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Golden Sections of Triangle Centers in the Golden Triangles

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Abstract. A golden triangle is one whose vertices are among the vertices of a regular pentagon. There are two kinds of golden triangles, short and tall, which are isosceles triangles with vertical angles 108° and 36° respectively. We consider some basic triangle centers of a short and tall golden triangles sharing one vertex and with the same circumcircle, and exhibit pairs of basic triangle centers divided in the golden ratio by another triangle center.

As is well known, the golden ratio naturally occurs in the regular pentagon, as the ratio of the length of a diagonal d and a side a: $\varphi := \frac{d}{a} = \frac{1}{2}(\sqrt{5}+1)$. The intersection of two diagonals divides each in the golden ratio. If $ABCDE$ is a regular pentagon, and the diagonals AD and BE intersect at P (see Figure 1), then

$$\frac{BE}{BP} = \frac{BP}{PE} = \varphi, \qquad \frac{DA}{DP} = \frac{DP}{PA} = \varphi.$$

For later use, we note the following simple trigonometric ratios from Figure 1:

$$\cos 36^\circ = \frac{d/2}{a} = \frac{\varphi}{2}, \qquad \sin 18^\circ = \frac{a/2}{d} = \frac{1}{2\varphi}.$$

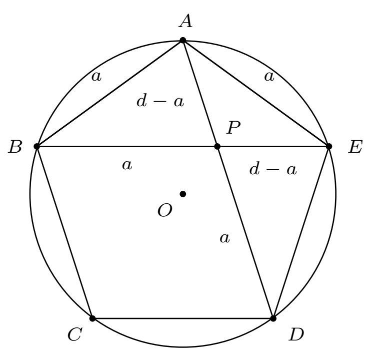

Figure 1. The golden section

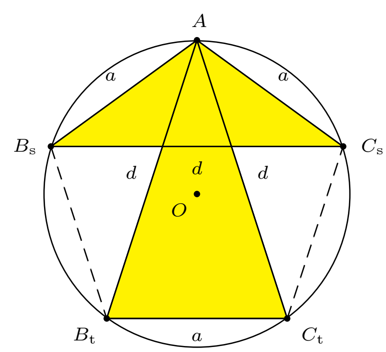

Figure 2. Short and tall golden triangles

Given a regular pentagon, the subtriangles with vertices among those of the pentagon are all isosceles. They fall into two types:
(i) those with three adjacent vertices of the pentagon have angles 108°, 36°, 36°, which we call *short golden triangles*,

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(ii) those with only two adjacent vertices of the pentagon have angles 36°, 72°, 72°, which we call *tall golden triangles*.

In this note we consider golden sections in the two kinds of golden triangles. For purpose of comparison, we consider a pair of short and tall golden triangles inscribed in the same regular pentagon $AB_\mathrm{s}B_\mathrm{t}C_\mathrm{t}C_\mathrm{s}$ (see Figure 2). The short golden triangle $\mathbf{T}_\mathrm{s} := AB_\mathrm{s}C_\mathrm{s}$ has sides d, a, a; the tall golden triangle $\mathbf{T}_\mathrm{t} := AB_\mathrm{t}C_\mathrm{t}$ has sides a, d, d. They share the same circumcenter O. Denote by R their common circumradius. Note that the areas Δ_i, $i = \mathrm{s}, \mathrm{t}$, of the golden triangles are in the golden ratio:

$$\frac{\Delta_\mathrm{t}}{\Delta_\mathrm{s}} = \frac{\frac{1}{2}ad\sin 72^\circ}{\frac{1}{2}a^2\sin 108^\circ} = \frac{d}{a} = \varphi.$$

For $i = \mathrm{s}, \mathrm{t}$, since the golden triangle $\mathbf{T}_i$ is isosceles, its triangle centers are all on the (common) perpendicular bisector of the side B_iC_i. We shall call this the *center line* of the golden triangles; It contains the midpoints F_i of B_iC_i (see Figure 3).

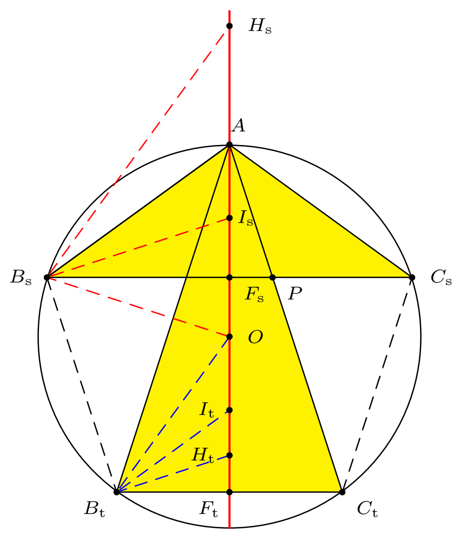

Figure 3

Here are some simple constructions of the basic triangle centers of $\mathbf{T}_\mathrm{s}$ and $\mathbf{T}_\mathrm{t}$.

