lemmas.

Lemma 1. *All six median planes obtained from a side of a tetrahedron and the midpoint of its opposite side are concurrent.*

Proof.
As shown in Figure 2 (a), let M_1 and M_2 be the midpoints of the opposite sides $\overline{CD}$ and $\overline{AB}$, respectively. The two median planes constructed from M_1 and $\overline{AB}$, and from M_2 and $\overline{CD}$ intersect at the line containing the points M_1 and M_2. Similarly, the other median planes constructed from $\overline{AC}$ and M_3, $\overline{BD}$ and M_4 contain $\overline{M_3M_4}$, and the planes formed with $\overline{BC}$ and M_5, and $\overline{AD}$ and M_6, contain $\overline{M_5M_6}$, where M_3, M_4, M_5, and M_6, are the midpoints of $\overline{BD}, \overline{AC}, \overline{AD}$, and $\overline{BC}$, respectively. Thus it is enough to show that the line segments $\overline{M_1M_2}$, $\overline{M_3M_4}$, and $\overline{M_5M_6}$ are concurrent. This can be shown by noticing that $\overline{M_1M_4}$ and $\overline{M_2M_3}$ are parallel to $\overline{AD}$, and $\overline{M_2M_4}$ and $\overline{M_1M_3}$ are both parallel to $\overline{BC}$. Thus the quadrilateral $M_1M_3M_2M_4$ is a parallelogram. It follows that the diagonals $\overline{M_3M_4}$ and $\overline{M_1M_2}$ cross each other at their midpoints. Similar argument shows that the quadrilateral $M_3M_5M_4M_6$ ia a parallelogram with diagonals $\overline{M_5M_6}$ and $\overline{M_3M_4}$ crossing each other at their midpoints. The desired result follows.$\square$

Lemma 2. *Consider the tetrahedron $ABCD$.*

(i): *If L and T are two points on two isogonal planes (P_1) and (P_2), respectively, through $\overline{AB}$, and if $\overline{LR}, \overline{LS}, \overline{TP}, \overline{TQ}$, are the perpendiculars from L and T to the triangles ABC, and ABD, respectively, then*

$$\frac{LR}{LS} = \frac{TQ}{TP} \tag{1}$$

(ii): *If L is on (P_1) and $LR/LS = TQ/TP$, then T is on (P_2), where (P_1) and (P_2) are isogonal planes through $\overline{AB}$.*

Proof.
To show (i) it is enough to show that the two triangles LRS and TQP are similar (see Figure 2 (b)). In fact, $\angle RLS = \angle PTQ = 180^\circ - \angle(\triangle ABC, \triangle ABD)$, where $\angle(\triangle ABC, \triangle ABD)$ is the dièdre angle between the planes of $\triangle ABC$ and $\triangle ABD$. Also, $\angle TPQ = \angle TNQ = \angle(\triangle ABD, (P_2))$, where $\angle(\triangle ABD, (P_2))$ is the dièdre angle between (P_2) and the plane of the triangle ABD, and N is the projection of P onto $\overline{AB}$. To see why notice that $\overline{TP}$ and $\overline{PN}$ are both perpendicular to $\overline{AB}$. Thus $\overline{AB}$ is also perpendicular to $\overline{TN}$. But, $\overline{AB}$ is also perpendicular to $\overline{TQ}$. Hence $\overline{AB}$ is perpendicular to the planes of the triangles QNT and PNT, and so these two triangles are in the same plane. Since the angles at its vertices P and Q are 90°, the quadrilateral $TPNQ$ is a circumscribed quadrilateral (vertices are located on the same circle) and so the equality $\angle TPQ = \angle TNQ$ holds. Similarly, $\angle LSR = \angle LOR = \angle(\triangle ABC, (P_1))$, where O is the projection of R onto $\overline{AB}$. But $\angle(\triangle ABD, (P_2)) = \angle(\triangle ABC, (P_1))$. So $\angle TPQ = \angle LSR$. The similarity of the triangles TQP, and LRS now follows. (ii) follows easily since in the triangles LRS and TQP, $\angle(PTQ) = \angle(RLS)$ and $LR/LS = TQ/TP$.$\square$

