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Locus of Centroids of Similar Inscribed Triangles

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Abstract. We study the locus of the centroids of families of similar triangles inscribed in a given triangle.

1. Miquel circles

Given a triangle ABC and three points X, Y, Z on the sidelines BC, CA, AB respectively, the three Miquel circles are the circumcircles of the triangles AYZ, BZX, and CXY. According to Miquel's theorem, the three Miquel circles concur at a point, the Miquel point of X, Y, Z.

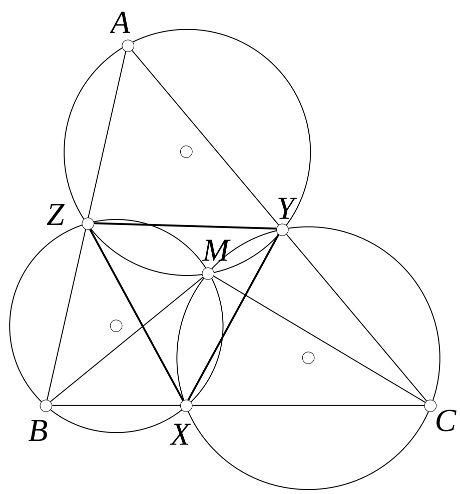

Figure 1

It is well known that if XYZ remains similar to a given triangle, the point M is fixed. The converse is also true: given a point M with homogeneous barycentric coordinates (u, v, w) with reference to ABC, if for X, Y, Z on the lines BC, CA, AB respectively, the circumcircles of AYZ, BZX, and CXY pass through M, then all such triangles XYZ are mutually similar. If $X_0Y_0Z_0$ is the pedal triangle of M, then every such triangle XYZ satisfies

$$\angle X_0MX = \angle Y_0MY = \angle Z_0MZ \ (= \ \theta).$$

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If $t=\tan\theta$, the vertices of the triangle are

$$\begin{aligned} X_t = &\ (0,\ (a^2+b^2-c^2)u+2a^2v-2Stu,\ (c^2+a^2-b^2)u+2a^2w+2Stu),\\ Y_t = &\ (2b^2u+(a^2+b^2-c^2)v+2Stv,\ 0,\ (b^2v+c^2-a^2)v+2b^2w-2Stv),\\ Z_t = &\ (2c^2u+(c^2+a^2-b^2)w-2Stw,\ 2c^2v+(b^2+c^2-a^2)w+2Stw,\ 0) \end{aligned} \tag{1}$$

in homogeneous barycentric coordinates. For basic formulas in barycentric coordinates, see [2].

2. The locus of the centroid of triangles XYZ

Note that in (1) above, the coordinate sums of X_t, Y_t, Z_t are respectively $2a^2(u+v+w)$, $2b^2(u+v+w)$, $2c^2(u+v+w)$. The centroid of $X_tY_tZ_t$ is the point

$$G_t = G_0 + 2t\cdot a^2b^2c^2S\left(\frac{v}{b^2}-\frac{w}{c^2},\ \frac{w}{c^2}-\frac{u}{a^2},\ \frac{u}{a^2}-\frac{v}{b^2}\right),$$

where

$$\begin{aligned} G_0 \ = \ &(a^2(4b^2c^2u+c^2(a^2+b^2-c^2)v+b^2(c^2+a^2-b^2)w),\\ &b^2(c^2(a^2+b^2-c^2)u+4a^2c^2v+a^2(b^2+c^2-a^2)w),\\ &c^2(b^2(c^2+a^2-b^2)u+a^2(b^2+c^2-a^2)v+4a^2b^2w)) \end{aligned}$$

is the centroid of the pedal triangle of M. Note that for $M=K=(a^2,b^2,c^2)$, the symmedian point of triangle ABC, $G_t=G_0=K$.

For $M\neq K$, the coordinates of G_t are linear functions of t, the locus of G_t is a straight line $\ell(M)$. The line $\ell(M)$ clearly contains G_0 and the infinite point

$$J(M) := \left(\frac{v}{b^2}-\frac{w}{c^2},\ \frac{w}{c^2}-\frac{u}{a^2},\ \frac{u}{a^2}-\frac{v}{b^2}\right). \tag{2}$$

This is the infinite point of the line

$$\frac{u}{a^2}x+\frac{v}{b^2}y+\frac{w}{c^2}z=0,$$

the trilinear polar of $M^* := \left(\frac{a^2}{u},\ \frac{b^2}{v},\ \frac{c^2}{w}\right)$, the isogonal conjugate of M. We summarize this in the following theorem.

