Forum Geometricorum
Volume 16 (2016) 233–239.

FORUM GEOM
ISSN 1534-1178

Area of the Orthic Quadrilaterals of a Convex Cyclic Orthodiagonal Quadrilateral

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Abstract. Among all orthic quadrilaterals inscribed in a given convex cyclic orthodiagonal quadrilateral, the orthic quadrilateral of the side midpoint rectangle has the largest area. We give here a short and simple proof of this recently established fact by describing the orthic quadrilaterals, inscribed or not, as a symmetric difference of orthic triangles and computing their area.

1. Introduction

We consider a convex cyclic orthodiagonal quadrilateral $\mathcal{Q} = ABCD$ (Figure 1). Its perpendicular diagonals AC and BD intersect at O. We set $OA = a$, $OB = b$, $OC = c$, and $OD = d$. Let $\mathcal{R} = KLMN$ be a rectangle with sides parallel to the diagonals of $\mathcal{Q}$ and vertices K, L, M, and N on the sidelines AB, BC, CD, and DA of $\mathcal{Q}$, respectively. The orthogonal projections of the vertices of $\mathcal{R}$ on the opposite sidelines of $\mathcal{Q}$ generate the *orthic quadrilateral* $\mathcal{R}^* = K^*L^*M^*N^*$ of $\mathcal{R}$. $\mathcal{R}$ and $\mathcal{R}^*$ are concyclic [1]. The *principal* orthic quadrilateral $\mathcal{R}_p^*$ is the orthic quadrilateral of the side midpoint rectangle. It was conjectured in [1] and proven recently in [2] that the principal orthic quadrilateral has the largest area among the orthic quadrilaterals $\mathcal{R}^*$ *inscribed* in $\mathcal{Q}$ (this restriction is missing in [2] although it is necessary since the area of $\mathcal{R}^*$ tends to infinity as K moves away from $\mathcal{Q}$ on the sideline AB). We give here another (simpler, shorter, and more enlightening) proof of this result by describing the orthic quadrilaterals, inscribed or not, as a symmetric difference of orthic triangles and computing their area.

We consider the diagonals as the axes of a Cartesian coordinate system with origin O and $A = (a, 0)$, $B = (0, b)$. We set $K = (x, y)$ on the line AB, which implies $y = b(1 - x/a)$, and write $\mathcal{R} = \mathcal{R}(x)$, $\mathcal{R}^* = \mathcal{R}^*(x)$. The solution for $a = b = c = d$ is immediate: $\mathcal{Q}$ is a square, $\mathcal{R}^*(x)$ is $\mathcal{R}(x)$ rotated by $\pi/2$ of area $4|x(x - a)|$. From now on we suppose $a < c$ and $b \leq d$ without loss of generality.

2. The orthic quadrilateral of a diagonal

Let $\mathcal{R}_v^* = K_v^*L_v^*M_v^*N_v^*$ be the orthic quadrilateral of the *vertical* diagonal $\mathcal{R}_v = \mathcal{R}(0) = BBDD$ (Figure 2). Triangle BCD is acute as $a < c$. The orthic triangle of BCD is $K_v^*ON_v^*$. By a well-known property of the orthocenter, A

Publication Date: June 2, 2016. Communicating Editor: Paul Yiu.

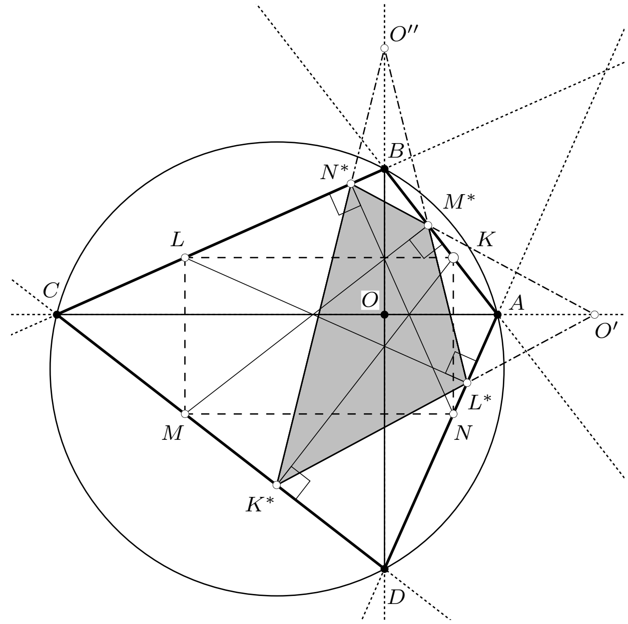

Figure 1. Inscribed orthic quadrilateral $\mathcal{R}^* = K^*L^*M^*N^*$ generated by rectangle $\mathcal{R} = KLMN$

