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Archimedes' Arbelos to the n-th Dimension

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Abstract. The *arbelos* was introduced in Proposition 4 of Archimedes' Book of Lemmas. It is the plane figure bounded by three pairwise tangent semicircles with diameters lying on the same line. This figure has several interesting properties that have been studied over time. For example, the area of the arbelos equals the area of the circle whose diameter is the portion inside the arbelos of the common tangent to the smaller circles. In this paper we consider n-dimensional analogues of this latter property.

1. Introduction

The *arbelos* ($\acute{\alpha}\rho\beta\eta\lambda o\varsigma$, literally "shoemaker's knife") was introduced in Proposition 4 of Archimedes' Book of Lemmas [2, p. 304]. It is the plane figure bounded by three pairwise tangent semicircles with diameters lying on the same line (see the left-hand side of Figure 1). In addition to the properties proved by Archimedes himself, there is a long list of properties satisfied by this figure. Boas's paper [3] presents some of them and is a good source for references.

It is quite surprising to discover that for 23 centuries no generalizations of this figure were introduced. Sondow [5] extended the original construction considering latus rectum arcs of parabolas instead of semicircles (see the center of Figure 1). In his paper, Sondow proves several interesting properties of his construction (named *parbelos*) that are, in some sense, counterparts of properties of the arbelos. More recently, the author [4] has considered a more general situation where the figure is bounded by the graphs of three functions that are similar, thus extending many of the known properties of the arbelos and parbelos. An example of this general construction (named f-belos) can be seen in the right-hand side of Figure 1.

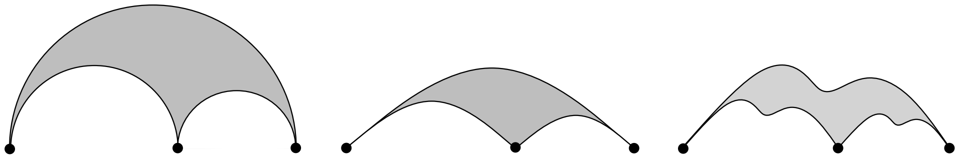

Figure 1. An arbelos (left), a parbelos (center) and an f-belos (right)

The idea of a 3-dimensional arbelos has already been introduced by Abu-Saymed and Hajja [1]. These authors define a 3-dimensional arbelos as the figure bounded by three hemispheres such that two are externally tangent to each other, and internally tangent to the third and whose equatorial circles lie on the same plane (see

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Figure 2). Nevertheless, not much attention has been paid to this possible generalization.

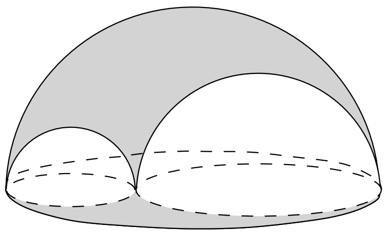

Figure 2. 3-dimensional arbelos

The following result was proved by Archimedes [2, Proposition 4]. We will refer to it as the *fundamental property* of the arbelos.

Proposition 1. *The area of the arbelos* (see Figure 3) *equals the area of the circle whose diameter AB is the portion inside the arbelos of the common tangent to the smaller circles. In other words, the area S of the arbelos is:*

$$S = \pi\left(\frac{\overline{AB}}{2}\right)^2.$$

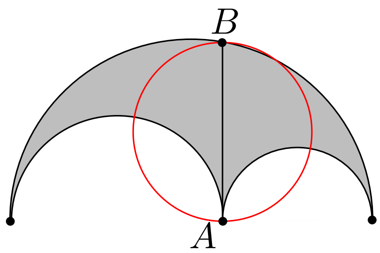

Figure 3. The fundamental property.

Remark. Observe that if R_1 and R_2 are the radii of the inner circles of the arbelos, then $\overline{AB} = 2\sqrt{R_1R_2}$.

In this paper we will present the analogue of the fundamental property in the 3-dimensional case and we will search for a possible generalization in the n-dimensional case.