(1) The incenter I_s of $\mathbf{T}_\mathrm{s}$ is the intersection of the center line with the perpendicular of $B_\mathrm{s}B_\mathrm{t}$ at B_s; it is also the reflection of O in the side $B_\mathrm{s}C_\mathrm{s}$. From this, the inradius of $\mathbf{T}_\mathrm{s}$ is

$$r_\mathrm{s} = F_\mathrm{s}I_\mathrm{s} = OF_\mathrm{s} = R\cos 72^\circ = \frac{R}{2\varphi}.$$

(2) The incenter I_{t} of $\mathbf{T}_{\mathrm{t}}$ is the intersection of the diagonals $B_{\mathrm{t}}C_{\mathrm{s}}$ and $C_{\mathrm{t}}B_{\mathrm{s}}$. It is also the reflection of A in the side $B_{\mathrm{s}}C_{\mathrm{s}}$. From this, the inradius of $\mathbf{T}_{\mathrm{t}}$ is

$$\begin{aligned} r_{\mathrm{t}} = & \ \frac{a}{2}\tan 36^\circ = R\sin 36^\circ\tan 36^\circ = R\cdot\frac{\sin^2 36^\circ}{\cos 36^\circ} \\ = & \ R\cdot\frac{1-\left(\frac{\varphi}{2}\right)^2}{\frac{\varphi}{2}} = R\cdot\frac{4-\varphi^2}{2\varphi} = \frac{R}{2}(3\varphi-4). \end{aligned}$$

(3) Let H_{s} be the orthocenter of $\mathbf{T}_{\mathrm{s}}$. Clearly $\angle H_{\mathrm{s}}OB_{\mathrm{s}} = \angle I_{\mathrm{s}}OB_{\mathrm{s}} = 72^\circ$. Since H_{s} is the isogonal conjugate of O in $\mathbf{T}_{\mathrm{s}}$, $\angle H_{\mathrm{s}}B_{\mathrm{s}}O = 2\angle I_{\mathrm{s}}B_{\mathrm{s}}O = 2\cdot 36^\circ = 72^\circ$. Therefore, triangle $H_{\mathrm{s}}B_{\mathrm{s}}O$ is a (tall) golden triangle, and

$$\frac{OH_{\mathrm{s}}}{OB_{\mathrm{s}}} = \frac{d}{a} = \varphi \implies OH_{\mathrm{s}} = \varphi R.$$

Also, by the angle bisector theorem,

$$\frac{H_{\mathrm{s}}I_{\mathrm{s}}}{I_{\mathrm{s}}O_{\mathrm{s}}} = \frac{B_{\mathrm{s}}H_{\mathrm{s}}}{B_{\mathrm{s}}O_{\mathrm{s}}} = \varphi.$$

This shows that I_{s} *divides* $H_{\mathrm{s}}O$ *in the golden ratio*.

Since $B_{\mathrm{s}}A$ bisects angle $H_{\mathrm{s}}B_{\mathrm{s}}I_{\mathrm{s}}$, the same reasoning shows that A *divides* $H_{\mathrm{s}}I_{\mathrm{s}}$ *in the golden ratio*.

(4) In the tall golden triangle $\mathbf{T}_{\mathrm{t}}$, the orthocenter H_{t} is the intersection of the center line with the perpendicular to $B_{\mathrm{s}}B_{\mathrm{t}}$ at B_{t}. Note that $B_{\mathrm{t}}H_{\mathrm{t}} = 2R\cos 72^\circ = 2R\cdot\frac{a/2}{d} = \frac{R}{\varphi}$.