is the reflection of the orthocenter H in the line BD. The reflection in the line BD maps thus the lines of the altitudes BK_v^* and DN_v^* of triangle BCD to the sidelines BA and DA of triangle BAD, respectively, and the sidelines BC and DC of BCD to the altitudes BL_v^* and DM_v^* of BAD. The reflections of K_v^* and N_v^* in the line BD are hence M_v^* and L_v^*, respectively. As the altitudes of a triangle are the angle bisectors of its orthic triangle, the points K_v^*, O, and L_v^* are collinear, as are the points M_v^*, O, and N_v^*. The following theorem is proven (Figures 2 and 3).

Theorem 1. *The orthic quadrilateral $\mathcal{R}_v^* = K_v^*L_v^*M_v^*N_v^*$ of the diagonal $\mathcal{R}_v = BBDD$ is symmetric in the diagonal BD. The sides $K_v^*L_v^*$ and $M_v^*N_v^*$ intersect at O. Triangles $K_v^*ON_v^*$ and $L_v^*OM_v^*$ are the orthic triangles of BCD and BAD, respectively. The orthic quadrilateral of the other diagonal has similar properties.*

By well-known formulæ, $\mathcal{Q}$ and triangle BCD have the circumdiameter

$$2\rho = \sqrt{AB^2 + CD^2} = \sqrt{BC^2 + DA^2} = \sqrt{a^2 + b^2 + c^2 + d^2} \tag{1}$$

and the area of the orthic triangle of (any triangle) BCD is

$$\frac{1}{2\rho} BC \cdot CD \cdot DB \cdot \left| \cos \widehat{CDB} \cos \widehat{DBC} \cos \widehat{BCD} \right| . \tag{2}$$

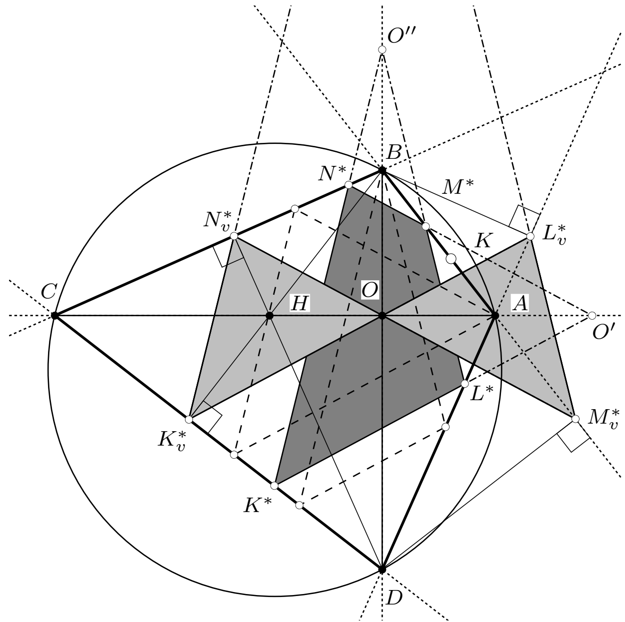

Figure 2. Orthic quadrilateral $\mathcal{R}_v^* = K_v^* L_v^* M_v^* N_v^*$ generated by the diagonal $BBDD$; first and last inscribed orthic quadrilaterals (dashed)

Remember that $ac = bd$ by the inscribed angle theorem. By expressing the sides of triangle BCD with b, c, d and using the cosine rule, one obtains

$$\cos \widehat{BCD} = \frac{c^2 - bd}{BC \cdot CD} = \frac{c(c-a)}{BC \cdot CD}. \tag{3}$$

Using (1)–(3), one finds easily for $a < c$

$$\operatorname{area}(\mathcal{R}_v^*) = \frac{2ac^2(c-a)(b+d)}{\sqrt{a^2+b^2+c^2+d^2}\sqrt{b^2+c^2}\sqrt{c^2+d^2}}. \tag{4}$$

3. Construction of $\mathcal{R}^*$ from homothetic copies of the halves of $\mathcal{R}_v^*$

We denote the homothety of ratio λ about P by $\mathsf{h}(P, \lambda)$ and refer to Figures 1 and 2.

