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A Distance Property of the Feuerbach Point and Its Extension

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Abstract. We prove that among the distances from the inner Feuerbach point of a triangle to the midpoints of the three sides, one is equal to the sum of the remaining two. The same is true if the inner Feuerbach point is replaced by any one of the outer Feuerbach points.

1. Introduction

The famous Feuerbach theorem asserts that the nine-point circle of a triangle is tangent to the incircle and each of the excircles. The point of tangency with the incircle is the Feuerbach point ; it is in the interior of the triangle. We call it the *inner* Feuerbach point. The points of tangency with the excircles are exterior to the triangle, and are called the *exterior* Feuerbach points. In this note we give an interesting distance property of F_e (Theorem 1 and Figure 1 below), and its analogues for the exterior Feuerbach points.

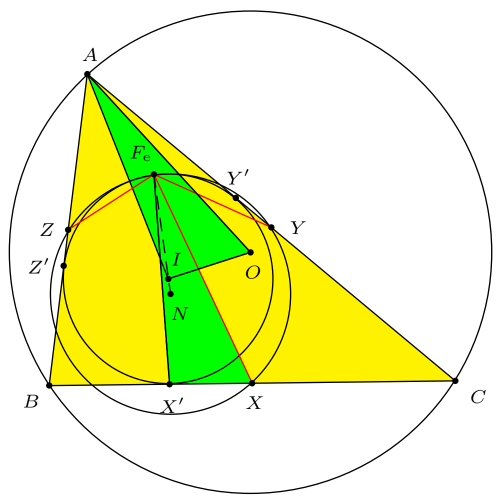

Figure 1

Theorem 1. *If F_e is the inner Feuerbach point of triangle ABC, and X, Y, Z are the midpoints of the sides BC, CA, AB, respectively, then one of the distances F_eX, F_eY, F_eZ is equal of the sum of the two others.*

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We make use of standard notations in triangle geometry (see [3]). Given triangle ABC, denote by a, b, c the lengths of the sides BC, CA, AB respectively, s the semiperimeter, Δ the area, and R, r the circumradius and inradius respectively. In homogeneous barycentric coordinates, the inner Feuerbach point is

$$F_{\rm e} = ((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2). \tag{1}$$

We shall simplify calculations in this paper by employing the notations

$$u := s-a = \frac{b+c-a}{2}, \quad v := s-b = \frac{c+a-b}{2}, \quad w := s-c = \frac{a+b-c}{2}.$$

In terms of u, v, w,

$$F_{\rm e} = (u(v-w)^2 : v(w-u)^2 : w(u-v)^2), \tag{2}$$

with coordinate sum

$$\begin{aligned}\sigma_{\rm e} :=&\ u(v-w)^2 + v(w-u)^2 + w(u-v)^2 \\ =&\ (v+w)(w+u)(u+v) - 8uvw \qquad (3)\\ =&\ abc - 8r^2 s = 4Rrs - 8r^2 s \\ =&\ 4rs(R-2r) = \frac{4\Delta}{R}\cdot R(R-2r) = \frac{4\Delta}{R}\cdot OI^2 \qquad (4)\end{aligned}$$

by Euler's formula ([1, Theorem 297]), where O and I are the circumcenter and incenter of the triangle.

Working with the distance formula, we also make use of

$$S_A := \frac{b^2+c^2-a^2}{2}, \quad S_B := \frac{c^2+a^2-b^2}{2}, \quad S_C := \frac{a^2+b^2-c^2}{2}.$$

Lemma 2. (1) $v+w=a$, $w+u=b$, $u+v=c$, *and* $u+v+w=s$.
(2) *The inradius and the exradii are*

$$r = \frac{\Delta}{s}, \quad r_a = \frac{\Delta}{u}, \quad r_b = \frac{\Delta}{v}, \quad r_c = \frac{\Delta}{w}.$$

(3) $uvw = r^2 s = r\Delta$.
(4) $vw + wu + uv = r(4R+r)$.
(5) $S_A = us - vw$, $S_B = vs - wu$, $S_C = ws - uv$.

Proof. (4)

$$\begin{aligned}vw + wu + uv =&\ \frac{uvw}{u} + \frac{uvw}{v} + \frac{uvw}{w} = \frac{r\Delta}{u} + \frac{r\Delta}{v} + \frac{r\Delta}{w} \\ =&\ r(r_a + r_b + r_c) = r(4R+r),\end{aligned}$$

since $r_a + r_b + r_c = 4R + r$ (see [1, §298 (c)]).
(5) $S_A = \frac{b^2+c^2-a^2}{2} = \frac{(w+u)^2+(u+v)^2-(v+w)^2}{2} = u(u+v+w) - vw = us - vw.$ □

Proposition 3. $F_{\rm e}X = \frac{R}{2\cdot OI} \cdot |b-c|$.