Since H_{t} is the isogonal conjugate of O in $\mathbf{T}_{\mathrm{t}}$, by the angle bisector theorem,

$$\frac{OI_{\mathrm{t}}}{I_{\mathrm{t}}H_{\mathrm{t}}} = \frac{B_{\mathrm{t}}O}{B_{\mathrm{t}}H_{\mathrm{t}}} = \varphi.$$

Therefore, I_{t} *divides* OH_{t} *in the golden ratio*

(5) Since O and I_{t} are the reflections of I_{s} and A in $B_{\mathrm{s}}C_{\mathrm{t}}$, $OI_{\mathrm{t}} = AI_{\mathrm{s}}$, and

$$\frac{I_{\mathrm{s}}O}{OI_{\mathrm{t}}} = \frac{I_{\mathrm{s}}O}{AI_{\mathrm{s}}} = \varphi.$$

Therefore, O *divides* $I_{\mathrm{s}}I_{\mathrm{t}}$ *in the golden ratio.*

(6) In the tall golden triangle, O *divides* AH_{t} *in the golden ratio*.

$$\frac{AH_{\mathrm{t}}}{AO} = \frac{2\cdot R\cos 36^\circ}{R} = 2\cos 36^\circ = \varphi.$$

We summarize these results in the following proposition.

Proposition 1. *Let* $\mathbf{T}_i$, $i = \mathrm{s}, \mathrm{t}$ *be golden triangles sharing a common vertex* A *and the same circumcircle with center* O. *Let* H_i *and* I_i *denote the orthocenter and incenter of* $\mathbf{T}_i$.

(a) *The incenter* I_i *divides* H_iO *or* OH_i *in the golden ratio, according as* $i = \mathrm{s}, \mathrm{t}$.

(b) *The circumcenter* O *divides each of* $I_{\mathrm{s}}I_{\mathrm{t}}$ *and* AH_{t} *in the golden ratio.*

(c) A *divides* $H_{\mathrm{s}}I_{\mathrm{s}}$ *in the golden ratio.*

Some observations by Nikolaos Dergiades:
(i) O is the midpoint of $I_{\rm s}H_{\rm t}$.
(ii) If A' is the antipode of A on the circumcircle, the triangles $I_{\rm s}H_{\rm s}B_{\rm s}$ and $OB_{\rm s}A'$ are similar to $\mathbf{T}_{\rm s}$, and since $I_{\rm s}B_{\rm s} = OB_{\rm s} = R$, we have $B_{\rm s}H_{\rm s} = B_{\rm s}A' = R\varphi$.
(iii) The triangle $I_{\rm t}H_{\rm s}B_{\rm s}$ has a right angle at $B_{\rm s}$, and since $I_{\rm s}H_{\rm s} = I_{\rm s}B_{\rm s}$, $I_{\rm s}$ is the midpoint of $H_{\rm s}I_{\rm t}$.
(iv) The segments $OI_{\rm s} = \frac{R}{\varphi}$, $I_{\rm s}B_{\rm s} = R$, $B_{\rm s}H_{\rm s} = R\varphi$ are in geometric progression (with common ratio φ). Since $\varphi = 1 + \frac{1}{\varphi}$, we have $B_{\rm s}H_{\rm s} = B_{\rm s}I_{\rm s} + OI_{\rm s}$. This means that the circles $B_{\rm s}(H_{\rm s})$, $I_{\rm s}(H_{\rm s})$ and $O(I_{\rm s})$ are concurrent at a point D which lies on the line $OB_{\rm s}$ (see Figure 4).

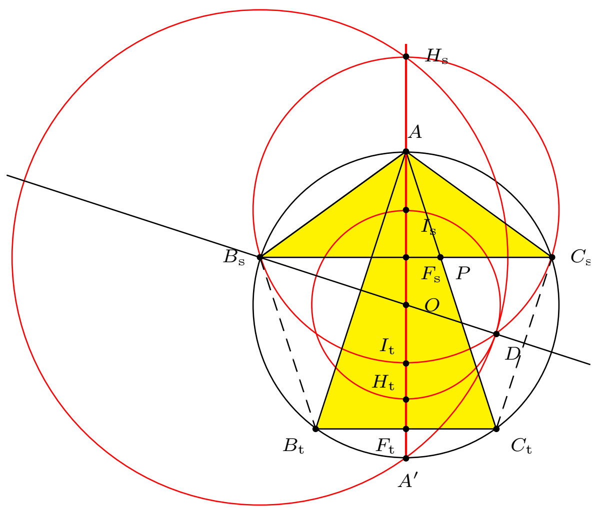

Figure 4

For $i = {\rm s}, {\rm t}$, the incircle of $\mathbf{T}_i$ is tangent to the side B_iC_i at its midpoint F_i. Since this midpoint also lies on the nine-point circle of $\mathbf{T}_i$, it is the Feuerbach point of $\mathbf{T}_i$. The nine-point circle of $\mathbf{T}_i$, $i = {\rm s}, {\rm t}$, also contains the midpoints $M_{i,b}$. $M_{i,c}$ of the sides AC_i and AB_i.

