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Euclidean Figures and Solids without Incircles or Inspheres

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Abstract. All classical convex planar Euclidean figures that possess incircles have areas $A = pr/2$, where p is the perimeter, r is the radius of the incircle, and the factor of 2 represents the dimension of the space. Similarly, all classical convex Euclidean solids that possess inspheres have volumes $V = Sr/3$, where S is the total surface area, r is the radius of the insphere, and the factor of 3 represents the dimension of the space. Elementary figures such as parallelograms and trapezoids without an incircle still obey the same area relation, but then r is the harmonic mean of the radii of the two internally tangent circles to opposite sides. Similarly, common solids without an insphere (notably cylinders and prisms) still obey the same volume relation, but then r is the harmonic mean of the three internally tangent spheres to their faces.

1. Introduction

Coxeter and Greitzer [2] show early in their book that the area of a triangle can be expressed as $A = sr$, where s is the semiperimeter and r is the inradius. This result can also be written as [1]

$$A = \frac{1}{2}pr \ , \tag{1}$$

where $p = 2s$ is the perimeter of the triangle. The use of s by [2] implies that the authors did not intend to compare their result to other classical results obtained for regular polygons (e.g., $A = pa/2$, where a is the apothem), probably because the equation $A = sr$ is valid for any triangle, regular or not. On the other hand, some comparisons have been made in print between objects and figures that are regular, such as regular polygons and regular polyhedra (e.g., Rowland 2015), where it was noticed that the equations $A = pa/2$ and

$$V = \frac{1}{3}Sr \ , \tag{2}$$

where V is the volume and S is the total surface area, are precisely similar except for the factors of 2 and 3 that appear because of the dimension of the space in each case.

This investigation started with the following question: Which global property does the ratio A/p or V/S represent in any Euclidean figure or solid, respectively?

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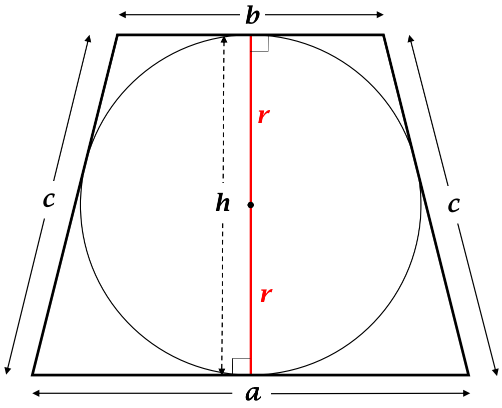

Figure 1. Isosceles trapezoid with bases a, b, legs c, inradius r, and altitude $h = 2r$.

These ratios represent a property with dimensions of length that should be characteristic of the entire object and not of any particular side and altitude. The above classical results indicate that for figures with an inscribed circle, the ratio A/p is related to the inradius; similarly, for three-dimensional solids with an insphere, the ratio V/S is related to the inradius of the insphere. This includes any rhombus, square, regular polygon, and one special isosceles trapezoid in two dimensions; and any cube, cone, and regular pyramid in three dimensions, respectively.

Notably absent from the above lists are parallelograms, rectangles, and arbitrary trapezoids; as well as rectangular prisms and right cylinders in three dimensions. So we set out to investigate such common cases in Euclidean geometry in which an incircle or an insphere cannot be drawn. For these cases, the above equations (1) and (2) are still valid, except that the inradius r has to be replaced by the harmonic mean of the radii of internally tangent circles or spheres.

In $\S$ 2, we analyze the ratio A/p in two-dimensional Euclidean figures and in $\S$ 3, we analyze the ratio V/S in three-dimensional Euclidean solids. Finally, in $\S$ 4, we discuss briefly and interpret our results in terms of the mean curvature [3, 5] defined by such tangent circles and spheres for these objects.

2. Two-Dimensional Figures

Eq. (1) can be derived quite easily for two-dimensional figures such as squares, rhombuses, and circles. But it is not obvious how it applies to trapezoids. Symmetry tells us that there exists a special isosceles trapezoid with bases a, b, congruent sides c, and altitude h between the bases that possesses an incircle (Fig. 1). Then:

Theorem 1. *If eq. (1) holds for an isosceles trapezoid, then* $c = (a + b)/2$.

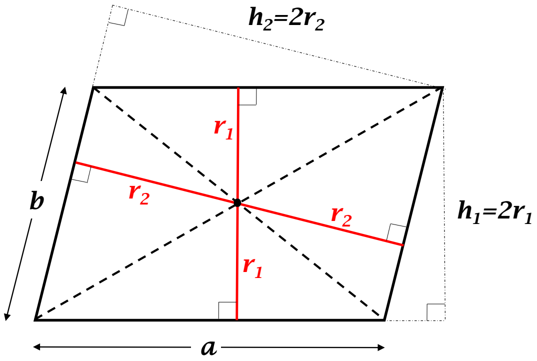

Figure 2. Parallelogram with sides a, b, altitudes $h_1 = 2r_1$, $h_2 = 2r_2$, and without an incircle.

