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The Feuerbach Point and the Fuhrmann Triangle

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Abstract. We establish a few results on circles through the Feuerbach point of a triangle, and their relations to the Fuhrmann triangle. The Fuhrmann triangle is perspective with the circumcevian triangle of the incenter. We prove that the perspectrix is the tangent to the nine-point circle at the Feuerbach point.

1. Feuerbach point and nine-point circles

Given a triangle ABC, we consider its intouch triangle $X_0Y_0Z_0$, medial triangle $X_1Y_1Z_1$, and orthic triangle $X_2Y_2Z_2$. The famous Feuerbach theorem states that the incircle $(X_0Y_0Z_0)$ and the nine-point circle (N), which is the common circumcircle of $X_1Y_1Z_1$ and $X_2Y_2Z_2$, are tangent internally. The point of tangency is the Feuerbach point F_e. In this paper we adopt the following standard notation for triangle centers: G the centroid, O the circumcenter, H the orthocenter, I the incenter, N_a the Nagel point. The nine-point center N is the midpoint of OH.

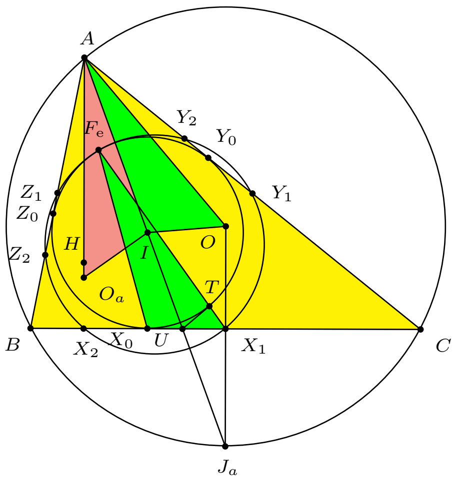

Figure 1

Proposition 1. *Let ABC be a non-isosceles triangle.*

(a) *The triangles $F_\mathrm{e}X_0X_1$, $F_\mathrm{e}Y_0Y_1$, $F_\mathrm{e}Z_0Z_1$ are directly similar to triangles AIO, BIO, CIO respectively.*

(b) *Let O_a, O_b, O_c be the reflections of O in IA, IB, IC respectively. The lines IO_a, IO_b, IO_c are perpendicular to $F_\mathrm{e}X_1$, $F_\mathrm{e}Y_1$, $F_\mathrm{e}Z_1$ respectively.*

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Proof. (a) It is enough to prove the direct similarity of triangles $F_eX_0X_1$ and AOI. We work with the notion of directed angles (see [2, §§16–19]). Assume that $AB < AC$. Let U and J_a be the intersections of the line AI with BC and the circumcircle (O) respectively. Draw a tangent UT to the incircle (I) (see Figure 1). The points F_e, T, X_1 are collinear (see [1, Theorem 215]). Hence, modulo π,

$$\begin{aligned}(X_0X_1,\ X_0T) &\equiv \frac{\pi}{2} - (X_0T,\ X_0I) \equiv\ (IX_0,\ IJ_a)\\ &\equiv\ (J_aO,\ J_aA) \equiv\ (AJ_a,\ AO) \equiv (AI,\ AO) \equiv -(AO,\ AI).\end{aligned}$$

On the other hand,

$$\frac{X_0T}{X_0X_1} = \frac{2r\sin X_0IJ_a}{\frac{b-c}{2}} = \frac{2r}{R}\cdot\frac{\sin\frac{B-C}{2}}{\sin B-\sin C} = \frac{r}{R\sin\frac{A}{2}} = \frac{AI}{AO}.$$

Therefore, triangles X_0TX_1 and AIO are inversely similar.

Since $(F_eX_0,\ F_eX_1) \equiv\ -(X_0T,\ X_0X_1) \pmod{\pi}$, and $(X_1F_e,\ X_1X_0) \equiv -(X_1X_0, X_1T) \pmod{\pi}$, triangles $F_eX_0X_1$ and X_0TX_1 are oppositely similar. It follows that $F_eX_0X_1$ and AIO are *directly* similar.

