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Regular Polygons and the Golden Section

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Abstract. Given an isosceles triangle and its circumcircle, we construct a chord parallel to the base so that the endpoints and the intersections with the other two sides of the triangle are divided in the golden ratio. This generalizes results of Hagge and Odom for equilateral triangles, and Tran for a "half-square". We apply this construction to an isosceles triangle formed by two consecutive sides of a regular n-gon.

1. The golden section line of an isosceles triangle

This note is on an extension to regular polygons of elegant results on the golden section of segments associated with an equilateral triangle (Hagge [2] and Odom [4]) and a square (Tran [5]) with its circumcircle (see Figures 1 and 2). In each case, Y divides ZP and Z divides YQ in the golden ratio.

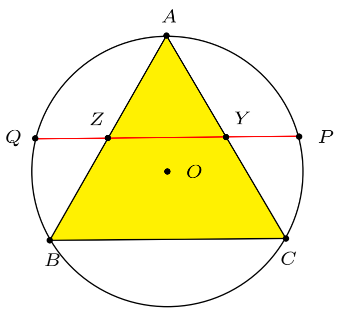

Figure 1

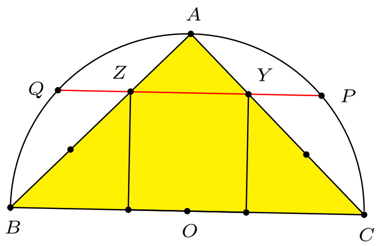

Figure 2

Let AB and AC be two consecutive sides of a regular n-gon inscribed in a circle, so that triangle ABC is isosceles with base angle $\theta = \frac{\pi}{n}$. Consider the problem of constructing a line parallel to BC, intersecting AB at Z, AC at Y, and the circle at P and Q (with P on the same side of OA as Y), such that Y divides ZP and Z divides YQ in the golden ratio $\varphi := \frac{\sqrt{5}+1}{2}$ (see Figure 3). The point Y divides ZP in the golden ratio if and only if $PY \cdot PZ = YZ^2$. Now, $PY \cdot PZ = PY \cdot YQ = CY \cdot YA$ by the intersecting chords theorem. Therefore, Y divides ZP in the golden ratio if and only if

$$\frac{CY}{YA} = \frac{CY \cdot YA}{YA^2} = \frac{YZ^2}{YA^2} = \frac{BC^2}{CA^2} = \frac{BC^2}{AB^2}.$$

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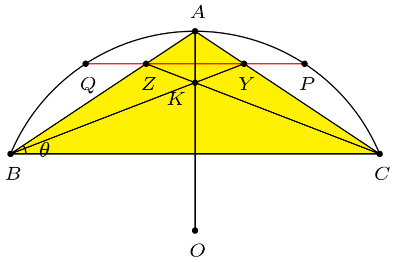

Figure 3

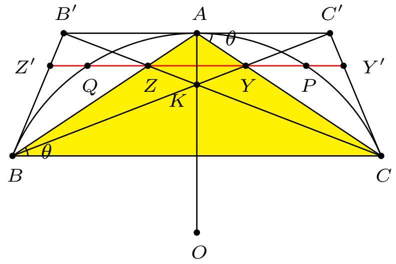

Figure 4

By symmetry, also $\frac{BZ}{ZA} = \frac{BC^2}{CA^2}$. Denote by K the intersection of the lines BY and CZ. With reference to triangle ABC, the point K has homogeneous barycentric coordinates $(BC^2 : CA^2 : AB^2)$. This is the symmedian point (see [7, §4.5.1]), and the problem is solved easily by constructing the tangents to the circle at the vertices A, B, C. Let the tangents at B and C intersect the tangent at A at B' and C' respectively. Then Y is the intersection of AC with BC' and Z that of AB and CB'. These points Y and Z determine the chord PQ of the circle such that Y divides ZP in the golden ratio (see Figure 4). Furthermore, if the chord PQ is extended to intersect BB' at Z' and CC' at Y', then P divides YY' and Q divides ZZ' in the golden ratio. We prove this by establishing a more general result on a symmetric trapezoid (see Figure 4).