Proof. We make use of the distance formula (see [3, §7.1]) for two points $P = (u,v,w)$ and $Q=(u',v',w')$ in *absolute* barycentric coordinates:

$$PQ^2 = S_A(u-u')^2 + S_B(v-v')^2 + S_C(w-w')^2.$$

The absolute barycentric coordinates of $F_{\rm e}$ are

$$(x_{\rm e}, y_{\rm e}, z_{\rm e}) = \frac{1}{\sigma_{\rm e}}(u(v-w)^2, v(w-u)^2, w(u-v)^2).$$

Since the midpoint X of BC has absolute barycentric coordinates $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, and

$$\begin{aligned}
y_{\rm e} - \frac{1}{2} =&\ \frac{1}{2\sigma_{\rm e}}\left(2v(w-u)^2 - \sigma_{\rm e}\right)\\
=&\ \frac{-1}{2\sigma_{\rm e}}\left(u(v-w)^2 - v(w-u)^2 + w(u-v)^2\right)\\
=&\ \frac{1}{2\sigma_{\rm e}}(v-w)(w+u)(u-v);\\
z_{\rm e} - \frac{1}{2} =&\ \frac{1}{2\sigma_{\rm e}}(v-w)(w-u)(u+v),
\end{aligned}$$

the distance of $F_{\rm e}X$ is given by

$$\begin{aligned}
F_{\rm e}X^2 =&\ S_A \cdot x_{\rm e}^2 + S_B\left(y_{\rm e} - \frac{1}{2}\right)^2 + S_C\left(z_{\rm e} - \frac{1}{2}\right)^2\\
=&\ \frac{(v-w)^2}{4\sigma_{\rm e}^2}\big(4(us-vw)u^2(v-w)^2 + (vs-wu)(w+u)^2(u-v)^2\\
&\ +(ws-uv)(w-u)^2(u+v)^2\big)\\
=&\ \frac{(v-w)^2}{4\sigma_{\rm e}^2}\big(s(4u^3(v-w)^2 + v(w+u)^2(u-v)^2 + w(w-u)^2(u+v)^2)\\
&\ -u(4uvw(v-w)^2 + w(w+u)^2(u-v)^2 + v(w-u)^2(u+v)^2)\big).
\end{aligned}$$

It turns out that both $4u^3(v-w)^2 + v(w+u)^2(u-v)^2 + w(w-u)^2(u+v)^2$ and $4uvw(v-w)^2 + w(w+u)^2(u-v)^2 + v(w-u)^2(u+v)^2$ are equal to

$$(w+u)(u+v)((v+w)(w+u)(u+v) - 8uvw) = (w+u)(u+v)\sigma_{\rm e}.$$

Therefore,

$$\begin{aligned}
F_{\rm e}X^2 =&\ \frac{(v-w)^2}{4\sigma_{\rm e}^2}\cdot(s-u)(w+u)(u+v)\sigma_{\rm e}\\
=&\ \frac{(v+w)(w+u)(u+v)}{4\sigma_{\rm e}}\cdot(v-w)^2\\
=&\ \frac{abc}{4\sigma_{\rm e}}\cdot(b-c)^2 = \frac{R^2}{4\cdot OI^2}\cdot(b-c)^2,
\end{aligned}$$

with σ_e given by (4) and $abc = 4R\Delta$. This gives $F_{\rm e}X = \frac{R}{2\cdot OI}\cdot|b-c|$. □

Proof of Theorem 1. If Y and Z are the midpoints of CA and AB respectively, then analogous to the result of Proposition 3,

$$F_{\mathrm{e}}Y = \frac{R}{2 \cdot OI} \cdot |c-a| \quad \text{and} \quad F_{\mathrm{e}}Z = \frac{R}{2 \cdot OI} \cdot |a-b|.$$

Thus, $F_{\mathrm{e}}X$, $F_{\mathrm{e}}Y$, $F_{\mathrm{e}}Z$ are in the proportions of $|b-c|$, $|c-a|$, $|a-b|$. It is clear that in the latter triad, one of the terms (the greatest) is the sum of the remaining two. The same holds for the former triad $F_{\mathrm{e}}X$, $F_{\mathrm{e}}Y$, $F_{\mathrm{e}}Z$. This completes the proof of Theorem 1.