Theorem 2. *Suppose $a < c$.*

(1) *The vertices of $\mathcal{R}(x)$ are the images of the vertices B and D of $\mathcal{R}_v = \mathcal{R}(0)$ under the homotheties $\mathsf{h}(A, 1 - x/a)$ for K and N and $\mathsf{h}(C, 1 - x/a)$ for L and M.*

(2) *The vertices K^* and N^* of $\mathcal{R}^*(x)$ are the images of the vertices K_v^* and N_v^* of $\mathcal{R}_v^*$ under the homothety $\mathsf{h}\big(C, 1 + 2x/(c-a)\big)$. The vertices L^* and M^* are the images of L_v^* and M_v^* under $\mathsf{h}\big(A, 1 - 2cx/(a(c-a))\big)$. The self-intersection point O of $\mathcal{R}_v^*$ has the same image O' under both homotheties: the sidelines K^*L^* and M^*N^* intersect thus at O'.*

(3) *The orthic quadrilateral* $\mathcal{R}^*$ *is the closure of the symmetric difference of triangles* $K^*O'N^*$ *and* $L^*O'M^*$*, which are the orthic triangles of the images of* BCD *and* BAD *under the respective homotheties.*

Proof. We only prove the assertion about K^* (the other assertions have similar proofs or are almost immediate). Using (3), one obtains

$$CK_v^* = BC\cos\widehat{BCD} = \frac{c(c-a)}{CD}. \tag{5}$$

The cosine angle difference identity gives

$$\begin{aligned}&\cos\left(\widehat{BAO}-\widehat{OCD}\right)\\ &= \cos\left(\arctan\frac{b}{a}-\arctan\frac{d}{c}\right) = \frac{ac+bd}{AB\cdot CD} = \frac{2ac}{AB\cdot CD}.\end{aligned} \tag{6}$$

As $BK = BA\cdot x/a$, one has by (6)

$$K_v^*K^* = BK\cos\left(\widehat{BAO}-\widehat{OCD}\right) = \frac{2cx}{CD} \tag{7}$$

and by (5) and (7)

$$\frac{CK^*}{CK_v^*} = 1+\frac{K_v^*K^*}{CK_v^*} = 1+\frac{2x}{c-a}. \qquad \square$$

Theorem 3. *Suppose* $a < c$ *and* $b \le d$*. The orthic quadrilateral* $\mathcal{R}^*(x)$ *is then inscribed in the convex cyclic orthodiagonal quadrilateral* $\mathcal{Q}$ *if and only if*

$$\left(1-\frac{a}{c}\right)\frac{a}{2} \le x \le \left(1+\frac{b}{d}\right)\frac{a}{2}.$$

On this interval, the area of $\mathcal{R}^*(x)$ *is*

$$\operatorname{area}(\mathcal{R}_v^*)\,\frac{2(c+a)}{a^2(c-a)}x(a-x) \tag{8}$$

– see (4) *– and attains its maximal value for the principal orthic quadrilateral* $\mathcal{R}_p^* = \mathcal{R}^*(a/2)$:

$$\operatorname{area}\left(\mathcal{R}_p^*\right) = \frac{c+a}{2(c-a)}\operatorname{area}(\mathcal{R}_v^*) = \frac{ac^2(a+c)(b+d)}{\sqrt{a^2+b^2+c^2+d^2}\sqrt{b^2+c^2}\sqrt{c^2+d^2}}. \tag{9}$$

Proof. We set $\operatorname{area}(\mathcal{R}_v^*) = \mu_v$ and refer to Figure 2. As long as $\mathcal{R}^*$ is inscribed in $\mathcal{Q}$, its area equals

$$\operatorname{area}\left(K^*O'N^*\right) - \operatorname{area}\left(L^*O'M^*\right) = \frac{\mu_v}{2}\left(1+\frac{2x}{c-a}\right)^2 - \frac{\mu_v}{2}\left(1-\frac{2cx}{a(c-a)}\right)^2$$

by Theorems 1 and 2. $\mathcal{R}^*$ is inscribed for the first time in $\mathcal{Q}$ when $L^*O'M^*$ collapses to A, that is, for $x = (1-a/c)\cdot a/2$. And $\mathcal{R}^*$ is inscribed for the last time in $\mathcal{Q}$ when M^* and N^* coincide with B: this is the case for $y = (1-b/d)\cdot b/2$, which is the vertical version of $x = (1-a/c)\cdot a/2$, where $y = b(1-x/a)$ is the ordinate of K, that is, when $x = (1+b/d)\cdot a/2$. $\square$