Proposition 1. *Given an isosceles triangle ABC with $AB = AC$, let $BB'C'C$ be the symmetric trapezoid such that BB', $B'C'$ and $C'C$ are tangent to the circumcircle of ABC at B, A, C respectively. Let Y be the intersection of AC with BC' and Z that of AB and CB'. Extend the line YZ to intersect BB' at Z', CC' at Y', and the circle at P and Q so that P is between Y, Y', and Q is between Z and Z'. Then*
(a) *Y and Z are trisection points of the segment $Y'Z'$,*
(b) *Y divides ZP and Z divides YQ in the golden ratio,*
(c) *P divides YY' and Q divides ZZ' in the golden ratio.*

Proof. (a) By the similarity of triangles BYZ and $BC'A$ and that of triangles $CY'Y$ and $CC'A$, we have

$$\frac{YZ}{C'A} = \frac{BZ}{BA} = \frac{CY}{CA} = \frac{Y'Y}{C'A}.$$

Therefore, $YZ = Y'Y$. The same reasoning shows that $YZ = ZZ'$. Therefore, Y and Z trisect the segment $Y'Z'$.

(b) Let K be the intersection of BC' and CB'. Since B' is the intersection of the tangents at A and B to the circumcircle of triangle ABC, and C' that of the tangents at A and C, K is the symmedian point of triangle ABC. With reference to this triangle, K has homogeneous barycentric coordinates $(BC^2 : CA^2 : AB^2)$. The point Y is the trace of K on the side CA. It has coordinates $(BC^2 : 0 : AB^2)$.

Therefore,

$$CY : YA = BC^2 : AB^2 = 4\cos^2\theta : 1,$$

where θ is the base angle of the isosceles triangle ABC.[1] From the similarity of triangles CYZ and CAB',

$$\begin{aligned}
\frac{YZ}{AB'} = \frac{CY}{CA} = \frac{CY}{CY+YA} &= \frac{\frac{CY}{YA}}{1+\frac{CY}{YA}} = \frac{4\cos^2\theta}{1+4\cos^2\theta} \\
\Longrightarrow\ YZ = \frac{4\cos^2\theta\cdot AB'}{1+4\cos^2\theta} &= \frac{2AB'\cos\theta\cdot 2\cos\theta}{1+4\cos^2\theta} = \frac{AC\cdot 2\cos\theta}{1+4\cos^2\theta} \\
\Longrightarrow\ YZ^2 = \frac{AC\cdot 4\cos^2\theta}{1+4\cos^2\theta}&\cdot\frac{AC}{1+4\cos^2\theta} = CY\cdot YA.
\end{aligned}$$

By the intersecting chords theorem, $CY\cdot YA = PY\cdot YQ = PY\cdot PZ$. Therefore, $PY\cdot PZ = YZ^2$, and $\frac{PZ}{YZ} = \frac{YZ}{PY}$. This shows that Y divides PZ in the golden ratio.

(c) Let M be the midpoint of YZ. It also bisects PQ and $Y'Z'$. By (a), $YY' = 2MY$. By (b), $YP = \frac{2MY}{\varphi}$. Therefore, $\frac{YY'}{YP} = \varphi$, and P divides YY' in the golden ratio. □

We call the line containing Y, Z, P, Q, Y', Z' the *golden section line* of the isosceles triangle ABC. With reference to a Cartesian coordinate system with origin O, so that A is the point $(0, R)$, the points B and C are respectively $B(-R\sin 2\theta,\ R\cos 2\theta)$ and $C(R\sin 2\theta,\ R\cos 2\theta)$. Since $CY : YA = 4\cos^2\theta : 1 = 2(1+\cos 2\theta) : 1$, Y is the point

$$\begin{aligned}
(x,y) &= \frac{(R\sin 2\theta,\ R\cos 2\theta) + 2(1+\cos 2\theta)(0,R)}{3+2\cos 2\theta} \\
&= \frac{(R\sin 2\theta,\ R(2+3\cos 2\theta))}{3+2\cos 2\theta}. \qquad (1)
\end{aligned}$$