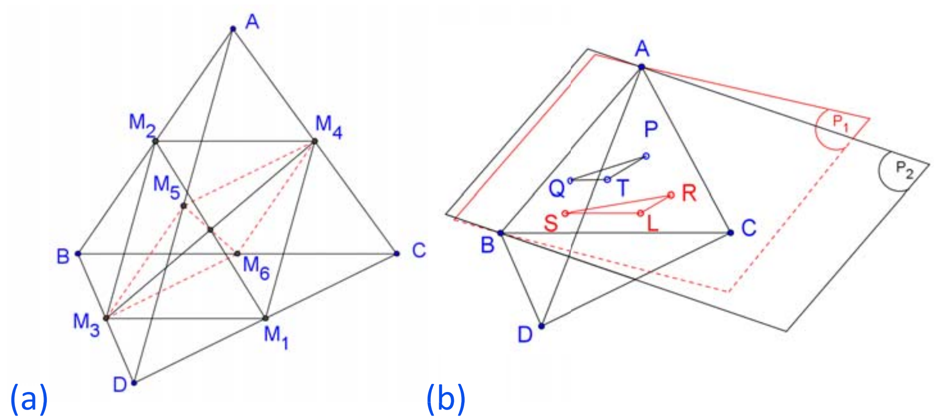


Figure 2. *(a) All six median planes obtained from a side of the $ABCD$ tetrahedron and the midpoint $M_i(1 \leq i \leq 6)$ of its opposite side are concurrent. (b) A representation of the isogonal planes (P_1) and (P_2), and the perpendicular lines from L and T to the triangles ABC and ABD, respectively.*

Now we are ready to show the existence of the symmedian point of a tetrahedron.

Theorem 3. *The symmedian planes are concurrent at a unique point K, the symmedian point of the tetrahedron.*

Proof.
Using Lemma 1, let M be the intersection point of all six median planes. Denote by $S_{\overline{EF}}$ the symmedian plane through a side $\overline{EF}$ and by P^X_{EFG} the orthogonal projection of a point X onto the plane formed by the three points $E, F,$ and G (no three vertices are located on the same line). Let K be the intersection point of the symmedian planes $S_{\overline{AB}}$, $S_{\overline{BC}}$, and $S_{\overline{AC}}$, and let W be the intersection of $S_{\overline{AB}}$ with $S_{\overline{BC}}$ and $S_{\overline{AD}}$. We will show that $W \in S_{\overline{AC}}$. In view of (ii) of Lemma 2, it suffices to show that

$$\frac{WP^W_{ACD}}{WP^W_{ABC}} = \frac{MP^M_{ABC}}{MP^M_{ACD}}.$$

Since W is in $S_{\overline{AD}}$, $S_{\overline{AB}}$, Lemma 2 (i) implies

$$\frac{WP^W_{ACD}}{WP^W_{ABD}} = \frac{MP^M_{ABD}}{MP^M_{ACD}}, \tag{2}$$

and

$$\frac{WP^W_{ABD}}{WP^W_{ABC}} = \frac{MP^M_{ABC}}{MP^M_{ABD}}, \tag{3}$$

Using (2), (3) we have

$$\frac{WP^W_{ACD}}{WP^W_{ABC}} = \frac{WP^W_{ACD}}{WP^W_{ABD}} \times \frac{WP^W_{ABD}}{WP^W_{ABC}} = \frac{MP^M_{ABD}}{MP^M_{ACD}} \times \frac{MP^M_{ABC}}{MP^M_{ABD}} = \frac{MP^M_{ABC}}{MP^M_{ACD}}.$$

Thus W coincides with K. Similar argument shows that the symmedian planes through $\overline{BD}$ and $\overline{CD}$ also pass through K.$\square$

Remark. An identical argument to the proof of Theorem 1 shows that if six planes are concurrent at X, where X is a point in the tetrahedron $ABCD$, then the six conjugate planes are also concurrent at a point X^*, the conjugate of X. In addition, as is in the triangle case, the restriction that X is a point *inside* $ABCD$ is unnecessary.