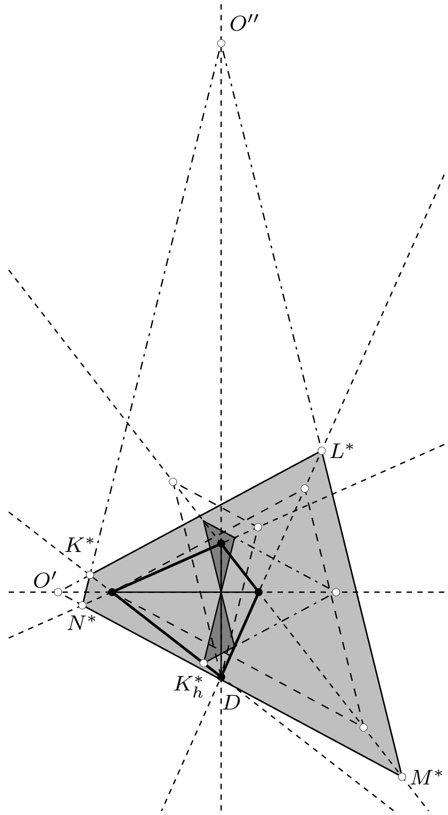

Figure 3. $\mathcal{R}_h^* = K_h^* L_h^* M_h^* N_h^*$ generated by the diagonal $ACCA$ and three other orthic quadrilaterals

Remark. $\mathcal{R}_v^*$ has a larger area than the principal orthic quadrilateral if and only if $c > 3a$.

4. Area of the orthic quadrilaterals

We show that the area of $\mathcal{R}^*(x)$ is a piecewise quadratic polynomial in x. We suppose $a < c$, $b < d$ and set $\mathcal{R}_h^* = \mathcal{R}^*(a)$, the orthic quadrilateral of the *horizontal* diagonal $\mathcal{R}_h = \mathcal{R}(a) = ACCA$ (Figure 3): $\mu_h = \operatorname{area}(\mathcal{R}_h^*)$ is obtained from (4) by interchanging a and b as well as c and d. As $ac = bd$, Theorem 3 and a direct calculation starting from the two versions of (4) show that

$$2\operatorname{area}\left(\mathcal{R}_p^*\right) = \frac{c+a}{c-a}\operatorname{area}\left(\mathcal{R}_v^*\right) = \frac{d+b}{d-b}\operatorname{area}\left(\mathcal{R}_h^*\right). \tag{10}$$

Theorem 2 can be reformulated for $\mathcal{R}^*(x)$, $\mathcal{R}_h$, $\mathcal{R}_h^*$, and $y = b(1 - x/a)$ with homotheties

$$\mathsf{h}\big(D,\, 1 + 2y/(d-b)\big) \quad \text{and} \quad \mathsf{h}\big(B,\, 1 - 2dy/(b(d-b))\big)$$

and a common image O'' of O on the line BD (Figures 2 and 3). $\mathcal{R}^*$ is the closure of the symmetric difference of triangles $K^*O''L^*$ and $M^*O''N^*$ as well as of $K^*O'N^*$ and $L^*O'M^*$.

As long as O' is on the left of C, for $x \leq (a-c)/2$, triangle $K^*O'N^*$ is the tip of triangle $L^*O'M^*$ (Figure 3): the area of their symmetric difference is thus given by (8) with opposite sign.

For x from $(a-c)/2 = (1-c/a)\cdot a/2$ to $(1-a/c)\cdot a/2$, where $O' = A$ (Figure 2), the two triangles share only O' and their areas have to be added. Then, until $O'' = B$ (Figure 2), the orthic quadrilateral is inscribed in $\mathcal{Q}$ and its area is given by (8).

For O'' below B (Figure 3), the area of the symmetric difference of $K^*O'N^*$ and $L^*O'M^*$ is

$$\operatorname{area}\left(K^*O''L^*\right)+\operatorname{area}\left(M^*O''N^*\right) = \frac{\mu_h}{2}\left(1+\frac{2y}{d-b}\right)^2+\frac{\mu_h}{2}\left(1-\frac{2dy}{b(d-b)}\right)^2$$

from $x = (1+b/d)\cdot a/2$, $y = (1-b/d)\cdot b/2$ to $y = (b-d)/2 = (1-d/b)\cdot b/2$, $x = (1+d/b)\cdot a/2$ when O'' reaches D (Figure 3).