Proposition 2. *Let AB_1C_1 and AB_2C_2 be isosceles triangles inscribed in the same circle, center O, with points Y_i, Z_i, P_i, Q_i, Y_i', Z_i', $i = 1, 2$, on their golden section lines. The lines Y_1Y_2, Z_1Z_2, P_1P_2, Q_1Q_2, $Y_1'Y_2'$, $Z_1'Z_2'$ are concurrent at a point on the line OA.*

Proof. Let the line Y_1Y_2 intersect OA at T. By symmetry, the line Z_1Z_2 also intersects OA at the same point. Triangles TY_1Z_1 and TY_2Z_2 are similar. Since Y_i divides P_iZ_i in the golden ratio, $Z_iP_i = \varphi Z_iY_i$. It follows that

$$\frac{Z_1P_1}{Z_2P_2} = \frac{Z_1Y_1}{Z_2Y_2} = \frac{TZ_1}{TZ_2}.$$

Therefore, triangles TZ_1P_1 and TZ_2P_2 are similar, $\angle P_1TZ_1 = \angle P_2TZ_2$. This shows that P_1, P_2, and T are collinear. Since $Z_iY_i' = 2Z_iY_i$, the same reasoning shows that Y_1', Y_2', and T are collinear. By symmetry in OA, each of the lines

[1] We do not assume $\theta = \frac{\pi}{n}$ for some positive integer $n \geqslant 3$.

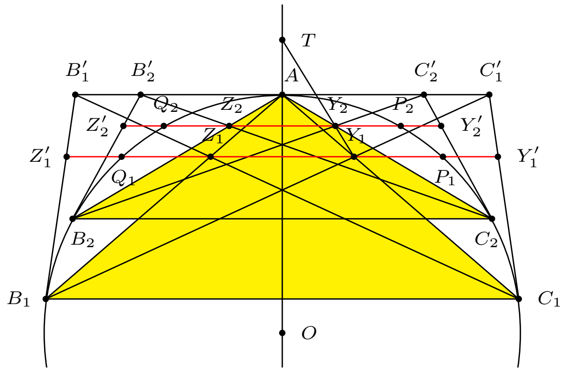

Figure 5

Q_1Q_2, and $Z_1'Z_2'$ also contains the point T. Therefore, the six lines are concurrent at T on OA. □

Remark. In a Cartesian coordinate system with origin O in which A is the point $(0, R)$, the coordinates of Y_i, $i = 1, 2$, are

$$Y_i = \left(\frac{R\sin 2\theta_i}{3 + 2\cos 2\theta_i}, \frac{R(2 + 3\cos 2\theta_i)}{3 + 2\cos\theta_i}\right),$$

where θ_i is the base angle of the isosceles triangle AB_iC_i. The line Y_1Y_2 intersects OA at

$$T = \left(0, \frac{2\cos(\theta_1 + \theta_2) + 3\cos(\theta_1 - \theta_2)}{3\cos(\theta_1 + \theta_2) + 2\cos(\theta_1 - \theta_2)}R\right) = \left(0, \frac{5 + \tan\theta_1\tan\theta_2}{5 - \tan\theta_1\tan\theta_2}R\right). \tag{2}$$

2. Regular polygons

By taking $\theta = \frac{\pi}{n}$ for a positive integer $n \geqslant 3$, we apply Proposition 1 to regular n-gons.

Proposition 3. *Let $A_1A_2\cdots A_n$ and $A_1'A_2'\cdots A_n'$ be regular n-gons such that A_k is the midpoint of $A_k'A_{k+1}'$ (for $k = 1, 2, \dots, n$ and indices taken modulo n). Let Y_k be the intersection of A_kA_{k-1} and $A_{k+1}A_k'$, and Z_k that of A_kA_{k+1} and $A_{k-1}A_{k+1}'$. Construct the line Y_kZ_k to intersect*
(i) *$A_{k-1}A_k'$ at Y_k', $A_{k+1}A_{k+1}'$ at Z_k',*
(ii) *the circumcircle of the regular n-gon $A_1A_2\cdots A_n$ at P_k and Q_k so that P_k is between Y_k and Y_k', and Q_k is between Z_k and Z_k'.*
Then
(a) *Y_k and Z_k are trisection points of the segment $Y_k'Z_k'$,*
(b) *Y_k divides Z_kP_k and Z_k divides Y_kQ_k in the golden ratio,*
(c) *P_k divides Y_kY_k' and Q_k divides Z_kZ_k' in the golden ratio.*