Now we explore the relationship between the barycentric coordinates of a point X and its isogonal conjugate X^*. Recall that in general, if $x_1, \cdots, x_n$ are the vertices of a simplex in affine space $\mathcal{A}$ and if $(a_1 + \cdots + a_n)X = a_1x_1 + \cdots + a_nx_n$ and at least one of the $a_i's$ does not vanish, then we say that the coefficients $(a_1 : \cdots : a_n)$ are *barycentric coordinates* of X, where $x \in\in \mathcal{A}$ [7]. Also, the barycentric coordinates are homogeneous:

$$(a_1, \cdots, a_n) = (\mu a_1 : \cdots : \mu a_n) \quad \mu \neq 0.$$

Analogous to the triangle case [2], we have the following property for the tetrahedron. Let X be a point in the space. Joining X to each vertex A, B, C, and D, four tetrahedra can be constructed. Let $X = (u : v : w : t)$ and $X^* = (u^* : v^* : w^* : t^*)$ be the barycentric coordinates of X and X^*, respectively, with respect to $ABCD$. Since *the volumes of these tetrahedra are proportional to the barycentric coordinates of* X, using lemma 2, and an argument similar to the proof of Theorem 1, one can establish the following

$$\frac{u^*u}{|\Delta BDC|^2} = \frac{w^*w}{|\Delta ABC|^2} = \frac{v^*v}{|\Delta ADC|^2} = \frac{t^*t}{|\Delta ABD|^2} = \mu,$$

where $|\Delta XYZ|$ denote the area of ΔXYZ. It follows that

$$\begin{aligned} X^* = (u^* : v^* : w^* : t^*) &= (\mu\frac{|\Delta BDC|^2}{u} : \mu\frac{|\Delta ABC|^2}{w} : \mu\frac{|\Delta ADC|^2}{v} : \mu\frac{|\Delta ABD|^2}{t}) \\ &= (\frac{|\Delta BDC|^2}{u} : \frac{|\Delta ABC|^2}{w} : \frac{|\Delta ADC|^2}{v} : \frac{|\Delta ABD|^2}{t}). \quad (4) \end{aligned}$$

(4) gives an extension of isogonal conjugates to tetrahedra with a simple formula in barycentric coordinates. Applying (4) to the centroid $(1 : 1 : 1 : 1)$, we obtain the coordinates of the symmedian point $(|\Delta BDC|^2 : |\Delta ABC|^2 : |\Delta ADC|^2 : |\Delta ABD|^2)$.

3. Concurrency of the Symmedian Point and the Least Squares Point

The LSP of a given tetrahedron $ABCD$ is the point from which the sum of the squares of the perpendicular distances to the four sides of the tetrahedron $ABCD$ is minimized. Now we show that the symmedian point and the LSP of a tetrahedron are concurrent. We start with the following lemma.

Lemma 4. *For a tetrahedron $ABCD$, let M be the midpoint of $\overline{CD}$ and $\overline{MP}$ and $\overline{MQ}$ be the perpendicular line segments from M to ΔABC and ΔABD, respectively. Then we have*

$$\frac{MQ}{MP} = \frac{\textit{area}(\triangle ABC)}{\textit{area}(\triangle ABD)} \tag{5}$$

Similar equalities hold if M is replaced with the midpoints of the other sides of the tetrahedron $ABCD$.

Proof.
Let $\overline{AH}$ be the perpendicular from A to $\triangle BCD$. Now note that

$$\begin{aligned} \frac{1}{3}MQ \times \text{area}(\triangle ABD) &= \text{volume}(ABMD) \\ &= \frac{1}{3}AH \times \text{area}(\triangle BMD) \\ &= \frac{1}{3}AH \times \text{area}(\triangle BMC) \\ &= \text{volume}(ABCM) \\ &= \frac{1}{3}MP \times \text{area}(\triangle ABC), \end{aligned}$$

which gives rise to equation (5). □

Now we can prove the concurrency of the symmedian point and the LSP of a tetrahedron.

Theorem 5. *The symmedian point K of tetrahedron $ABCD$ coincides with its LSP.*

Proof.
First, Lemma 2 (i) together with Lemma 4 imply

$$\frac{x}{\text{area}(\triangle ABC)} = \frac{y}{\text{area}(\triangle ABD)} = \frac{z}{\text{area}(\triangle ACD)} = \frac{w}{\text{area}(\triangle BCD)}, \tag{6}$$

where x, y, z, w are the distances from the the symmedian point to the triangles ABC, ABD, ACD, BCD, respectively.
Second, let $\text{area}(\triangle ABC) = a$, $\text{area}(\triangle ABD) = b$, $\text{area}(\triangle ACD) = c$, $\text{area}(\triangle BCD) = d$. By Lagrange's identity,

$$\begin{aligned} (x^2+y^2+z^2+w^2)&(a^2+b^2+c^2+d^2) - (ax+by+cz+dw)^2 \\ &= (bx-ay)^2 + (cx-az)^2 + (dx-aw)^2 \\ &\quad + (cy-bz)^2 + (dy-bw)^2 + (dz-cw)^2. \end{aligned} \tag{7}$$