Proof. An inscribed circle with radius r must be tangent to the bases, so $r = h/2$ (Fig. 1). Substituting this equation and the perimeter $p = a + b + 2c$ into eq. (1):

$$A = \frac{1}{2}pr = \frac{1}{2}(a+b+2c)\frac{h}{2} = \frac{1}{2}\left(\frac{a+b}{2} + c\right)h \,. \tag{3}$$

Comparing this result to the well-known formula $A = (a + b)h/2$, we find that $\frac{a+b}{2} + c = a + b$ which implies that $c = (a + b)/2$. □

It is also interesting to note that for this special isosceles trapezoid, the altitude is the geometric mean of the bases, i.e., $h = \sqrt{ab}$.

Next we prove the converse of Theorem 1:

Theorem 2. *If $c = (a + b)/2$, then eq. (1) holds for this isosceles trapezoid.*

Proof. We begin with the well-known formula $A = (a + b)h/2$. An inscribed circle with radius r must be tangent to the bases, so $h = 2r$ (Fig. 1). Then

$$A = \frac{1}{2}(a+b)h = \frac{1}{2}(a+b)2r \,. \tag{4}$$

Then, we rewrite $(a + b)2 = (a + b) + (a + b) = (a + b) + 2c = p$, using the hypothesis. Therefore, $(a + b)2 = p$ and by substitution to eq. (4), we find eq. (1). □

Rectangles and parallelograms do not possess an incircle. Nevertheless, we can draw two different circles that will be internally tangent between opposite parallel sides. Let r_1 and r_2 be the radii of these circles (Fig. 2). Then:

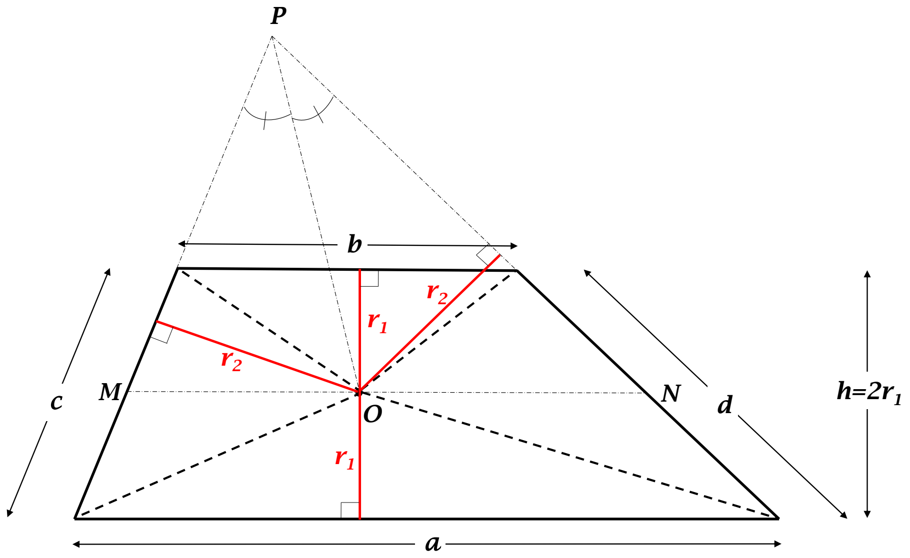

Figure 3. Trapezoid with bases a, b, legs c, d, altitude $h = 2r_1$, and without an incircle.

Theorem 3. *In rectangles and parallelograms with two internally tangent circles to opposite sides, eq. (1) holds, where r is the harmonic mean of r_1 and r_2.*

Proof. For rectangles of dimensions L and W, the proof is trivial given that $r_1 = L/2$, $r_2 = W/2$, and $p = 4(r_1 + r_2)$:

$$\frac{A}{p} = \frac{LW}{2(L+W)} = \frac{r_1 r_2}{r_1 + r_2} = \frac{1}{2}\left(\frac{2r_1 r_2}{r_1 + r_2}\right) . \tag{5}$$

For parallelograms of dimensions a and b and altitudes h_1 and h_2, we find that $h_1 = 2r_1$ and $h_2 = 2r_2$ (Fig. 2). We draw the diagonals to partition the figure into four triangles of equal area, and we sum up their areas to obtain the total area:

$$A = 2 \cdot \frac{1}{2} a r_1 + 2 \cdot \frac{1}{2} b r_2 = a r_1 + b r_2 . \tag{6}$$