It follows from Proposition 3 that the golden section line of an isosceles triangle formed by two adjacent sides of a regular n-gon is constructible with ruler and compass if and only if the regular n-gon is constructible. By Gauss' theorem, this is the case precisely when n is the product of a power of 2 and distinct Fermat primes of the form $2^{2^k}+1$.

For $n=5$, we consider a regular pentagon $AB_\mathrm{s}B_\mathrm{t}C_\mathrm{t}C_\mathrm{s}$. The isosceles triangles $AB_\mathrm{s}C_\mathrm{s}$ and $AB_\mathrm{t}C_\mathrm{t}$ are called the *short* and *tall golden triangles* respectively, with base angles $\theta_\mathrm{s}=36°$, $\theta_\mathrm{t}=72°$ (see [1]). The lines $Y_\mathrm{s}Y_\mathrm{t}$, $Z_\mathrm{s}Z_\mathrm{t}$, $P_\mathrm{s}P_\mathrm{t}$, $Q_\mathrm{s}Q_\mathrm{t}$, $B_\mathrm{s}B'_\mathrm{s}$, and $C_\mathrm{s}C'_\mathrm{s}$ are concurrent (Proposition 2) at the point

$$T=\left(0,\ \frac{2\cos 108°+3\cos(-36°)}{3\cos 108°+2\cos(-36°)}\right)=\left(0,\ \frac{2\left(-\frac{1}{2\varphi}\right)+3\cdot\frac{\varphi}{2}}{3\left(-\frac{1}{2\varphi}\right)+2\cdot\frac{\varphi}{2}}\right)=(0,(\varphi+1)R)$$

according to formula (2). The point A divides TO in the golden ratio: $TA:AO=\varphi:1$ (see Figure 6).

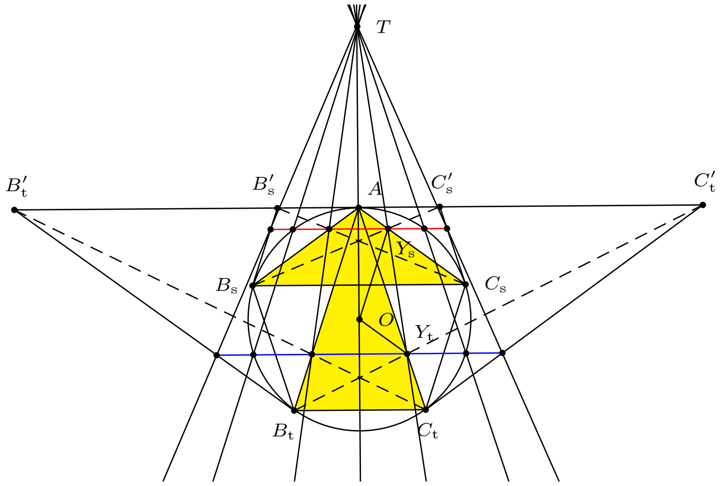

Figure 6

Figure 6 also shows a simple alternative to the construction of the golden section lines for the golden triangles. For the short golden triangle $AB_\mathrm{s}C_\mathrm{s}$, the point Y_s can be constructed as the intersection of AC_s and the parallel through O to $C_\mathrm{s}C_\mathrm{t}$. The reflection of Y_s in OA is the point Z_s, and $Y_\mathrm{s}Z_\mathrm{s}$ is the golden section line for the isosceles triangle $AB_\mathrm{s}C_\mathrm{s}$. For the tall golden triangle $AB_\mathrm{t}C_\mathrm{t}$, Y_t is the intersection of AC_t with the parallel through O to AC_s. The reflection of Y_t in OA is the point Z_t, and $Y_\mathrm{t}Z_\mathrm{t}$ is the golden section line for the isosceles triangle $AB_\mathrm{t}C_\mathrm{t}$.

