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An Improvement of Bîrsan's Inequalities for the Sides of a Triangle

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Abstract. In terms of the distance d between the circumcenter and the incenter of a triangle, Bîrsan's inequalities can be rewritten as $4((R-d)^2-r^2) \leq a^2 \leq 4((R+d)^2-r^2)$, where R and r are respectively the circumradius and inradius, and a is the length of a side of the triangle. We improve these inequalities by finding the best lower and upper bounds of a^2.

1. Introduction

Let R and r be the circumradius and the inradius of an arbitrary triangle ABC, O and I be the circumcenter and the incenter of the triangle ABC, $d = OI$ (see Figure 1).

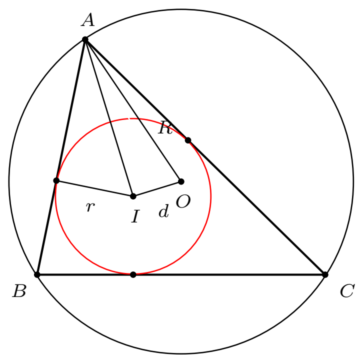

Figure 1

T. Bîrsan [1] has established the following inequalities

$$8R^2-8Rr-4r^2-8R\sqrt{R^2-2Rr} \leq a^2 \leq 8R^2-8Rr-4r^2+8R\sqrt{R^2-2Rr}, \tag{1}$$

where a is the length of a side of the triangle ABC. By Euler's theorem, $d = \sqrt{R^2-2Rr}$ (see, for example, [2, Theorem 295] or [3]). Hence, Bîrsan inequalities (1)) may been written as

$$4((R-d)^2-r^2) \leq a^2 \leq 4((R+d)^2-r^2). \tag{2}$$

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In this paper, we improve these inequalities by finding the best lower and upper bounds of a^2 for the lengths of the sides. Here is the main result.

Theorem 1. *Let a be the length of a side of a triangle with circumradius R, inradius r, and distance d between the circumcenter and the incenter.*

(a) *If $(\sqrt{2}+1)r \le R$, then*

$$4((R-d)^2-r^2) < 16R^2r^2\frac{(R+d)^2-r^2}{(R+d)^4} \le a^2 \le 4R^2 < 4((R+d)^2-r^2).$$

(b) *If $2r \le R < (\sqrt{2}+1)r$, then*

$$4((R-d)^2-r^2) \le 16R^2r^2\frac{(R+d)^2-r^2}{(R+d)^4} \le a^2 \le 16R^2r^2\frac{(R-d)^2-r^2}{(R-d)^4} \le 4((R+d)^2-r^2).$$

(c) *The improved lower bounds and upper bounds in* (a) *and* (b) *over* (2) *are best possible.*

We shall make use of the following lemma.

Lemma 2. *For any triangle ABC, the circumcenter O is in the interior of the incircle if and only if $2r \le R < (\sqrt{2}+1)r$.*

Proof. The first inequality $2r \le R$ follows from $0 \le d = \sqrt{R(R-2r)}$.

The circumcenter O is in the interior of the incircle of the triangle ABC if and only if $d^2 < r^2$. This condition is equivalent to $R^2 - 2Rr - r^2 < 0$, i.e.,

$$(R-(\sqrt{2}+1)r)(R+(\sqrt{2}+1)r) < 0.$$

Since the second factor is positive, we have $2r \le R < (\sqrt{2}+1)r$. □

Proof of Theorem 1

Let R be the radius of some circle ω, O be the center of ω, r be a number such that $2r \le R$. Denote $d = \sqrt{R^2-2Rr}$. Also, let MN be a diameter of ω. Consider a point I such that I lies in the diameter MN and $|IO| = d$ (see Figure 2).

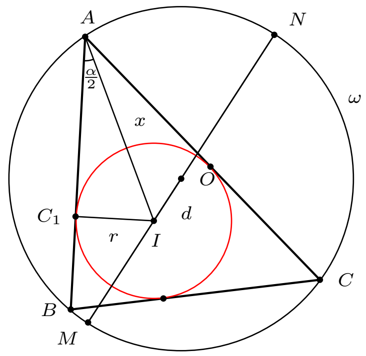

Figure 2

Take an arbitrary point A of the circle ω and draw the tangents AB and AC to the circle with radius r and center I. By Poncelet's closure theorem, the segment BC is also tangent to the circle. Moreover, every triangle with circumradius R and inradius r can be constructed in this way (see also [3]).

For an arbitrary point A on ω, let $x = |AI|$. Assume $|IM| < |IN|$. Since the two circles center I, passing through M and N respectively are tangent to ω, $|MI| \leq x \leq |IN|$, i.e., $x \in [R-d; R+d]$.