Theorem 1. *Let $M\neq K$ be a point with homogeneous barycentric coordinates (u,v,w) with reference to ABC. The locus of the centroids of triangles XYZ with Miquel point M is the line $\ell(M)$ through the centroid G_0 of the pedal triangle of M parallel to the trilinear polar of the isogonal conjugate of M.*

The line $\ell(M)$ has barycentric equation

$$\sum_{\text{cyclic}}\left(\begin{array}{r} b^2c^2u^2-2c^2a^2v^2-2a^2b^2w^2-a^2(b^2+c^2-a^2)vw\\ +b^2(-a^2+b^2+2c^2)wu+c^2(-a^2+2b^2+c^2)uv \end{array}\right)x=0. \tag{3}$$

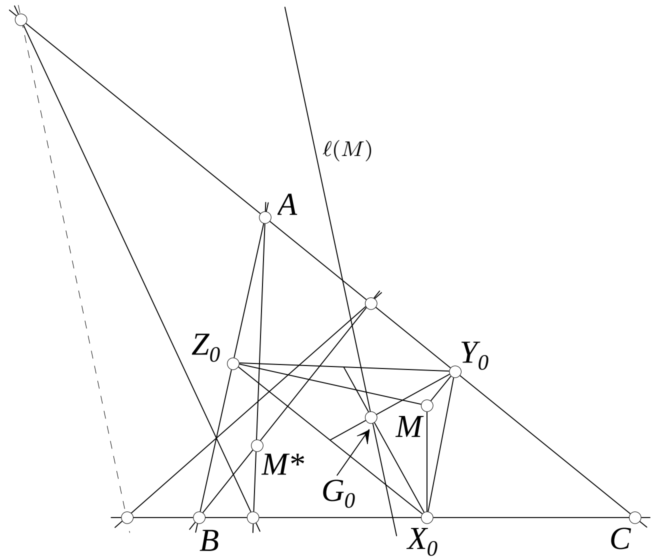

Figure 2

Example 1.

	M	$\ell(M)$	$J(M)$
(i)	G	$\sum_{\text{cyclic}}(a^4 - 4a^2(b^2+c^2) + b^4 + 5b^2c^2 + c^4)x = 0$	$X(512)$
(ii)	O	$\sum_{\text{cyclic}}(b^2 + c^2 - 2a^2)x = 0$	$X(523)$
(iii)	H	line joining O and	$X(520)$
(iv)	I	$\sum_{\text{cyclic}}(a^2 - 2a(b+c) + b^2 + bc + c^2)x = 0$	$X(513)$
(v)	$X(55)$	line joining $X(7)$ and	$X(514)$
(vi)	$X(56)$	line joining $X(8)$ and	$X(522)$
(vii)	$X(99)$	$\sum_{\text{cyclic}} \frac{x}{a^2(b^2+c^2)-2b^2c^2} = 0$	$X(888)$
(viii)	$X(110)$	$\sum_{\text{cyclic}} \frac{x}{b^2+c^2-2a^2} = 0$	$X(690)$

3. Parallelism and orthogonality

Proposition 2. *The lines $\ell(M)$ and $\ell(M')$ are parallel if and only if the line MM' passes through the symmedian point of triangle ABC.*

Proof. Let $M = (u, v, w)$ and $M' = (u', v', w')$ in homogeneous barycentric coordinates. The lines $\ell(M)$ and $\ell(M')$ are parallel if and only if the trilinear polars of M and M' are parallel. Equivalently, $J(M)$ lies on the line $\frac{u'}{a^2}x + \frac{v'}{b^2}y + \frac{w'}{c^2}z = 0$:

$$\begin{aligned} 0 =& \ \frac{u'}{a^2}\left(\frac{v}{b^2} - \frac{w}{c^2}\right) + \frac{v'}{b^2}\left(\frac{w}{c^2} - \frac{u}{a^2}\right) + \frac{w'}{c^2}\left(\frac{u}{a^2} - \frac{v}{b^2}\right) \\ =& \ \frac{1}{a^2b^2c^2}\left(a^2(v'w - vw') + b^2(w'u - wu') + c^2(u'v - uv')\right). \end{aligned}$$