Since $a^2+b^2+c^2+d^2$ is constant for all x, y, z, w, and $ax+by+cz+dw = 3\text{vol}(ABCD)$, $(x^2+y^2+z^2+w^2)$ is minimum if and only if the right hand side of (7) is zero. This occurs only when

$$bx = ay,\ cx = az,\ dx = aw,\ cy = bz,\ dy = bw,\ dz = cw.$$

In view of (6), this occurs at the symmedian point K. So the symmedian point coincides with the LSP. □

4. Concurrency of the Symmedian Point and the Centroid of the Corresponding Petal Tetrahedron

In this section we show that the symmedian point of a tetrahedron coincides with the centroid of the corresponding pedal tetrahedron.

Theorem 6. *The symmedian point of a tetrahedron coincides with the centroid of the corresponding pedal tetrahedron.*

Proof.
Let K be the symmedian point of the tetrahedron $ABCD$. Drop the perpendiculars from K to the four sides of the tetrahedron $ABCD$ and let their intersection with $\Delta ABC, \Delta ABD, \Delta ACD, \Delta BCD$ be the points V_1, V_2, V_3, V_4, respectively. Let $\widehat{C}$ be the centroid of the pedal tetrahedron $V_1V_2V_3V_4$ of K. It is well known that $\widehat{C}$ minimizes the sum of the squares of the distances to four vertices V_1, V_2, V_3, V_4. So we have

$$\sum_{i=1}^{4}(\widehat{C}V_i)^2 \le \sum_{i=1}^{4}(XV_i)^2 \ \text{ for any } \ X \in R^3. \tag{8}$$

Suppose $\widehat{C} \neq K$. Drop the perpendiculars from $\widehat{C}$ to the four sides of the tetrahedron $ABCD$ and let their intersection with ΔABC, ΔABD, ΔACD, ΔBCD be the points W_1, W_2, W_3, W_4, respectively. Since K is also the LSP of the tetrahedron $ABCD$

$$\sum_{i=1}^{4}(KV_i)^2 < \sum_{i=1}^{4}(\widehat{C}W_i)^2. \tag{9}$$

Note also that we have $\widehat{C}W_i \le \widehat{C}V_i$ for each i. So using (8) with $X = K$, we have

$$\sum_{i=1}^{4}(\widehat{C}W_i)^2 \le \sum_{i=1}^{4}(\widehat{C}V_i)^2 \le \sum_{i=1}^{4}(KV_i)^2,$$

which contradicts (9). So we must have $\widehat{C} = K$. □

Corollary 7. *The symmedian point and hence the LSP of a tetrahedron belongs to its interior.*

Proof.
Since $K = \widehat{C}$ and $\widehat{C}$ is in the interior of the petal tetrahedron and the pertal tetrahedron is in the interior of the given tetrahedron, the symmedian point K of the given tetrahedron belongs to its interior. □

5. Discussion

In this section we show that our symmedian point of a tetrahedron $ABCD$ is different from the symmedian point defined by Marr [5]. Marr's symmedian point of an equiharmonic tetrahedron (that is, $AD \times BC = AB \times CD = AC \times BD$)

can be defined as the point of intersection of the lines joining the vertices to the symmedian points of the opposite faces. Now we give an example that shows that our symmedian point is different from Marr's symmedian point.

Example 1. Consider the tetrahedron $ABCD$ such that $A(0,0,0)$, $B(1,0,0$, $C(0,1,0)$ and $D(0,0,1)$. Note that the tetrahedron $ABCD$ is equiharmonic and one can compute Marr's symmedian point $\tilde{K} = (1/5, 1/5, 1/5)$. Our symmedian point is $K(1/6, 1/6, 1/6)$. So $\tilde{K} \neq K$.

In summary, the merit of the present work is twofold. First, the definition of the symmedian point of a tetrahedron is a true generalization of the symmedian point of a triangle, because they both coincide with their corresponding least square points. Second, the notion of isogonal conjugate has been extended to tetrahedra, with a simple formula in barycentric coordinates. In particular, a formula for the symmedian point of a tetrahedron has been given in terms of the barycentric coordinates.

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