Since $p = 2(a + b)$, we can write

$$\frac{A}{p} = \frac{a r_1 + b r_2}{2(a+b)} . \tag{7}$$

But $a r_1 = b r_2$ $(= A/2)$, in which case $b/a = r_1/r_2$. Substituting into eq. (7), we find after some algebra that

$$\frac{A}{p} = \frac{1}{2}\left(\frac{2r_1 r_2}{r_1 + r_2}\right) . \tag{8}$$

□

The same result (eq. [8]) can be proven for the arbitrary trapezoid without an incircle. First, we need to find the common center of the two circles that are internally tangent to the bases and to the legs, respectively. Let P be the point at which the legs meet when they are extended (Fig. 3). This center O is located at the intersection of the midsegment $\overline{MN}$ (so that it is equidistant from the bases) and the bisector of $\angle$P (so that it is equidistant from the legs). By construction, O is interior to the trapezoid. Let r_1 be the distance of O from the bases and r_2 be the distance of O from the legs. Then:

Theorem 4. *In trapezoids with two internally tangent circles to opposite sides, eq. (1) holds, where r is the harmonic mean of r_1 and r_2.*

Proof. For a trapezoid with bases a and b, legs c and d, and altitude h, we find that $h = 2r_1$ (Fig. 3). We draw segments from O to the vertices of the trapezoid to partition the figure into four triangles, and we sum up their areas to obtain the total area:

$$A = \frac{1}{2}(a+b)r_1 + \frac{1}{2}(c+d)r_2 \ . \tag{9}$$

The area of the trapezoid can also be expressed as

$$A = \frac{1}{2}(a+b)h = \frac{1}{2}(a+b)2r_1 = (a+b)r_1 \ . \tag{10}$$

Equating the two expressions for A, we find that $c + d = (a + b)r_1/r_2$. Using this equation to eliminate $c + d$ from the perimeter, we find that $p = a + b + c + d = (a + b)(r_1 + r_2)/r_2$. Then using this equation for p and eq. (10), we find that

$$\frac{A}{p} = \frac{(a+b)r_1}{(a+b)(r_1+r_2)/r_2} = \frac{r_1 r_2}{r_1+r_2} \ , \tag{11}$$

which is equivalent to eq. (1) with $r = 2r_1r_2/(r_1 + r_2)$. □

3. Three-Dimensional Solids

Eq. (2) can be derived quite easily for three-dimensional solids such as cubes and other regular polyhedra [4], and spheres. But it is not obvious how it applies to cones, cylinders, regular polygonal pyramids, and rectangular prisms. We give below only the proofs for cones and cylinders; the other proofs are similar.

Consider a cone of base radius R, height H, and slant height L (Fig. 4). A sphere of radius r can always be inscribed into this cone. Then:

Theorem 5. *Eq. (2) holds for any cone.*

Proof. Certain radii of the sphere are normal to the base of the cone and to its lateral surface (Fig. 4). A vertical cross-section through the vertex V and the center of the base C contains two similar triangles (VDO and VCB) from which we derive the proportion

$$\frac{R}{r} = \frac{L}{x} \ , \tag{12}$$

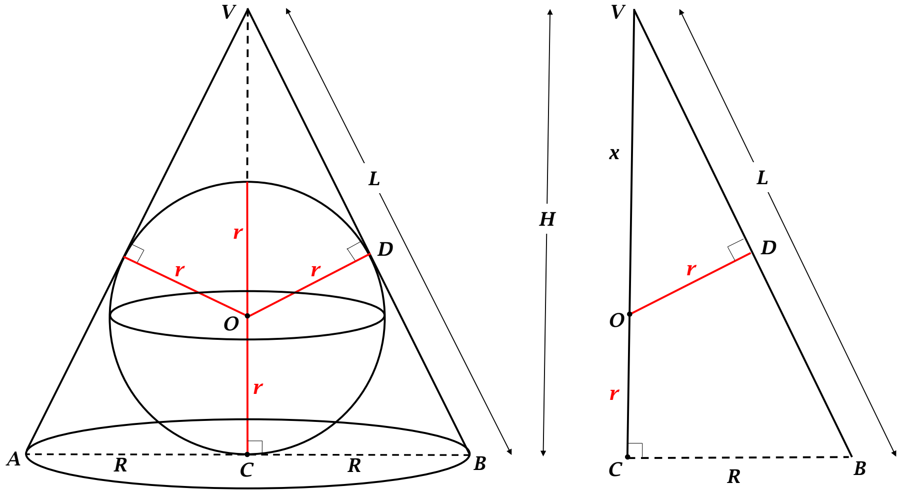

Figure 4. Cone of radius R, height H, and slant height L with an insphere of radius r. Triangles VDO and VCB on the cross-section VAB are similar (AA Similarity Postulate).