Since $(v'w - vw')x + (w'u - wu')y + (u'v - uv')z = 0$ represents the line MM', the condition is equivalent to the line MM' containing (a^2, b^2, c^2), the symmedian point of triangle ABC. □

Corollary 3. *If M lies on the Brocard axis, the line $\ell(M)$ is perpendicular to Euler line.*

Proof. If M lies on the Brocard axis, by Theorem 1,

$$\begin{aligned} J(M) = &\ J(O) = \left(\frac{b^2 S_B}{b^2} - \frac{c^2 S_C}{c^2}, \frac{c^2 S_C}{c^2} - \frac{a^2 S_A}{a^2}, \frac{a^2 S_A}{a^2} - \frac{b^2 S_B}{b^2}\right) \\ = &\ (S_B - S_C, S_C - S_A, S_A - S_B), \end{aligned}$$

which is the triangle center $X(523)$ in [1], the infinite point of the perpendicular to the Euler line. Therefore, $\ell(M)$ is perpendicular to the Euler line. □

Corollary 4. *The locus of the centroids of equilateral inscribed triangles is formed by two lines perpendicular to Euler line.*

Proof. This follows from the fact that equilateral inscribed triangles have Miquel points the isodyanamic points $X(15)$ or $X(16)$ on the Brocard axis. The locus is formed by the lines

$$\sum_{\text{cyclic}} (\sqrt{3}(a^2(b^2 + c^2) - (b^4 + c^4)) \pm 2(b^2 + c^2 - 2a^2)S)x = 0,$$

S being twice the area of triangle ABC. □

Proposition 5. *Let $M = (u, v, w)$. The locus of M' for which $\ell(M') \perp \ell(M)$ is the line*

$$\sum_{\text{cyclic}} (-2b^2c^2u + c^2(a^2 + b^2 - c^2)v + b^2(c^2 + a^2 - b^2)w)x = 0. \tag{4}$$

Proof. For $M' = (x, y, z)$, $\ell(M) \perp \ell(M')$ if and only if

$$S_A\left(\frac{v}{b^2} - \frac{w}{c^2}\right)\left(\frac{y}{b^2} - \frac{z}{c^2}\right) + S_B\left(\frac{w}{c^2} - \frac{u}{a^2}\right)\left(\frac{z}{c^2} - \frac{x}{a^2}\right) + S_C\left(\frac{u}{a^2} - \frac{v}{b^2}\right)\left(\frac{x}{a^2} - \frac{y}{b^2}\right) = 0.$$

Rearrangement with the substitutions $S_A = \frac{b^2 + c^2 - a^2}{2}$ etc leads to (4) above. □

Example 2.

	M	locus of M' for which $\ell(M') \perp \ell(M)$	inf. point
(i)	G	$\sum_{\text{cyclic}} (a^2(b^2 + c^2) - (b^4 + c^4))x = 0$	$X(523)$
(ii)	O	$\sum_{\text{cyclic}} \frac{2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2}{a^2} x = 0$	$X(8675)$
(iii)	I	$\sum_{\text{cyclic}} \frac{a^2(b+c) - 2abc - (b+c)(b-c)^2}{a} x = 0$	$X(9001)$
(iv)	$X(55)$	$\sum_{\text{cyclic}} \frac{2a^3 - a^2(b+c) - (b+c)(b-c)^2}{a^2} x = 0$	$X(9000)$
(v)	$X(56)$	$\sum_{\text{cyclic}} \frac{2a^4 - a^3(b+c) - a^2(b-c)^2 + a(b+c)(b-c)^2 - (b^2-c^2)^2}{a^2} x = 0$	$X(8999)$
(vi)	$X(110)$	$\sum_{\text{cyclic}} \frac{2a^6 - 2a^4(b^2+c^2) + a^2(b^4+c^4) - (b^2+c^2)(b^2-c^2)^2}{a^2} x = 0$	$X(526)$

Proposition 6. *Let $\mathscr{L}_i: \ p_ix+q_iy+r_iz=0$, $i=1,2$, be two lines through the symmedian point K, and J_1, J_2 the infinite points of $\ell(M_1$, $\ell(M_2)$ for M_i on $\mathscr{L}_i$ respectively. Then points J_1 and J_2 correspond to perpendicular lines if and only if*

$$(q_1+r_1)p_2+(r_1+p_1)q_2+(p_1+q_1)r_2=0.$$

Remark. In other words, the point $Q=(q_1+r_1:r_1+p_1:p_1+q_1)$ lies on the line $\mathscr{L}_2$.