Proposition 2. (a) $F_{\rm s}$ *divides* $F_{\rm t}A$ *in the golden ratio.*
(b) *The incenter* $I_{\rm t}$ *divides* $F_{\rm s}F_{\rm t}$ *in the golden ratio.*

Proof. (a) Let P be the intersection of the diagonals $AC_{\rm t}$ and $B_{\rm s}C_{\rm s}$ (see Figure 3). Since $B_{\rm s}C_{\rm s}$ and $B_{\rm t}C_{\rm t}$ are parallel,

$$\frac{F_{\rm t}A}{F_{\rm t}F_{\rm s}} = \frac{C_{\rm t}A}{C_{\rm t}P} = \varphi.$$

Therefore, $F_{\rm s}$ divides $F_{\rm t}A$ in the golden ratio.

(b) Since $I_{\rm t}$ is the intersection of the diagonals $B_{\rm s}C_{\rm t}$ and $B_{\rm t}C_{\rm s}$, $\frac{F_{\rm s}I_{\rm t}}{I_{\rm t}F_{\rm t}} = \frac{B_{\rm s}C_{\rm s}}{B_{\rm s}C_{\rm t}} = \varphi$. □

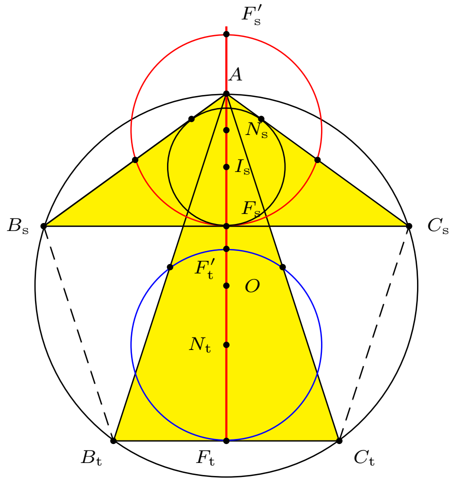

Figure 5

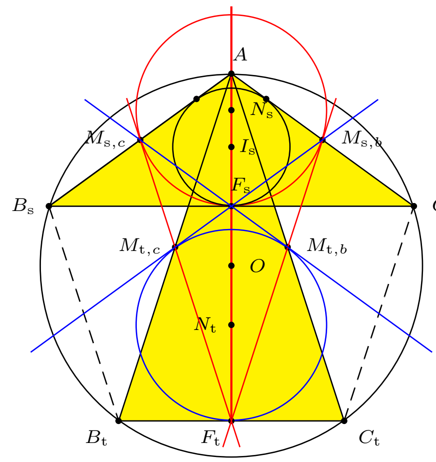

Figure 6

Proposition 3. (a) *For the short golden triangle $\mathbf{T}_\mathrm{s}$ with nine-point center N_s, the incenter I_s divides $F_\mathrm{s}N_\mathrm{s}$ in the golden ratio.*

(b) *For the tall golden triangle $\mathbf{T}_\mathrm{t}$, the nine-point center N_t divides $F_\mathrm{t}O$ in the golden ratio* (See Figure 5).

Proof. (a) The inradius of $\mathbf{T}_\mathrm{s}$ is $r_\mathrm{s} = \frac{R}{2\varphi}$. Therefore, $\frac{F_\mathrm{s}N_\mathrm{s}}{F_\mathrm{s}I_\mathrm{s}} = \frac{\frac{R}{2}}{r_\mathrm{s}} = \varphi$, and I_s divides $F_\mathrm{s}N_\mathrm{s}$ in the golden ratio.

(b) $F_\mathrm{t}O = R\cos 36^\circ = \frac{R}{2}\cdot\varphi$. Therefore, $\frac{F_\mathrm{t}O}{F_\mathrm{t}N_\mathrm{t}} = \varphi$, and N_t divides $F_\mathrm{t}O$ in the golden ratio. □

Proposition 4. *For $\{i,j\} = \{\mathrm{s},\mathrm{t}\}$, the nine-point center N_i of $\mathbf{T}_i$ is the reflection of F_j in the center O.*

Proof. (a) Since I_s is the reflection of O in $B_\mathrm{s}C_\mathrm{s}$,

$$OF_\mathrm{s} = F_\mathrm{s}I_\mathrm{s} = r_\mathrm{s} = \frac{R}{2\varphi},$$

$$ON_\mathrm{s} = OF_\mathrm{s} + F_\mathrm{s}N_\mathrm{s} = \frac{R}{2\varphi} + \frac{R}{2} = \frac{R}{2}\left(\frac{1}{\varphi}+1\right) = R\cdot\frac{\varphi}{2} = R\cos 36^\circ = F_\mathrm{t}O.$$

Therefore, N_s is the reflection of F_t in O.