(b) Triangle AIO_a is oppositely similar to triangle $F_eX_0X_1$. Since $AO_a \perp X_0X_1$, it follows that $IO_a \perp F_eX_1$. Similarly, $IO_b \perp F_eY_1$ and $IO_c \perp F_eZ_1$. □

The Feuerbach point F_e is also the Poncelet point of the quadrilateral $ABCI$. This means that F_e is the common point of the nine-point circles of the four triangles IBC, ICA, IAB, and ABC. The circles $(F_eX_0X_1)$, $(F_eY_0Y_1)$, $(F_eZ_0Z_1)$ are therefore the nine-point circles of triangles IBC, ICA, IAB respectively. Each of them passes through the midpoints of two of the segments AI, BI, CI. Denote by N_a, N_b, N_c the nine-point centers of the triangles IBC, ICA, IAB respectively. We shall prove in Theorem 5 below that N_a, N_b, N_c are equidistant from N, the nine-point center of ABC.

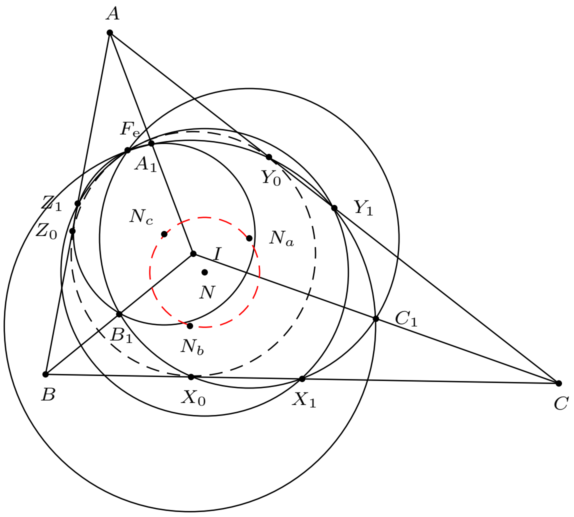

Figure 2

2. Fuhrmann triangle and Fuhrmann circle

The triangle $N_aN_bN_c$ is closely related to the Fuhrmann triangle. Let $J_aJ_bJ_c$ be the circumcevian triangle of the incenter I, and J'_a, J'_b, J'_c the reflections of J_a in BC, J_b in CA, J_c in AB respectively. These reflections form the Fuhrmann triangle $J'_aJ'_bJ'_c$. Now, J_a is the center of the circumcircle of IBC, which also passes through the excenter I_a. The nine-point center of IBC is the midpoint between I and the reflection of its circumcenter in the side BC. [1] Therefore, N_a is the midpoint of IJ'_a. Similarly, N_b and N_c are the midpoints of IJ'_b and IJ'_c. In other words, $N_aN_bN_c$ is the image of the Fuhrmann triangle under the homothety h with center I and ratio $\frac{1}{2}$. [2]

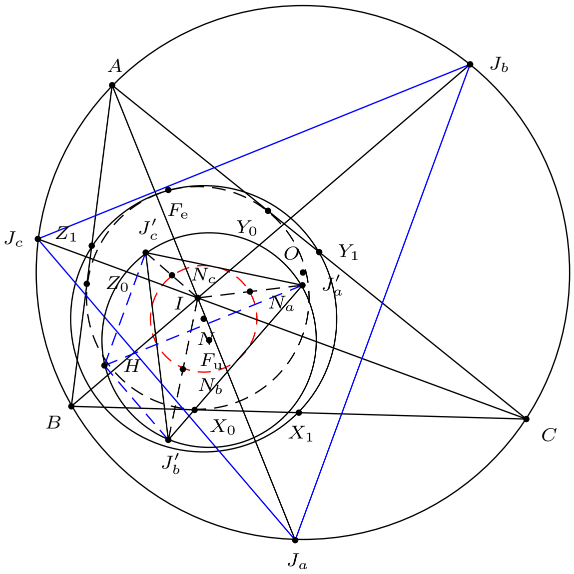

Figure 3

Basic results about the Feuerbach point and the Fuhrmann triangle can be found in [1, §§215–216] and [2, §§320–324, 367–372]. A proof of the Feuerbach theorem is given in [4].

The Fuhrmann circle is the circumcircle of the Fuhrmann triangle. It contains $HN_{\rm a}$ as a diameter ([2, Theorem 369]). The center of the Fuhrmann circle is the midpoint $F_{\rm u}$ of $HN_{\rm a}$. Here is an alternative description.

Proposition 2. *The center of the Fuhrmann circle is the reflection of I in N.*

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[1] If ABC is a triangle with centroid G and circumcenter O, its nine-point center is $N = \frac{3G-O}{2} = \frac{A+((B+C)-O)}{2} = \frac{A+O'}{2}$, where $O' = B + C - O = 2 \cdot \frac{B+C}{2} - O$ is the reflection of O in (the midpoint of) BC.