Since AY_s is parallel to OY_t, it follows that Y_s divides TY_t in the golden ratio; similarly for the other pairs of corresponding points on the golden section lines.

3. A diagonal of a regular n-gon as a golden section line

Consider for an isosceles triangle $A_1A_{k+1}A_{n-k+1}$, $k+1\le\lfloor\frac{n}{2}\rfloor$, from a regular n-gon $A_1A_2\cdots A_n$ with circumradius R, the possibility of its golden section line being a diagonal of the regular n-gon. The golden section line is at a distance

$$\frac{2R\sin\frac{k\pi}{n}\cdot\sin\frac{k\pi}{n}}{1+4\cos^2\frac{k\pi}{n}}=R\cdot\frac{1-\cos\frac{2k\pi}{n}}{3+2\cos\frac{2k\pi}{n}}$$

from the vertex A_1. This line is the diagonal $A_{h+1}A_{n-h+1}$ if and only if this distance is $2R\sin\frac{h\pi}{n}\cdot\sin\frac{h\pi}{n}=2R\sin^2\frac{h\pi}{n}=R\left(1-\cos\frac{2h\pi}{n}\right)$. Cancelling the common factor R, and simplifying, this condition becomes

$$\frac{2+3\cos\frac{2k\pi}{n}}{3+2\cos\frac{2k\pi}{n}}=\cos\frac{2h\pi}{n}. \tag{3}$$

For $n=10$, $k=2$, we have

$$\frac{2+3\cos\frac{2\pi}{5}}{3+2\cos\frac{2\pi}{5}}=\frac{2+\frac{3}{2\varphi}}{3+\frac{1}{\varphi}}=\frac{4\varphi+3}{2(3\varphi+1)}=\frac{\varphi}{2}=\cos\frac{2\pi}{10}.$$

The condition (3) is satisfied with $h=1$.

Also, for $n=10$ and $k=4$,

$$\frac{2+3\cos\frac{4\pi}{5}}{3+2\cos\frac{4\pi}{5}}=\frac{2-\frac{3\varphi}{2}}{3-\varphi}=\frac{4-3\varphi}{2(3-\varphi)}=-\frac{1}{2\varphi}=\cos\frac{2\cdot 3\pi}{10}.$$

The condition (3) is satisfied with $h=3$.

We summarize these in the following proposition.

Proposition 4. *In a regular decagon $A_1A_2\cdots A_{10}$, the golden section line of the isosceles triangles $A_1A_3A_9$ and $A_1A_5A_7$ are respectively the diagonals A_2A_{10} and A_4A_8* (see Figure 7).

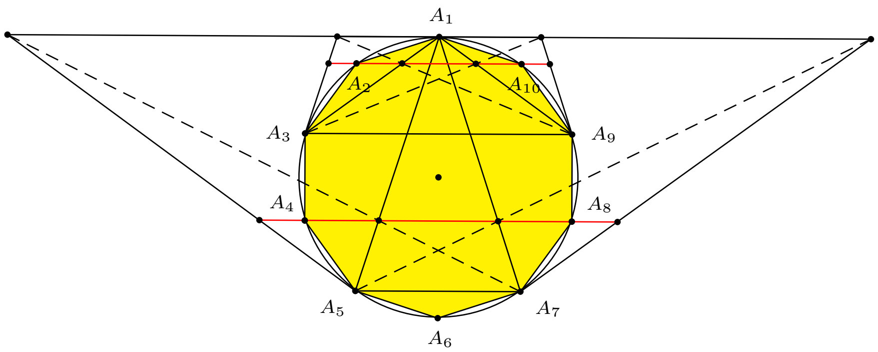

Figure 7

4. Golden section line as a side or diagonal of an inscribed regular n-gon

Proposition 3 generalizes results of Hagge [2] and Odom [4] for equilateral triangles and Tran [5] for squares to regular polygons. In Figures 8 and 9, the segment YZ is a side of a regular n-gon, $n = 3, 4$, inscribed in the isosceles triangle $A_nA_1A_2$. Figure 10, communicated by Gerhard Wanner [6], is the case of a regular hexagon, where YZ is a diagonal of a regular hexagon inscribed in the isosceles triangle. We say that a regular n-gon is inscribed in ABC if each side of ABC contains at a vertex or a side of the n-gon. We show that this is possible only for $n = 3, 4, 6$.