2. The fundamental property in 3 dimensions

Let us consider a 3-dimensional arbelos such that the radii of the inner hemispheres are R_1 and R_2. This implies that the outer hemisphere has radius $R_1 + R_2$. Hence, the volume of the arbelos is:

$$V = \frac{1}{2}\left[\frac{4}{3}\pi(R_1+R_2)^3 - \frac{4}{3}\pi R_1^3 - \frac{4}{3}\pi R_2^3\right] = 2\pi R_1R_2(R_1+R_2). \tag{1}$$

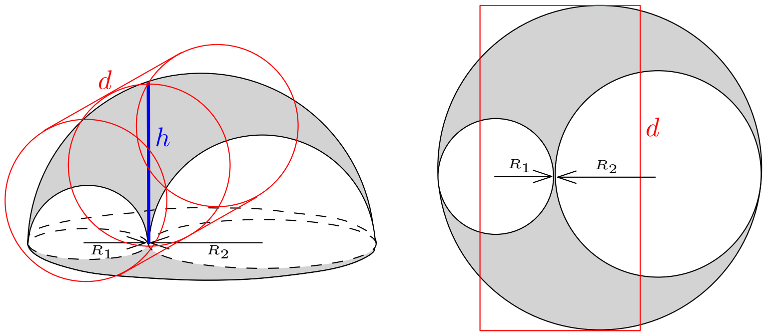

Figure 4. The fundamental property in dimension 3

Let us denote by h the length of the segment which is tangent to both inner hemispheres and perpendicular to the "base plane" (see Figure 4). Clearly we have that $h/2 = \sqrt{R_1 R_2}$. Also observe that $d = 2(R_1 + R_2)$ is precisely the diameter of the outer hemisphere (and also of its equatorial circle). The following result is an analogue of the fundamental property.

Proposition 2. *The volume of the 3-dimensional arbelos equals the volume of a cylinder whose base has diameter h and whose height is the diameter of the outer equatorial circle of the arbelos.*

Proof. The volume of such a cylinder is

$$V_c = \pi \left(\frac{h}{2}\right)^2 2(R_1 + R_2) = 2\pi R_1 R_2 (R_1 + R_2),$$

which coincides with the volume of the arbelos (1). □

3. The fundamental property in n dimensions

In order to extend the fundamental property to an arbitrary dimension we need to consider the volume of an n-dimensional ball. If we denote by $V_n(R)$ the bolumen of an n-dimensional ball of radius R, it is well-known that:

$$V_n(R) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} R^n,$$

where Γ denotes Euler Gamma function.

Hence, in order to extend the fundamental property of the arbelos, we are interested in the difference:

$$D_n = \frac{1}{2}\Big[V_n(R_1 + R_2) - V_n(R_1) - V_n(R_2)\Big].$$

In the previous sections we have seen that:

$$D_2 = \pi R_1 R_2 = V_2(h/2),$$
$$D_3 = 2\pi R_1 R_2 (R_1 + R_2) = V_2(h/2) V_1(d/2).$$

where $h/2 = \sqrt{R_1 R_2}$ and $d/2 = R_1 + R_2$.

Clearly we have that

$$D_n = \frac{\pi^{n/2}}{2\Gamma\left(\frac{n}{2}+1\right)}\Big[(R_1+R_2)^n - R_1^n - R_2^n\Big],$$

so we just have to analyze the behavior of $\delta_n(R_1, R_2) = (R_1+R_2)^n - R_1^n - R_2^n$. In particular, we want to express δ_n in terms of R_1R_2 and R_1+R_2.

To do so, for a fixed positive integer n, we recursively introduce a family of numbers $\{A_{p,q}(n) \mid 1 \leq p \leq n/2,\ 1 \leq q \leq n-2p+2\}$ in the following way:

$$\begin{aligned} A_{1,i}(n) &= \binom{n}{i}, \\ A_{k,i}(n) &= A_{k-1,i+1}(n) - A_{k-1,1}(n)\binom{n-2k+2}{i}. \end{aligned}$$

Obviously $A_{p,q}(n) \in \mathbb{Z}$ for every p, q for which $A_{p,q}(n)$ makes sense. Moreover, the following lemma gives a closed form for $A_{k,1}(n)$ that will be useful in the sequel. The proof is inductive and we omit.

Lemma 3.

$$A_{k,1}(n) = (-1)^{k+1}\frac{n}{k}\binom{n-k-1}{k-1}.$$

The following result shows how to express $\delta_n(R_1, R_2)$ as a polynomial in (R_1R_2) and (R_1+R_2).

Proposition 4. *For every integer $n \geq 2$, the following holds:*

$$\delta_n(x,y) = (x+y)^n - x^n - y^n = \sum_{k=1}^{\lfloor n/2 \rfloor} A_{k,1}(n)(xy)^k(x+y)^{n-2k}.$$

Proof. We will give a sketch of the proof. Due to its recursive nature, details are left to the reader.