where x is the hypotenuse of the smaller triangle (see Fig. 4). This is used in the total surface area S of the cone to eliminate L:

$$S = \pi R^2 + \pi RL = \pi R^2 \left(1 + \frac{x}{r}\right) = \frac{\pi R^2 H}{r}\,, \tag{13}$$

since $r + x = H$. This equation implies that $\pi R^2 H = Sr$. Using this equation into the volume formula of the cone $V = \pi R^2 H/3$, then eq. (2) is obtained. □

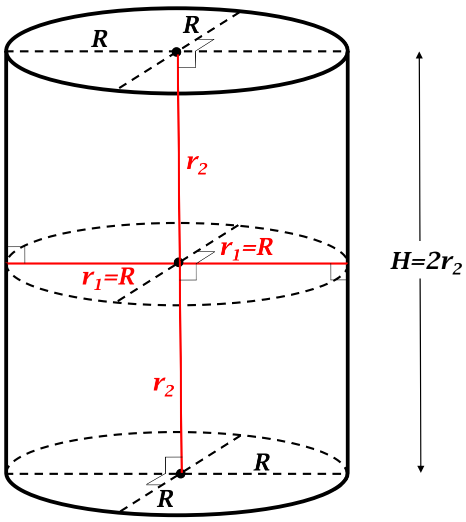

Figure 5. Cylinder of radius $R = r_1$, height $H = 2r_2$, and without an insphere.

There also exists one special cylinder in which a sphere of radius r can be inscribed. Its height then is $H = 2r$. Eq. (2) holds for that cylinder too, but the proof is trivial and is omitted. In general, a sphere cannot be inscribed in an arbitrary cylinder. In that case, we consider a cylinder with base radius R and height H in which we can draw two spheres: one internally tangent to the lateral surface with radius $r_1 = R$; and another internally tangent to the bases with radius $r_2 = H/2$ (Fig. 5). Then:

Theorem 6. *In cylinders with two such internally tangent spheres, eq. (2) holds, where r is the harmonic mean of r_1, r_1, and r_2.*

Proof. We use $r_2 = H/2$ to eliminate the height from the well-known formulae for V and S for a cylinder. Then we obtain $V = 2\pi r_1^2 r_2$, $S = 2\pi r_1(r_1 + 2r_2)$, and their ratio is

$$\frac{V}{S} = \frac{r_1 r_2}{r_1 + 2r_2} = \frac{1}{3}\left(\frac{3}{\frac{2}{r_1} + \frac{1}{r_2}}\right) . \tag{14}$$

□

Radius r_1 enters twice in the harmonic mean because the sphere that is internally tangent to the lateral surface covers two of the three principal directions in three-dimensional space. This is understood when eq. (14) is compared to the corresponding result for an arbitrary rectangular prism that has three different internally tangent spheres with radii r_1, r_2, and r_3:

$$\frac{V}{S} = \frac{1}{3}\left(\frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}\right) . \tag{15}$$

4. Discussion

We have shown that eqs. (1) and (2) hold for a variety of classical Euclidean figures and solids, respectively. In cases where an incircle or an insphere cannot be drawn, the radius r in these equations reverts to a harmonic mean of the radii of the circles or spheres that are internally tangent to the sides or faces of the figures or solids, respectively. We note that no such general results can be obtained by using the radii of the circumscribed circles and spheres. Therefore, it appears that the inradius r is the most important global property of these objects in any dimension; and that the ratio of "content to boundary" (A/p or V/S) is equal to r/N, where $N \geq 2$ is the dimension of space.

The concept of mean curvature

$$K_m \equiv \frac{1}{N}\sum_{i=1}^{N} K_i , \tag{16}$$

from differential geometry [3, 5] allows us to rewrite the equations in all cases as

$$A = \frac{p}{\frac{1}{r_1} + \frac{1}{r_2}} = \frac{p}{2K_m} , \tag{17}$$

and as

$$V = \frac{S}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} = \frac{S}{3K_m} \,, \tag{18}$$

where the reciprocals of the radii represent the principal curvatures K_i in the N dimensional space. We see now that the ratios A/p and V/S (of content to boundary) are equal to

$$\frac{\text{Content}}{\text{Boundary}} = \frac{\rho_m}{N} \,, \tag{19}$$

where

$$\rho_m \equiv \frac{1}{K_m} \,, \tag{20}$$

is the radius of the mean curvature K_m in each object. This answers the question posed in the Introduction.

Eq. (17) is only approximately valid for an ellipse with semi-axes $a = r_1$ and $b = r_2$. Similarly, in the case of a circular sector with an incircle, eq. (1) is approximate and becomes more accurate in the limit of small opening angles θ (i.e., for $\theta \to 0$). But the errors are remarkably small in both of these cases. We believe that the problem with the ellipse is that the circle about the two vertices is nowhere interior to the figure. And the problem with the circular sector is that its perimeter combines two linear segments with an arc. However, as $\theta \to 0$, the arc length approaches that of a straight line segment and this is why the error term then becomes minimal.

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