Proposition 4. *Let the incircle of triangle ABC touch the sides BC, CA, AB at X', Y', Z' respectively. The triangles $F_{\mathrm{e}}XX'$, $F_{\mathrm{e}}YY'$, $F_{\mathrm{e}}ZZ'$ are similar to the triangles AOI, BOI, COI respectively.*

Proof. It is enough to prove the similarity of triangles $F_{\mathrm{e}}XX'$ and AOI (see Figure 1). Since $BX' = s-b = \frac{c+a-b}{2}$, $XX' = \left|\frac{a}{2} - \frac{c+a-b}{2}\right| = \frac{1}{2}|b-c|$. By Proposition 3, $\frac{F_{\mathrm{e}}X}{AO} = \frac{XX'}{OI} = \frac{|b-c|}{2 \cdot OI}$. It remains to show that $\frac{F_{\mathrm{e}}X'}{AI} = \frac{|b-c|}{2 \cdot OI}$ also. For this, we compute the length of $F_{\mathrm{e}}X'$.

The absolute barycentric coordinates of X' are $\frac{1}{a}(0, s-c, s-b) = \frac{1}{v+w}(0, w, v)$. Now,

$$\begin{aligned} y_{\mathrm{e}} - \frac{w}{v+w} &= \frac{v(w-u)^2}{\sigma_{\mathrm{e}}} - \frac{w}{v+w} = \frac{v(v+w)(w-u)^2 - w\sigma_{\mathrm{e}}}{(v+w)\sigma_{\mathrm{e}}} \\ &= \frac{u(v-w)((v+w)(w+u) - 4vw)}{(v+w)\sigma_{\mathrm{e}}}, \end{aligned}$$

see Lemma 5(a) below. Similarly,

$$z_{\mathrm{e}} - \frac{v}{v+w} = \frac{-u(v-w)((u+v)(v+w) - 4vw)}{(v+w)\sigma_{\mathrm{e}}}.$$

Therefore,

$$F_{\mathrm{e}}X'^2 = S_A x_{\mathrm{e}}^2 + S_B\left(y_{\mathrm{e}} - \frac{w}{v+w}\right)^2 + S_C\left(z_{\mathrm{e}} - \frac{v}{v+w}\right)^2 = \frac{u^2(v-w)^2}{(v+w)^2\sigma_{\mathrm{e}}^2} \cdot \mathscr{F},$$

where

$$\begin{aligned} \mathscr{F} = \; & S_A(v+w)^2(v-w)^2 + S_B((v+w)(w+u) - 4vw)^2 \\ & + S_C((u+v)(v+w) - 4vw)^2. \end{aligned} \tag{5}$$

We shall establish in Lemma 5(b) below that $\mathscr{F} = 4vw(v+w)\sigma_{\mathrm{e}}$. From this,

$$\begin{aligned} F_{\mathrm{e}}X'^2 &= \frac{u^2(v-w)^2}{(v+w)^2\sigma_{\mathrm{e}}^2} \cdot 4vw(v+w)\sigma_{\mathrm{e}} = \frac{4u^2vw(v-w)^2}{(v+w)\sigma_{\mathrm{e}}} \\ &= \frac{4u \cdot r^2 s(b-c)^2}{(v+w) \cdot \frac{4rs}{R} \cdot OI^2} = \frac{u \cdot Rr(b-c)^2}{(v+w)OI^2}. \end{aligned}$$

Now,

$$AI^2 = \frac{r^2}{\sin^2 \frac{A}{2}} = \frac{uvw}{s} \cdot \frac{(w+u)(u+v)}{vw} = \frac{u(w+u)(u+v)}{s}.$$

Therefore,

$$\frac{F_{\rm e}X'^2}{AI^2} = \frac{u\cdot Rr(b-c)^2}{(v+w)OI^2}\cdot\frac{s}{u(w+u)(u+v)} = \frac{Rrs(b-c)^2}{4abc\cdot OI^2} = \frac{(b-c)^2}{4\cdot OI^2},$$

and $\frac{F_{\rm e}X'}{AI} = \frac{|b-c|}{2\cdot OI}$. The completes the proof of the similarity of triangles $F_{\rm e}XX'$ and AOI. □

Lemma 5. (a) $v(v+w)(w-u)^2 - w\sigma_{\rm e} = u(v-w)((v+w)(w+u)-4vw)$.
(b) *The polynomial* $\mathscr{F}$ *defined in* (5) *is* $4vw(v+w)\sigma_{\rm e}$.