Let $\angle BAC = \alpha$ and C_1 be the orthogonal projection of I on AB. Then $|AC_1| = \sqrt{x^2 - r^2} = x\cos\frac{\alpha}{2}$ and $|IC_1| = r = x\sin\frac{\alpha}{2}$. Hence,

$$a = 2R\sin\alpha = 4R\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 4R\cdot\frac{r}{x}\cdot\frac{\sqrt{x^2-r^2}}{x} = 4Rr\frac{\sqrt{x^2-r^2}}{x^2},$$

i.e.,

$$a^2(x) = 16R^2r^2\cdot\frac{x^2-r^2}{x^4}.$$

For the function $a^2(x)$ on the interval $[R-d; R+d]$, we have

$$\frac{d}{dx}(a^2(x)) = 32R^2r^2\cdot\frac{2r^2-x^2}{x^5}.$$

Let $x_0 = \sqrt{2}r$. Clearly, for any $x > 0$, $\frac{d}{dx}(a^2(x))$ is positive, zero, or negative according as $x < x_0$, $= x_0$, or $x > x_0$. Therefore, x_0 is a maximum of $a^2(x)$, provided $x_0 \in [R-d,\ R+d]$, and the function is increasing on $[R-d,\ x_0]$, and decreasing on $[x_0,\ R+d]$.

(a) Let $R - d \leq \sqrt{2}r$. Note that in this case $x_0 = \sqrt{2}r < 2r \leq R < R+d$. So $x_0 \in [R-d; R+d)$. Furthermore,

$$R-d \leq \sqrt{2}r \iff (R-\sqrt{2}r)^2 \leq d^2 \iff (\sqrt{2}+1)r \leq R.$$

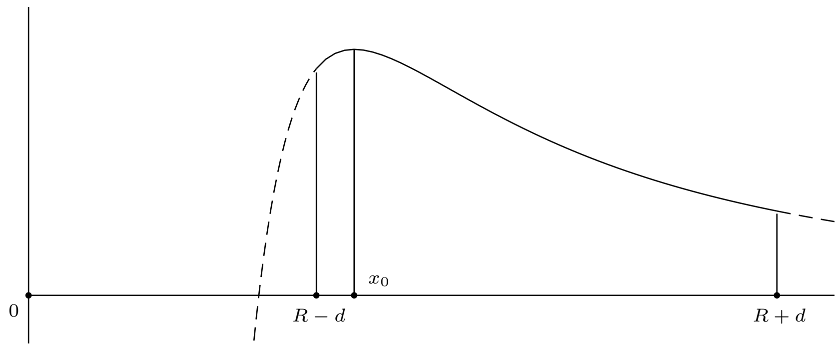

Figure 3

By Lemma 2, the circumcenter O is not in the interior of the incircle of the triangle ABC. Furthermore,

$$\frac{a^2(R-d)-a^2(R+d)}{16R^2r^2} = \frac{(R-d)^2-r^2}{(R-d)^4} - \frac{(R+d)^2-r^2}{(R+d)^4} = \frac{8Rr^2d(R^2-d^2)}{(R-d)^4(R+d)^4} > 0,$$

i.e., $a^2(R-d) > a^2(R+d)$. The maximal value $a^2(x_0) = a^2(\sqrt{2}r) = 4R^2$, and the minimal value $a^2(R+d) = 16R^2r^2\frac{(R+d)^2-r^2}{(R+d)^4}$.

Therefore, for $R \geq (\sqrt{2}+1)r$,

$$16R^2r^2\frac{(R+d)^2-r^2}{(R+d)^4} \leq a^2 \leq 4R^2. \tag{3}$$

(b) Let $R-d > \sqrt{2}r$.

By Lemma 2, the condition $R-d > \sqrt{2}r$ implies that the circumcenter O is in the interior of the incircle. In this case, x_0 is outside the interval $[R-d,\ R+d]$, and the function $a^2(x)$ is decreasing (see Figure 4).

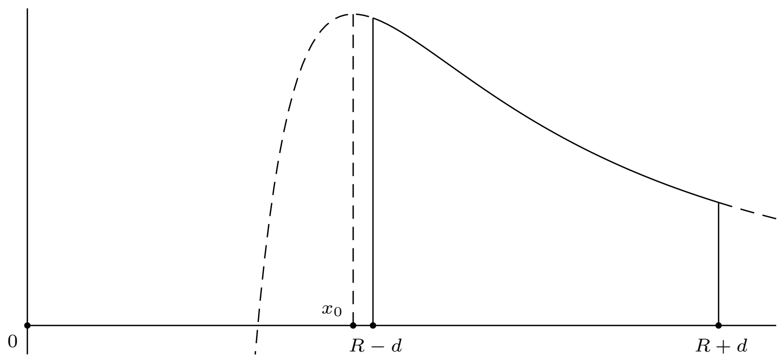

Figure 4

Therefore, $a^2(R-d) \geq a^2 \geq a^2(R+d)$, and we have

$$16R^2r^2\frac{(R+d)^2-r^2}{(R+d)^4} \leq a^2 \leq 16R^2r^2\frac{(R-d)^2-r^2}{(R-d)^4}. \tag{4}$$

(c) Clearly in (3) and (4), the lower and upper bounds cannot be improved.

This completes the proof of Theorem 1.

References

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