$$\begin{aligned} \delta_n(x,y) &= xy\sum_{k=1}^{n-1}\binom{n}{k}x^{n-k-1}y^{k-1} \\ &= xy\left[\binom{n}{1}(x+y)^{n-2} + \sum_{k=1}^{n-3}\left[\binom{n}{k+1} - \binom{n}{1}\binom{n-2}{k}\right]x^{n-k-2}y^k\right] \\ &= A_{1,1}xy(x+y)^{n-2} + xy\sum_{k=1}^{n-3}A_{2,k}x^{n-k-2}y^k \\ &= A_{1,1}xy(x+y)^{n-2} + (xy)^2\sum_{k=1}^{n-3}A_{2,k}x^{n-k-3}y^{k-1} \\ &= A_{1,1}xy(x+y)^{n-2} + (xy)^2\left[A_{2,1}(x+y)^{n-4} + \sum_{k=1}^{n-5}A_{3,k}x^{n-k-4}y^k\right] \end{aligned}$$

$$
\begin{aligned}
&= A_{1,1}xy(x+y)^{n-2} + (xy)^2 \sum_{k=1}^{n-3} A_{2,k} x^{n-k-3} y^{k-1} \\
&= A_{1,1}xy(x+y)^{n-2} + (xy)^2 \left[A_{2,1}(x+y)^{n-4} + \sum_{k=1}^{n-5} A_{3,k} x^{n-k-4} y^k \right] \\
&= A_{1,1}xy(x+y)^{n-2} + A_{2,1}(xy)^2(x+y)^{n-4} + (xy)^2 \sum_{k=1}^{n-5} A_{3,k} x^{n-k-4} y^k \\
&\vdots \\
&= \sum_{k=1}^{\lfloor n/2 \rfloor} A_{k,1}(n)(xy)^k (x+y)^{n-2k}.
\end{aligned}
$$

□

With all these ingredients, we can present the main result of the paper. Recall that $h = 2\sqrt{R_1 R_2}$ and $d = 2(R_1 + R_2)$.

Theorem 5. *Let $n \geq 2$ be any integer. Then:*

$$D_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha_k(n) \Big[V_2(h/2) \Big]^k V_{n-2k}(d/2),$$

where

$$\alpha_k(n) = \frac{(-2)^{k-1}(n-2k)!!(n-k-1)!}{(n-2)!!k!(n-2k)!}.$$

Proof.

$$
\begin{aligned}
D_n &= \frac{\pi^{n/2}}{2\Gamma\left(\frac{n}{2}+1\right)} \delta_n(R_1, R_2) = \frac{\pi^{n/2}}{2\Gamma\left(\frac{n}{2}+1\right)} \sum_{k=1}^{\lfloor n/2 \rfloor} A_{k,1}(R_1 R_2)^k (R_1 + R_2)^{n-2k} \\
&= \frac{\pi^{n/2}}{2\Gamma\left(\frac{n}{2}+1\right)} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \frac{n}{k} \binom{n-k-1}{k-1} (R_1 R_2)^k (R_1 + R_2)^{n-2k} \\
&= \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \frac{\Gamma\left(\frac{n-2k}{2}+1\right)}{k\Gamma\left(\frac{n}{2}\right)} \binom{n-k-1}{k-1} (\pi R_1 R_2)^k \frac{\pi^{\frac{n-2k}{2}}}{\Gamma\left(\frac{n-2k}{2}+1\right)} (R_1 + R_2)^{n-2k} \\
&= \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha_k(n) \Big[V_2(h/2) \Big]^k V_{n-2k}(d/2).
\end{aligned}
$$

So, to finish the proof we will have a closer look at $\alpha_k(n)$.

$$\begin{aligned}\alpha_k(n) &= (-1)^{k+1}\frac{\Gamma\left(\frac{n-2k}{2}+1\right)}{k\Gamma\left(\frac{n}{2}\right)}\binom{n-k-1}{k-1}\\ &= (-1)^{k-1}\frac{(n-2k)!!\sqrt{\pi}2^{\frac{n-1}{2}}}{2^{\frac{n-2k+1}{2}}k(n-2)!!\sqrt{\pi}}\frac{(n-k-1)!}{(k-1)!(n-2k)!}\\ &= \frac{(-2)^{k-1}(n-2k)!!(n-k-1)!}{(n-2)!!k!(n-2k)!}\end{aligned}$$

□

Remark. Theorem 5 above extends the known results in $n = 2, 3$. In fact:

- In the case $n = 2$ Theorem 5 implies that (recall Proposition 1):

$$D_2 = \alpha_1(2)V_2(h/2)V_0(d/2) = V_2(h/2) = \pi\left(\frac{h}{2}\right)^2 = \pi R_1 R_2.$$

- In the case $n = 3$ we have that (recall Proposition 2):

$$D_3 = \alpha_1(3)V_2(h/2)V_1(d/2) = V_2(h/2)V_1(d/2) = 2\pi R_1 R_2(R_1 + R_2).$$

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