Proof. (a) Using (3) for $\sigma_{\rm e}$, we have

$$\begin{aligned}
&v(v+w)(w-u)^2 - w\sigma_{\rm e}\\
=\ &v(v+w)(w-u)^2 - w((v+w)(w+u)(u+v)-8uvw)\\
=\ &(v+w)(v(w-u)^2 - w(w+u)(u+v)) + 8uvw^2\\
=\ &(v+w)(u^2v - uw^2 - u^2w - 3uvw) + 8uvw^2\\
=\ &u((v+w)(uv-uw-w^2-3vw) + 8vw^2)\\
=\ &u(u(v^2-w^2) - (v+w)w(3v+w) + 8vw^2)\\
=\ &u(u(v^2-w^2) - w((v+w)(3v+w) - 8vw))\\
=\ &u(u(v^2-w^2) - w(v-w)(3v-w))\\
=\ &u(v-w)(u(v+w) + w(w-3v))\\
=\ &u(v-w)((v+w)(w+u) - 4vw).
\end{aligned}$$

(b) The polynomial $\mathscr{F}$ defined in (5) is

$$\begin{aligned}
\mathscr{F} =\ &(us-vw)(v+w)^2(v-w)^2 + (vs-wu)((v+w)(w+u)-4vw)^2\\
&+ (ws-uv)((u+v)(v+w)-4vw)^2.
\end{aligned}$$

Note that the coefficient of s is

$$\begin{aligned}
&u(v+w)^2(v-w)^2 + v((v+w)(w+u)-4vw)^2 + w((u+v)(v+w)-4vw)^2\\
=\ &u(v+w)^2(v-w)^2 + v(v+w)^2(w+u)^2 - 8v^2w(v+w)(w+u) + 16v^3w^2\\
&+ w(u+v)^2(v+w)^2 - 8vw^2(u+v)(v+w) + 16v^2w^3\\
=\ &(v+w)\left(u(v+w)(v-w)^2 + v(v+w)(w+u)^2 + w(u+v)^2(v+w)\right.\\
&\left.-8v^2w(w+u) - 8vw^2(u+v) + 16v^2w^2\right)\\
=\ &(v+w)\left(u(v+w)(v-w)^2 + v(v+w)(w+u)^2 + w(u+v)^2(v+w)\right.\\
&\left.-8uvw(v+w)\right)\\
=\ &(v+w)^2\left(u(v-w)^2 + v(w+u)^2 + w(u+v)^2 - 8uvw\right)\\
=\ &(v+w)^2\left((v+w)(w+u)(u+v) - 8uvw\right)\\
=\ &(v+w)^2\sigma_{\rm e}.
\end{aligned}$$

The sum of the terms in $\mathscr{F}$ without s is

$$\begin{aligned}
&-vw(v+w)^2(v-w)^2 - wu((v+w)(w+u)-4vw)^2\\
&\qquad\qquad - uv((u+v)(v+w)-4vw)^2\\
=&\ -vw(v+w)^2(v-w)^2 - wu(v+w)^2(w+u)^2 + 8uvw^2(v+w)(w+u) - 16uv^2w^3\\
&\qquad\qquad - uv(u+v)^2(v+w)^2 + 8uv^2w(u+v)(v+w) - 16uv^3w^2\\
=&\ -(v+w)\left(vw(v+w)(v-w)^2 + wu(v+w)(w+u)^2 + uv(u+v)^2(v+w)\right.\\
&\qquad\qquad \left.-8uvw^2(w+u) - 8uv^2w(u+v) + 16uv^2w^2\right)\\
=&\ -(v+w)\left((v+w)\left(vw(v-w)^2 + wu(w+u)^2 + uv(u+v)^2\right)\right.\\
&\qquad\qquad \left.-8uvw(u(v+w)+(v-w)^2)\right)\\
=&\ -(v+w)\left((v+w)(w+u)(u+v)(u(v+w)+(v-w)^2)\right.\\
&\qquad\qquad \left.-8uvw(u(v+w)+(v-w)^2)\right)\\
=&\ -(v+w)(u(v+w)+(v-w)^2)\left((v+w)(w+u)(u+v)-8uvw\right)\\
=&\ -(v+w)(u(v+w)+(v-w)^2)\sigma_{\mathrm{e}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathscr{F} =&\ (v+w)^2\sigma_{\mathrm{e}}s - (v+w)(u(v+w)+(v-w)^2)\sigma_{\mathrm{e}}\\
=&\ (v+w)\sigma_{\mathrm{e}}\left((v+w)(u+v+w) - (u(v+w)+(v-w)^2)\right)\\
=&\ 4vw(v+w)\sigma_{\mathrm{e}}.
\end{aligned}$$