[2] Throughout this paper, h denotes this specific homothety. For every point P, $\mathsf{h}(P) = \frac{1}{2}(I+P)$, the midpoint of IP.

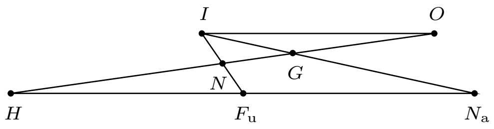

Figure 4

Proof.

$$F_{\rm u} = \frac{H + N_{\rm a}}{2} = \frac{H + (3G - 2I)}{2} = \frac{H + (H + 2 \cdot O) - 2I}{2} = H + O - I = 2N - I.$$

□

Proposition 3. *The Fuhrmann triangle $J'_aJ'_bJ'_c$ and the circumcevian triangle of I are oppositely similar.*

Proof. Since the circumcircle of $J'_aJ'_bJ'_c$ contains H, and $HJ'_a \parallel J_cJ_b$ etc. (see Figure 3),

$$(J'_aJ'_b,\ J'_aJ'_c) \equiv (HJ'_b,\ HJ'_c) \equiv (J_cJ_a,\ J_bJ_a) \equiv -(J_aJ_b,\ J_aJ_c) \pmod \pi.$$

Similarly, $(J'_bJ'_c,\ J'_bJ'_a) \equiv -(J_bJ_c,\ J_bJ_a) \pmod \pi$. The two triangles $J'_aJ'_bJ'_c$ and $J_aJ_bJ_c$ are oppositely similar. □

Since the vertices of $J_aJ_bJ_c$ are on the angle bisectors of ABC and the sides are perpendicular to these bisectors, the triangle $J_aJ_bJ_c$ has orthocenter I. This is also true for the Fuhrmann triangle.

Proposition 4. *The Fuhrmann triangle has orthocenter I.*

Proof. We begin with the excentral triangle $I_aI_bI_c$, where I_a, I_b, I_c are the excenters of triangle AC. It is well known that it has orthocenter I and circumcenter $I' = 2 \cdot O - I$, the reflection of I in O. Therefore, the centroid of the excentral triangle is

$$\frac{I_a + I_b + I_c}{3} = \frac{2I' + I}{3} = \frac{4 \cdot O - I}{3}.$$

From this we have

$$I_a + I_b + I_c = 4 \cdot O - I.$$

Since J_a is the center of the circle (IBC), which also passes through I_a, $J_a = \frac{I+I_a}{2}$. Now, for the Fuhrmann triangle, we have

$$J'_a = 2 \cdot \frac{B + C}{2} - J_a = \frac{2(B + C) - (I + I_a)}{2}.$$

and analogous expressions for J_b' and J_c'. The centroid of the Fuhrmann triangle is therefore

$$\begin{aligned} G' = \frac{J_a' + J_b' + J_c'}{3} &= \frac{4(A+B+C) - 3I - (I_a + I_b + I_c)}{6} \\ = \frac{12G - 3I - (4 \cdot O - I)}{6} &= \frac{6G - 2 \cdot O - I}{3} \\ = \frac{3G + (3G - 2 \cdot O) - I}{3} &= \frac{3G + H - I}{3}. \end{aligned}$$

Its orthocenter is

$$H' = 3G' - 2F_{\mathrm{u}} = (3G + H - I) - (H + N_{\mathrm{a}}) = 3G - N_{\mathrm{a}} - I = 2I - I = I.$$

□

Theorem 5. *The triangle $N_aN_bN_c$ has circumcenter N, circumradius $\frac{OI}{2}$, and orthocenter I.*

Proof. The triangle $N_aN_bN_c$ is the image of the Fuhrmann triangle under h. It has circumcenter $\mathsf{h}(2N - I) = \frac{I+(2N-I)}{2} = N$ and orthocenter $\mathsf{h}(I) = \frac{I+I}{2} = I$.

Since the Fuhrmann circle has diameter HN_{a}, which is parallel to and equal to twice OI (see Figure 4), its circumradius is OI. It follows that the circumradius of $N_aN_bN_c$ is $\frac{OI}{2}$. □

Corollary 6. *The circumcircle of $N_aN_bN_c$ is the nine-point circle of the Fuhrmann triangle.*

Proof. Since the Fuhrmann triangle has orthocenter I, the point N_a, being the midpoint of IJ_a', lies on its nine-point circle. Similarly, N_b and N_c are on the same nine-point circle. Therefore, the circumcircle of $N_aN_bN_c$ is the nine-point circle of the Fuhrmann triangle. □

Consider the midpoints A_1, B_1, C_1 of AI, BI, CI respectively.