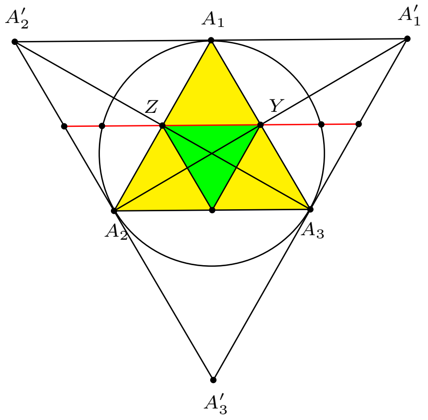

Figure 8

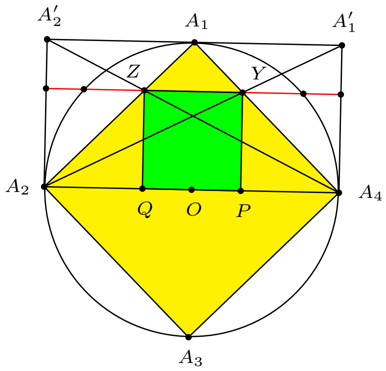

Figure 9

Proposition 5. *For an isosceles triangle formed by two consecutive sides of a regular n-gon, the golden section line is a side or a diagonal of an inscribed regular n-gon if and only if $n = 3, 4, 6$.*

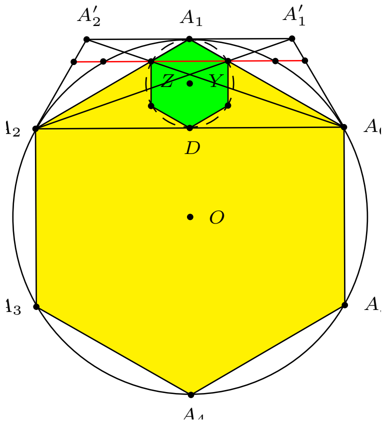

Figure 10

Proof. It is enough to prove the necessity part. The sufficiency part is indicated in Figures 8, 9, 10.

Let ℓ be the length of a side of a regular n-gon inscribed in the isosceles triangle $A_1A_2A_n$ with $A_1A_2 = A_1A_n = a$ and $\angle A_1A_2A_n = \angle A_1A_nA_2 = \frac{\pi}{n}$.

Suppose YZ is a side of the inscribed regular n-gon. Then its length is $\ell = 2a\frac{\cos\frac{\pi}{n}}{1+4\cos^2\frac{\pi}{n}}$, and its distance from A_2A_n is $A_nY\sin\frac{\pi}{n} = 4a\frac{\sin\frac{\pi}{n}\cos^2\frac{\pi}{n}}{1+4\cos^2\frac{\pi}{n}}$.

(i) If n is odd, A_2A_n contains a vertex of the n-gon. In this case

$$\begin{aligned}
&\frac{\ell}{2}\cot\frac{\pi}{2n} = 4a\frac{\sin\frac{\pi}{n}\cos^2\frac{\pi}{n}}{1+4\cos^2\frac{\pi}{n}} \implies a\frac{\cos\frac{\pi}{n}\cot\frac{\pi}{2n}}{1+4\cos^2\frac{\pi}{n}} = 4a\frac{\sin\frac{\pi}{n}\cos^2\frac{\pi}{n}}{1+4\cos^2\frac{\pi}{n}}\\
\implies\ &\cot\frac{\pi}{2n} = 4\sin\frac{\pi}{n}\cos\frac{\pi}{n} \implies \cos\frac{\pi}{2n} = 8\sin^2\frac{\pi}{2n}\cos\frac{\pi}{2n}\cos\frac{\pi}{n}\\
\implies\ &1 = 4\left(1-\cos\frac{\pi}{n}\right)\cos\frac{\pi}{n} \implies \left(2\cos\frac{\pi}{n}-1\right)^2 = 0 \implies \cos\frac{\pi}{n} = \frac{1}{2}.
\end{aligned}$$

This shows that the only possibility is $n = 3$ (see Figure 7).