Construction 7. *Given a line $\mathscr{L}_1$ containing the symmedian point K, construct*
(i) *$Q=$ the complement of the isotomic conjugate of the trilinear pole of $\mathscr{L}_1$,*
(ii) *the line $\mathscr{L}_2=KQ$.*
For arbitrary points M on $\mathscr{L}_1$ and M' on $\mathscr{L}_2$, $\ell(M)\perp\ell(M')$.

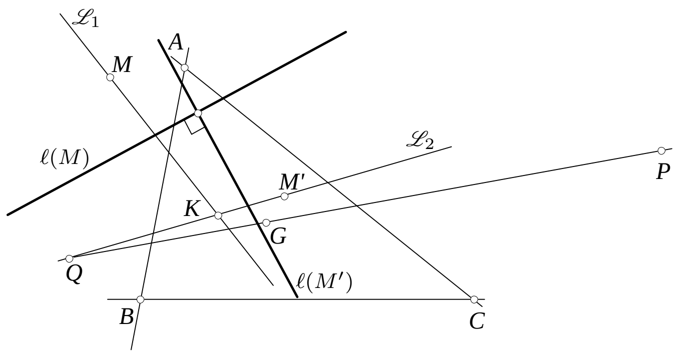

Figure 3

Proposition 8. *The locus of M for which $\ell(M)$ contains a given point $P(u,v,w)$ is the circle $\Gamma(P)$*

$$\begin{aligned}&(a^2+b^2+c^2)(u+v+w)(a^2yz+b^2zx+c^2xy)\\ &-(x+y+z)\left(\sum_{\text{cyclic}}b^2c^2(2v+2w-u)x\right)=0.\end{aligned}$$

with center

$$\begin{aligned}O(P)=\ &(a^2((a^4-2a^2(b^2+c^2)+b^4-8b^2c^2+c^4)u\\ &+(a^4-a^2(2b^2-c^2)+(b^2-c^2)(b^2+2c^2))v\\ &+(a^4+a^2(b^2-2c^2)-(b^2-c^2)(2b^2+c^2)w),\\ &\cdots,\cdots).\end{aligned}$$

and passing through the symmedian point K.

For $M=G$, the centroid, this is the circle

$$(a^2+b^2+c^2)(a^2yz+b^2zx+c^2xy)-(x+y+z)(b^2c^2x+c^2a^2+a^2b^2z)=0$$

with center $X(182)$, the midpoint of OK.

Proposition 9. *The tangent of $\Gamma(P)$ at K is parallel to $\ell(P)$.*

Proof. The tangent of $\Gamma(P)$ at K is the line[1]

$$\frac{(b^2+c^2)u-a^2v-a^2w}{a^2}x+\frac{-b^2u+(c^2+a^2)v-b^2w}{b^2}y+\frac{-c^2u-c^2v+(a^2+b^2)w}{c^2}z=0.$$

This has infinite point

$$\begin{aligned}&\left(\frac{-b^2u+(c^2+a^2)v-b^2w}{b^2}-\frac{-c^2u-c^2v+(a^2+b^2)w}{c^2},\right.\\
&\quad\frac{-c^2u-c^2v+(a^2+b^2)w}{c^2}-\frac{(b^2+c^2)u-a^2v-a^2w}{a^2},\\
&\quad\left.\frac{(b^2+c^2)u-a^2v-a^2w}{a^2}-\frac{-b^2u+(c^2+a^2)v-b^2w}{b^2}\right)\\
=&\left(\frac{(c^2+a^2+b^2)v}{b^2}-\frac{(a^2+b^2+c^2)w}{c^2},\frac{(a^2+b^2+c^2)w}{c^2}-\frac{(b^2+c^2+a^2)u}{a^2},\right.\\
&\quad\left.\frac{(b^2+c^2+a^2)u}{a^2}-\frac{(c^2+a^2+b^2)v}{b^2}\right)\\
=&(a^2+b^2+c^2)\left(\frac{v}{b^2}-\frac{w}{c^2},\frac{w}{c^2}-\frac{u}{a^2},\frac{u}{a^2}-\frac{v}{b^2}\right)\end{aligned}$$