Proposition 7. *The orthocenter of triangle $A_1B_1C_1$ lies on the circumcircle of $N_aN_bN_c$.*

Proof. Triangle $A_1B_1C_1$ is the image of ABC under the homothety h. Its orthocenter H' is the midpoint of IH. Since I is the orthocenter of the Fuhrmann triangle, and H lies on the Fuhrmann circle, it follows that H' lies on the nine-point circle of the Fuhrmann triangle, which is the circle $(N_aN_bN_c)$. □

Theorem 8. *The N_aX_1, N_bY_1, N_cZ_1 are concurrent at the Spieker center of triangle ABC.*

Proof. In triangle $J_a'IJ_a$, N_a and X_1 are the midpoints of the sides $J_a'I$ and $J_a'J_a$. Therefore, N_aX_1 is parallel to J_aI, the bisector of angle A. In the medial triangle $X_1Y_1Z_1$, the line N_aX_1 is the bisector of angle X_1. Similarly, N_bY_1 and N_cZ_1 are the bisectors of angles Y_1 and Z_1. The three lines N_aX_1, N_bY_1, N_cZ_1 are concurrent at the Spieker center, the incenter of the medial triangle $X_1Y_1Z_1$. □

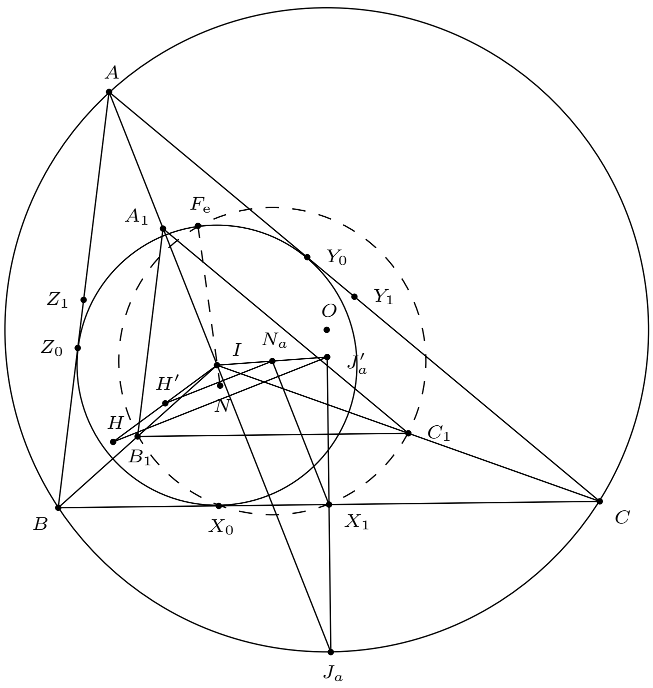

Figure 5

Remark. This point of concurrency is also the antipode of the orthocenter H' of $A_1B_1C_1$ on the circle $N_aN_bN_c$. Since this circle has center N and contains $H' = \frac{I+H}{2}$, the antipode of H' is

$$2N - H' = O + H - \frac{I+H}{2} = \frac{2 \cdot O + H - I}{2} = \frac{3G - I}{2},$$

which divides IG in the ratio $3 : -1$. This is the Spieker center.

3. The residual triangles of the orthic triangle

Consider the orthic triangle $X_2Y_2Z_2$. Let X_3, Y_3, Z_3 be the midpoints of its sides Y_2Z_2, Z_2X_2, X_2Y_2 respectively, and let
O_1, I_1, F_1 be the circumcenter, incenter and Feuerbach point of triangle AY_2Z_2,
O_2, I_2, F_2 those of BZ_2X_2, and
O_3, I_3, F_3 those of CX_2Y_2.

Note that the circumcenter O_1 is the midpoint of AH, and is a point on the nine-point circle of ABC.