(ii) If n is even, A_2A_n contains a side of the n-gon. In this case,

$$\frac{2a\cos\frac{\pi}{n}}{1+4\cos^2\frac{\pi}{n}}\cot\frac{\pi}{n} = \frac{4a\sin\frac{\pi}{n}\cos^2\frac{\pi}{n}}{1+4\cos^2\frac{\pi}{n}} \implies \sin^2\frac{\pi}{n} = \frac{1}{2}.$$

Therefore, $n = 4$ (see Figure 8).

Now suppose each of A_1A_2 and A_1A_n contains a side of the n-gon. In this case, $\ell = A_1Y = \frac{a}{1+4\cos^2\frac{\pi}{n}}$.

(iii) If n is even, then A_2A_n contains a vertex of the n-gon, and $\frac{\ell}{\sin\frac{\pi}{n}} = a\sin\frac{\pi}{n}$.

$$\frac{a}{\sin\frac{\pi}{n}\left(1+4\cos^2\frac{\pi}{n}\right)} = a\sin\frac{\pi}{n} \implies \cos^2\frac{\pi}{n} = \frac{3}{4}.$$

This is possible only when $n = 6$ (see Figure 10). In this case, $CY : YA = 3 : 1$. There is an easier construction of YZ: If D is the intersection of BC and OA, then the circle with diameter AD intersects A_1A_n and A_1A_2 at Y and Z.

(iv) If n is odd, then A_2A_n contains a side of the n-gon, and $\frac{\ell}{2}\cot\frac{\pi}{2n} = a\sin\frac{\pi}{n}$.

$$\begin{aligned}
&\frac{a\cot\frac{\pi}{2n}}{2\left(1+4\cos^2\frac{\pi}{n}\right)} = a\sin\frac{\pi}{n} \implies \cos\frac{\pi}{2n} = 2\sin\frac{\pi}{n}\sin\frac{\pi}{2n}\left(1+4\cos^2\frac{\pi}{n}\right)\\
\implies\ &1 = 4\sin^2\frac{\pi}{2n}\left(1+4\cos^2\frac{\pi}{n}\right) = 2\left(1-\cos\frac{\pi}{n}\right)\left(1+4\cos^2\frac{\pi}{n}\right)\\
\implies\ &8\cos^3\frac{\pi}{n} - 8\cos^2\frac{\pi}{n} + 2\cos\frac{\pi}{n} - 1 = 0. \qquad (4)
\end{aligned}$$

Therefore, $2\cos\frac{\pi}{n}$ is a root of the irreducible polynomial $f(x) = x^3-2x^2+x-1 \in \mathbb{Z}[x]$. But there is no integer n satisfying this condition. By Lehmer's theorem [3, Theorem 3.9], $2\cos\frac{\pi}{n}$ is an algebraic integer of degree $m := \frac{\phi(2n)}{2}$, where ϕ is the Euler totient function. If $f(x)$ is the minimal polynomial of $2\cos\frac{\pi}{n}$, then $x^m f(x+x^{-1})$ is the cyclotomic polynomial Φ_{2n}. Now there are only two integers

n for which $\frac{\phi(2n)}{2} = 3$. These are $n = 7, 9$. Since

$$x^3 f(x + x^{-1}) = x^6 - 2x^5 + 4x^4 - 5x^3 + 4x^2 - 2x + 1$$

is not the same as $\Phi_{14}(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$ or $\Phi_{18}(x) = x^6 - x^3 + 1$, this shows that there is no integer n satisfying (4) $\square$

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