equal to $J(P)$. Therefore the tangent is parallel to $\ell(P)$. □

Here is a construction of the center $O(P)$ of the circle $\Gamma(P)$.

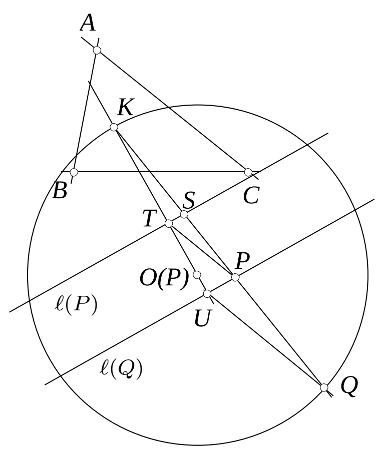

Figure 4

[1]Given the homogeneous (quadratic) equation of a conic, the tangent at a point (u,v,w) can be obtained by replacing x^2, y^2, z^2 by ux, vy, wz, and yz, zx, xy by $\frac{1}{2}(wy+vz)$, $\frac{1}{2}(uz+wx)$, $\frac{1}{2}(vx+uy)$ respectively.

Construction 10. *Given a point $P \neq K$, construct*
(1) *the line $\ell(P)$ to intersect KP at S;*
(2) *the orthogonal projections*
T of K on $\ell(P)$, and
U of P on KT;
(3) *the parallel of PT through U to intersect the line KP at Q, (the line $\ell(Q)$ passes through P);*
(4) *the perpendicular bisector of KQ to intersect KT at $O(P)$.*
$O(P)$ is the center of $\Gamma(P)$.

4. Envelopes

Proposition 11. *If M traverses a line $\mathscr{L}$, the lines $\ell(M)$ envelope a parabola whose axis is parallel to the trilinear polar of the isogonal conjugate of the infinite point of $\mathscr{L}$.*

Focus

$$\begin{aligned} F = \big(&a^4p^2 + b^2(c^2+a^2-b^2)q^2 + c^2(a^2+b^2-c^2)r^2 \\ &- (c^2+a^2-b^2(a^2+b^2-c^2)qr - c^2a^2rp - a^2b^2pq, \\ &\cdots, \cdots\big). \end{aligned}$$

Directrix

$$\begin{aligned} \sum_{\text{cyclic}} &(a^2((b^2+c^2-a^2)^2+8b^2c^2)p \\ &+ b^2(a^4+a^2(-2b^2+c^2)+(b^2-c^2)(b^2+2c^2))q \\ &+ c^2(a^4+a^2(b^2-2c^2)-(b^2-c^2)(2b^2+c^2))r)x \\ =& \ 0. \end{aligned}$$

For example, if $\mathscr{L}$ is the Lemoine axis $\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=0$, the parabola has barycentric equation

$$\sum_{\text{cyclic}}(b^2+c^2-2a^2)^2yz - (x+y+z)\left(\sum_{\text{cyclic}}(a^4-4b^2c^2)x\right) = 0.$$

It has focus the Parry point[2]

$$X(111) = \left(\frac{a^2}{b^2+c^2-2a^2}, \frac{b^2}{c^2+a^2-2b^2}, \frac{c^2}{a^2+b^2-2c^2}\right),$$

and directrix the line

$$\sum_{\text{cyclic}}(a^4-a^2(b^2+c^2)+4b^2c^2)x = 0$$

[2]The Parry point is the isogonal conjugate of the infinite point of the line GK.

which is the perpendicular to the line GK from the intersection of the Euler line and the Simson line of the Steiner point. [3]