(b) Since $N_\mathrm{t} - F_\mathrm{t} = N_\mathrm{s} - F_\mathrm{s}$,

$$N_\mathrm{t} = N_\mathrm{s} - F_\mathrm{s} + F_\mathrm{t} = 2\cdot O - F_\mathrm{t} - F_\mathrm{s} + F_\mathrm{t} = 2\cdot O - F_\mathrm{s}$$

is the reflection of F_s in O. □

Therefore, the nine-point center N_s is the antipode of F_t on the circle, center O, passing through F_t, which is the inscribed circle of the regular pentagon. It follows that $\angle N_\mathrm{s}M_{\mathrm{s},b}F_\mathrm{t}$ and $\angle N_\mathrm{s}M_{\mathrm{s},b}F_\mathrm{t}$ are right angles. This means that $F_\mathrm{t}M_{\mathrm{s},b}$ and $F_\mathrm{t}M_{\mathrm{s},c}$ are tangents to the nine-point circle of $\mathbf{T}_\mathrm{s}$ at $M_{\mathrm{s},b}$ and $M_{\mathrm{s},c}$ respectively (see Figure 6). The line $F_\mathrm{t}M_{\mathrm{s},b}$ passes through $M_{\mathrm{t},b}$, which divides $F_\mathrm{t}M_{\mathrm{s},b}$ in the golden ratio. Similarly, $F_{\mathrm{t}}M_{\mathrm{s},c}$ is the tangent at $M_{\mathrm{s},c}$ and is divided in the golden ratio by $M_{\mathrm{t},c}$.

The same reasoning also leads to the following.

(i) The points $M_{\mathrm{s},b}$, F_{s}, $M_{\mathrm{t},c}$ are collinear, and F_{s} divides $M_{\mathrm{s},b}M_{\mathrm{t},c}$ in the golden ratio. Furthermore, the line containing them is tangent to the nine-point circle of $\mathbf{T}_{\mathrm{t}}$ at $M_{\mathrm{t},c}$.

(ii) The points $M_{\mathrm{s},c}$, F_{s}, $M_{\mathrm{t},b}$ are collinear, and F_{s} divides $M_{\mathrm{s},c}M_{\mathrm{t},b}$ in the golden ratio. Furthermore, the line containing them is tangent to the nine-point circle of $\mathbf{T}_{\mathrm{t}}$ at $M_{\mathrm{t},b}$.

We conclude this note with a few more division in the golden ratio with points in Figure 5. The simple proofs are omitted.

For $i = \mathrm{s}, \mathrm{t}$, let F_i' be the antipode of F_i on the nine-point circle of $\mathbf{T}_i$. Then
(a) A divides $F_{\mathrm{s}}'N_{\mathrm{s}}$ in the golden ratio,
(b) F_{t}' divides each of the segments AN_{t} and $F_{\mathrm{t}}N_{\mathrm{s}}$ in the golden ratio,
(c) O divides $F_{\mathrm{s}}'F_{\mathrm{t}}$ in the golden ratio,
(d) I_{s} divides $F_{\mathrm{s}}'F_{\mathrm{t}}'$ in the golden ratio.

Statement (d) follows from Proposition 2(b) and a translation by R along the center line.

Figure 7 summarizes the golden sections in this note, each indicated by a longer solid segment followed by a shorter dotted segment. The endpoints and the division points are indicated on the "center line".

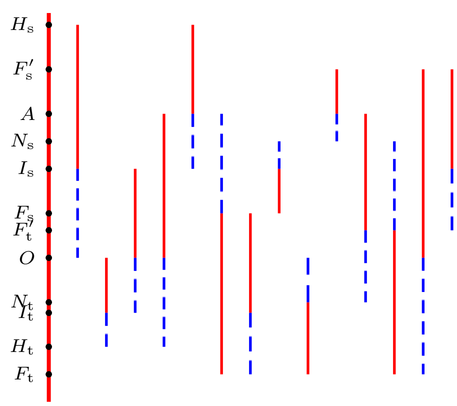

Figure 7

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