Forum Geometricorum
Volume 16 (2016) 185–186.

FORUM GEOM
ISSN 1534-1178

A Proof of the Butterfly Theorem Using Ceva's Theorem

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Abstract. A proof is given of the butterfly theorem by using a simple auxiliary construction and Ceva's theorem.

The two well-known theorems considered here are illustrated, for instance, in [2], each with a selected proof; see [2, p.45, Theorem 2.81] for the butterfly theorem and [2, p.5, Theorem 1.22] for Ceva's theorem. In [1] about twenty different proofs of the butterfly theorem are described, with comments on their features, related references and historical information.

In this note the butterfly theorem is proved by preliminarily adding some auxiliary lines to its usual illustrative diagram. Then Ceva's theorem is used as a lemma within this extended graph, and the butterfly theorem itself follows from elementary geometry.

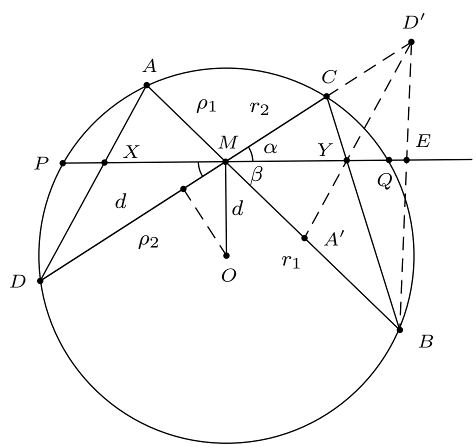

Figure 1

Theorem 1 (The Butterfly Theorem)**.** *Through the midpoint M of a chord PQ of a circle, two other chords AB and CD are drawn. Chords AD and BC intersect PQ at points X and Y, respectively. Then M is also the midpoint of XY.*

We introduce the points A' and D' that are the symmetric of A and D about M, respectively. Hence, $MA' = MA$ and $MD' = MD$. Next we connect the point

Publication Date: May 3, 2016. Communicating Editor: Paul Yiu.

D' to A' and B, and call E the intersection of $D'B$ with the line through P and Q (Figure 1). Thus we have constructed triangle MBD' with cevians $D'A'$, ME, and BC. We show that the segment $D'A'$ cuts the chord PQ at the same point Y as BC, i.e., that the three cevians are concurrent at Y. This property will be proved by applying Ceva's theorem to triangle MBD'.

Lemma 2. *In triangle MBD', the cevians $D'A'$, ME, and BC are concurrent at Y.*

Proof. We set $MA' = MA = \rho_1$, $MB = r_1$, $MC = r_2$, and $MD' = MD = \rho_2$. We observe that $\frac{BE}{ED'}$ is equal to the ratio $\frac{r_1 \sin\beta}{\rho_2 \sin\alpha}$ of the respective distances of B and D' from the line PQ. Moreover, $A'B = r_1 - \rho_1$, and $D'C = \rho_2 - r_2$ (see Figure 1). Now,

$$\frac{BE}{ED'} \cdot \frac{D'C}{CM} \cdot \frac{MA'}{A'B} = \frac{r_1 \sin\beta}{\rho_2 \sin\alpha} \cdot \frac{\rho_2 - r_2}{r_2} \cdot \frac{\rho_1}{r_1 - \rho_1} = \frac{(\rho_2 - r_2)\sin\beta}{(r_1 - \rho_1)\sin\alpha} \tag{1}$$

since $\rho_1 r_1 = \rho_2 r_2$ by the intersecting chords theorem (see, e.g., [2, p.28, Theorem 2.11]).

The differences appearing in (1) can be written in terms of the distance $d = OM$ of the circle center O to the chord PQ, and the angles α and β. Figure 1 shows that the projection of OM onto the chord CD has length $d \sin\alpha$, so that we get $\rho_2 = \frac{CD}{2} + d\sin\alpha$, and $r_2 = \frac{CD}{2} - d\sin\alpha$. Hence, $\rho_2 - r_2 = 2d\sin\alpha$. Similarly, we find $r_1 - \rho_1 = 2d\sin\beta$. Substituting these expressions into (1) we obtain

$$\frac{BE}{ED'} \cdot \frac{D'C}{CM} \cdot \frac{MA'}{A'B} = 1.$$

By Ceva's theorem, the cevians $D'A'$, ME, and BC are concurrent. The common point is clearly Y. □

Proof of the Butterfly Theorem. We observe that triangle $MA'D'$ is congruent by construction to triangle MAD, because two sides of the first (MA', MD') are equal to two sides of the second (MA, MD), and the included angles are equal. It follows that $\angle MD'Y = \angle MDX$. Consequently, triangles $MD'Y$ and MDX are also congruent, since they have equal two pairs of angles ($\angle MD'Y = \angle MDX$, and $\angle YMD' = \angle XMD$, vertical angles), as well as the included sides ($MD' = MD$). This congruence implies that the corresponding sides MY and MX are equal. Therefore, M is the midpoint of XY and the butterfly theorem is proved. □

References

[1] A. Bogomolny, The Butterfly Theorem, http://www.cut-the-knot.org/pythagoras/Butterfly.shtml.

[2] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, Washington, D.C., 1967.

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