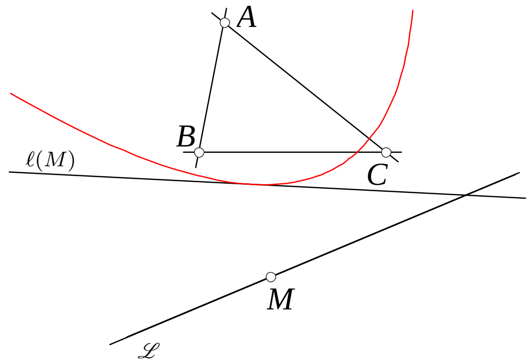

Figure 5

Proposition 12. *If M is a point on the circumcircle, the line $\ell(M)$ is tangent to the Steiner inellipse.*

Proof. If $M = \left(\frac{a^2}{(b^2-c^2)(a^2+\tau)}, \frac{b^2}{(c^2-a^2)(b^2+\tau)}, \frac{c^2}{(a^2-b^2)(c^2+\tau)}\right)$ on the circumcircle, the centroid of the (degenerate) pedal triangle of M is the point

$$\begin{aligned} G_0 = \ & (c^2a^2 + a^2b^2 - 2b^2c^2 - (b^2 + c^2 - 2a^2)\tau) \\ & \cdot (a^2(a^2(b^2 + c^2) - (b^4 + c^4)) + (2a^4 - a^2(b^2 + c^2) - (b^2 - c^2)^2)\tau), \\ & \cdots, \cdots). \end{aligned}$$

The trilinear polar of M^* is the line

$$\frac{x}{(b^2 - c^2)(a^2 + \tau)} + \frac{y}{(c^2 - a^2)(b^2 + \tau)} + \frac{z}{(a^2 - b^2)(c^2 + \tau)} = 0$$

with infinite point

$$\begin{aligned} J(\tau) = \ & ((b^2 - c^2)(a^2 + \tau)(a^2(b^2 + c^2) - 2b^2c^2 - (b^2 + c^2 - 2a^2)\tau), \\ & (c^2 - a^2)(b^2 + \tau)(b^2(c^2 + a^2) - 2c^2a^2 - (c^2 + a^2 - 2b^2)\tau), \\ & (a^2 - b^2)(c^2 + \tau)(c^2(a^2 + b^2) - 2a^2b^2 - (a^2 + b^2 - 2c^2)\tau)). \end{aligned}$$

The line $\ell(M)$ contains G_0 and $J(\tau)$. It has barycentric equation

$$\sum_{\text{cyclic}} \frac{x}{a^2(b^2 + c^2) - 2b^2c^2 - (b^2 + c^2 - 2a^2)\tau} = 0.$$

This is the tangent to the Steiner inellipse

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0$$

[3]This intersection is the triangle center $X(1513) = ((a^2(b^2 + c^2) - (b^4 + c^4))(3a^4 + (b^2 - c^2)^2), \cdots, \cdots)$.

at the point[4]

$$T(\tau) = ((a^2(b^2+c^2) - 2b^2c^2 - (b^2+c^2-2a^2)\tau)^2, \cdots, \cdots).$$

□

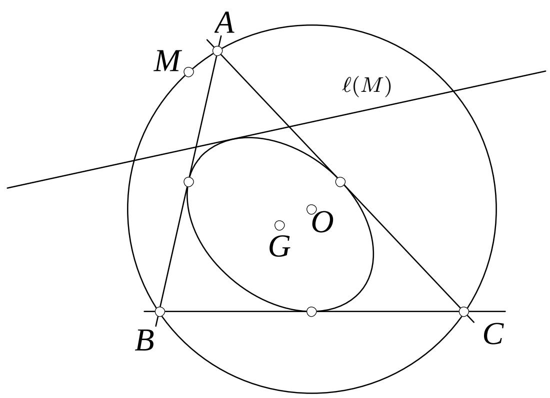

Figure 6

Corollary 13. *The Steiner inellipse is the envelope of $\ell(M)$ for M on the circumcircle of triangle ABC.*

Remark. If $S_{\rm t}$ is the Steiner point, the fourth intersection of the cirucumcircle and the Steiner circum-ellipse, and the line $M(\tau)S_{\rm t}$ intersects the Steiner circum-ellipse at $T'(\tau)$, then $T(\tau)$ is the midpoint of G and $T'(\tau)$.