For $y < (b-d)/2$, when O'' is below D, triangle $K^*O''L^*$ is the tip of $M^*O''N^*$ and $\operatorname{area}(\mathcal{R}^*) = -\operatorname{area}(K^*O''L^*) + \operatorname{area}(M^*O''N^*)$.

If $b = d$, the area of $\mathcal{R}^*(x)$ for $x \geq a$, after the inscribed cases, is again given by (8) with opposite sign. Using (8) and (10), the results can now be simplified and summarized. With the change of variable $x = \xi a/2$ one obtains a further simplification by considering the *normalized area* of $\mathcal{R}^*(\xi a/2)$ given by

$$\frac{\operatorname{area}\left(\mathcal{R}^*(\xi a/2)\right)}{\operatorname{area}\left(\mathcal{R}_p^*\right)}.$$

(We leave the details to the reader!)

Theorem 4. *For $a \leq c$ and $b \leq d$, the normalized area of $\mathcal{R}^*(\xi a/2)$ is* (Figure 4)

$$\frac{\operatorname{area}\left(\mathcal{R}^*(\xi a/2)\right)}{\operatorname{area}\left(\mathcal{R}^*(a/2)\right)} = \begin{cases} \dfrac{c-a}{c+a}\left(\dfrac{c^2+a^2}{(c-a)^2}\xi^2-2\xi+2\right), & 1-\dfrac{c}{a}<\xi<1-\dfrac{a}{c} \\ \dfrac{d+b}{d-b}\left(\dfrac{d^2+b^2}{(d+b)^2}\xi^2-2\xi+2\right), & 1+\dfrac{b}{d}<\xi<1+\dfrac{d}{b} \\ \left|\xi^2-2\xi\right|, & \textit{otherwise.} \end{cases}$$

(*The first interval, whose endpoints correspond to $O' = C$ and $O' = A$, is empty for $a = c$. The second interval, whose endpoints correspond to $O'' = B$ and $O'' = D$, is empty for $b = d$. The area of $\mathcal{R}^*(a/2) = \mathcal{R}_p^*$ is given by* (9).)

The area of $\mathcal{R}^*(\xi a/2)$ is thus a piecewise quadratic polynomial in ξ that is differentiable everywhere except at $\xi = 0$ when $a = c$ ($\mathcal{R}_v^*$ degenerates) and at $\xi = 2$ when $b = d$ ($\mathcal{R}_h^*$ degenerates). The normalized area is $2\xi - \xi^2$ exactly when the orthic quadrilateral is inscribed in $\mathcal{Q}$. The normalized area is further strictly greater than $\left|\xi^2 - 2\xi\right|$ on the intervals $1-c/a < \xi < 1-a/c$ and $1+b/d < \xi < 1+d/b$,

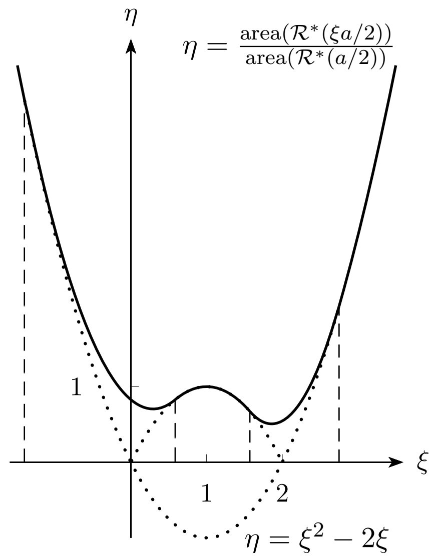

Figure 4. Normalized area of the orthic quadrilaterals for the convex orthodiagonal quadrilateral whose unit circumcircle is centered at $\left(-\frac{2}{5}, -\frac{1}{4}\right)$: $c,\ a = \frac{\sqrt{15}}{4} \pm \frac{2}{5}$ and $d,\ b = \frac{\sqrt{21}}{5} \pm \frac{1}{4}$

its two local minima are

$$\frac{c^2 - a^2}{c^2 + a^2} \quad \text{at} \quad \xi = \frac{(c-a)^2}{c^2 + a^2} \quad \text{and} \quad \frac{d^2 - b^2}{d^2 + b^2} \quad \text{at} \quad \xi = \frac{(d+b)^2}{d^2 + b^2},$$

and its unique local maximum corresponds to the principal orthic quadrilateral.

References

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