□

2. Outer Feuerbach points

The nine-point circle is also tangent to each of the excircles. If the excircle (I_a) touches BC at X_a, and the extensions of AC and AB at Y_a, Z_a respectively, then the outer Feuerbach point F_a is the intersection of the nine-point circle with the segment joining I_a to the nine-point center N (see Figure 2). In homogeneous barycentric coordinates,

$$F_a = (-s(b-c)^2 :\ (s-c)(c+a)^2 :\ (s-b)(a+b)^2).$$

Note that these can be obtained from the coordinates of $-F_{\mathrm{e}}$ by changing (a,b,c) into $(-a,b,c)$. Under this transformation, (u,v,w,s) becomes $(s,-w,-v,u)$. In terms of u, v, w,

$$F_a = (-s(v-w)^2 : w(s+v)^2 : v(s+w)^2)$$

with coordinate sum σ_a which can be obtained from

$$-\sigma_{\mathrm{e}} = -(v+w)(w+u)(u+v) + 8uvw$$

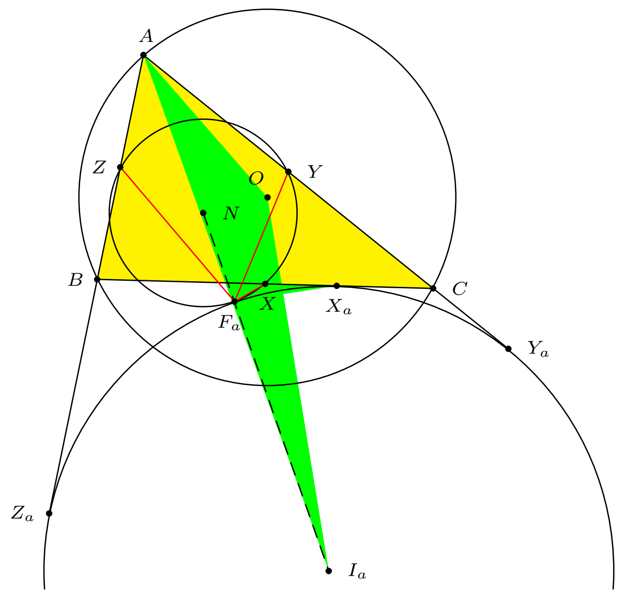

Figure 2

by the transformation (u, v, w, s) becomes $(s, -w, -v, u)$. Thus,

$$\begin{aligned}\sigma_a &= (v+w)(w+u)(u+v) + 8vws \\ &= 4\Delta(R + 2r_a) \\ &= \frac{4\Delta}{R} \cdot R(R+2r_a) = \frac{4\Delta}{R} \cdot OI_a^2,\end{aligned}$$

where I_a is the A-excenter of the triangle (see [1, §295]).

The transformations $(a, b, c) \to (-a, b, c)$ and $(u, v, w, s) \to (s, -w, -v, u)$ also wrap X' and X_a, Y' and Y_a, Z' and Z_a. Therefore, we can easily translate the results in the previous section about the incircle into results on the excircles.

First of all, the translation of the distance formulas:

$$\begin{array}{lcl} F_eX = \frac{R}{2 \cdot OI}|b-c| & \to & F_aX = \frac{R}{2 \cdot OI_a}|b-c|; \\ F_eY = \frac{R}{2 \cdot OI}|c-a| & \to & F_aY = \frac{R}{2 \cdot OI_a}(c+a); \\ F_eZ = \frac{R}{2 \cdot OI}|a-b| & \to & F_aZ = \frac{R}{2 \cdot OI_a}(a+b). \end{array}$$

Since $a + \max(b, c) = (a + \min(b, c)) + |b - c|$, we easily obtain the following analogue of Theorem 1.

Proposition 6. *If the nine-point circle touches the A-excircle at F_a, then one of F_aX, F_aY, F_aZ is the sum of the remaining two.*

The analogues of Proposition 4 also hold. From the relation

$$\frac{F_eX'}{AI} = \frac{|b-c|}{2 \cdot OI},$$

we obtain

$$\frac{F_aX_a}{AI_a} = \frac{|b-c|}{2\cdot OI_a} = \frac{XX_a}{OI_a} = \frac{F_aX}{AO}.$$

The similarity of triangles F_aXX_a and AOI_a follows (see Figure 2). Similar results hold for the other two excircles.

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