5. The inverse problem

We solve the inverse problem of finding the point $M(u, v, w)$ so that $\ell(M)$ is a given line $\mathscr{L}: px + qy + rz = 0$ not containing the symmedian point K. This has to satisfy two conditions:
(i) $J(M)$ is the infinite point of $\mathscr{L}$, and
(ii) the centroid of the pedal triangle of M lies on $\mathscr{L}$.

$$\begin{aligned}
&\frac{q-r}{a^2}u + \frac{r-p}{b^2}v + \frac{p-q}{c^2}w = 0,\\
&\frac{4a^2p + (a^2+b^2-c^2)q + (c^2+a^2-b^2)r}{a^2}u\\
&\quad + \frac{(a^2+b^2-c^2)p + 4b^2q + (b^2+c^2-a^2)r}{b^2}v\\
&\quad\quad + \frac{(c^2+a^2-b^2)p + (b^2+c^2-a^2)q + 4c^2r}{c^2}w = 0.
\end{aligned}$$

[4]If (u, v, w) is an infinite point, then (u^2, v^2, w^2) is a point on the Steiner inellipse, and the tangent at that point is $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$.

Solving these equations, we obtain

$$\begin{aligned}
&u : v : w\\
=&a^2(-a^2p^2+2b^2q^2+2c^2r^2+(b^2+c^2-a^2)qr-(b^2+2c^2-a^2)rp-(2b^2+c^2-a^2)pq)\\
&:b^2(2a^2p^2-b^2q^2+2c^2r^2-(2c^2+a^2-b^2)qr+(c^2+a^2-b^2)rp-(c^2+2a^2-b^2)pq)\\
&:c^2(2a^2p^2+2b^2q^2-c^2r^2-(a^2+2b^2-c^2)qr-(2a^2+b^2-c^2)rp+(a^2+b^2-c^2)pq).
\end{aligned}$$

Example 3.

	$\mathscr{L}$	M
(i)	orthic axis	$X(187)$
(ii)	Lemoine axis	$X(352)$

Remarks. (1) $X(187)$ is the midpoint of the isodynamic points .

(2) $X(352)$ is a point on the circle through the centroid and the isodynamic points.

We conclude with a construction of M from $\ell(M)$.

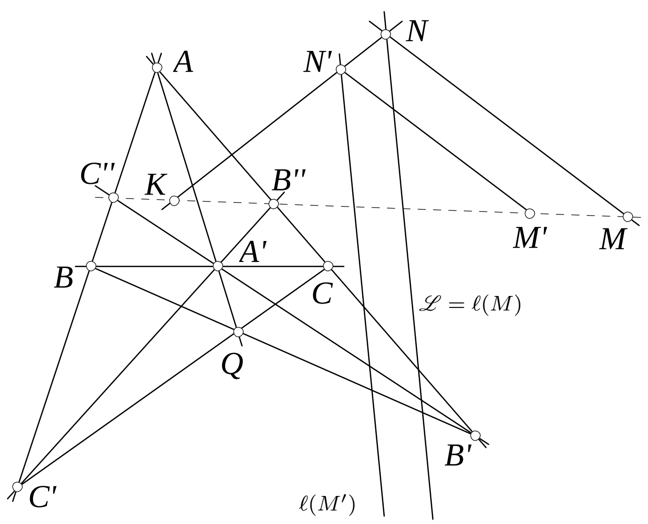

Figure 7

Construction 14. *Given a line $\mathscr{L}$, construct*
(i) *the isogonal conjugate Q of the infinite point of $\mathscr{L}$,*
(ii) *the cevian triangle $A'B'C'$ of Q and the points $B''=C'A'\cap CA$ and $C''=A'B'\cap AB$, (the line $B''C''$ passes through the symmedian point K),*
(iii) *the line $\ell(M')$ for any point M' on the line $B''C''$, (this line is parallel to $\mathscr{L}$),*
(iv) *any line through K intersecting $\mathscr{L}$ and $\ell(M')$ at N and N' respectively,*
(v) *the parallel through N to $M'N'$ to intersect $B''C''$ at M.*

The point M has $\ell(M)$ equal to the given line $\mathscr{L}$.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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