Theorem 9. *The lines F_1X_3, F_2Y_3, F_3Z_3 are perpendicular to OI.*

Proof. Let the line AI intersect BC at A'. Draw a line passing through A' parallel to Y_2Z_2, intersecting AC and AB at B' and C' respectively. Triangle $AB'C'$ is

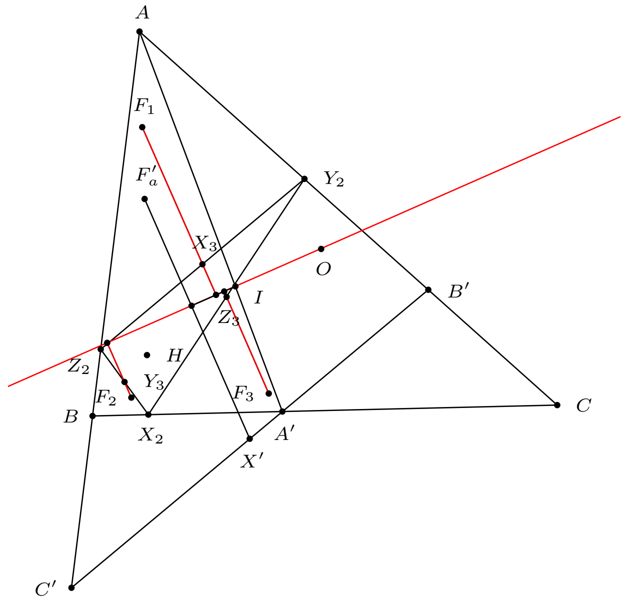

Figure 6

the reflection of ABC in AI, and is homothetic to triangle AY_2Z_2. Under this homothety, F_1 corresponds to the reflection F'_a of F_e in AI. Also, X_3 corresponds to the midpoint X' of $B'C'$. It follows that $F_1X_3 \parallel F'_aX'$. By Lemma 1(ii), $F'_aX' \perp OI$. Therefore, $F_1X_3 \perp OI$. Similarly, F_2Y_3 and F_3Z_3 are also perpendicular to OI. □

Theorem 10. *The lines O_1I_1, O_2I_2, O_3I_3 are concurrent at the Feuerbach point F_e.*

Proof. Since A, Y_2, H, Z_2 are concyclic, the circumcenter N_1 is the midpoint of AH. Let O' be the reflection of O in AI. By Proposition 1(b), $O'I \perp F_eX_1$. Now, N_1X_1 is a diameter of nine-point circle of ABC. This means that $N_1F_e \perp F_eX_1$. Therefore, $O'I$ and O_1F_e are parallel.

Since the reflection of triangle AY_2Z_2 in AI is homothetic to ABC, the incenter I_1 of AY_2Z_2 is the intersection of AI with the parallel to $O'I$ through O_1. This is the line N_1F_e. From this we conclude that I_1 is the intersection of N_1F_e with AI. The line O_1I_1 passes through F_e; similarly for the lines O_2I_2 and O_3I_3. □

4. Perspectivity and orthology of $J_aJ_bJ_c$ and $J'_aJ'_bJ'_c$

The triangles $J_aJ_bJ_c$ and $J'_aJ'_bJ'_c$ are clearly perspective at O. They are also axis-perspective. This means that the three points

$$X = J_bJ_c \cap J'_bJ'_c, \quad Y = J_cJ_a \cap J'_cJ'_a, \quad Z = J_aJ_b \cap J'_aJ'_b,$$

are collinear.

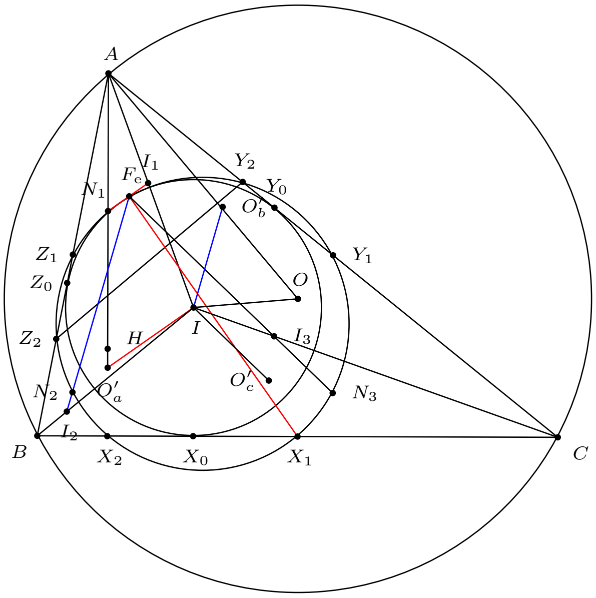

Figure 7

Theorem 11. *The line containing X, Y, Z is the tangent to the nine-point circle at the Feuerbach point.*

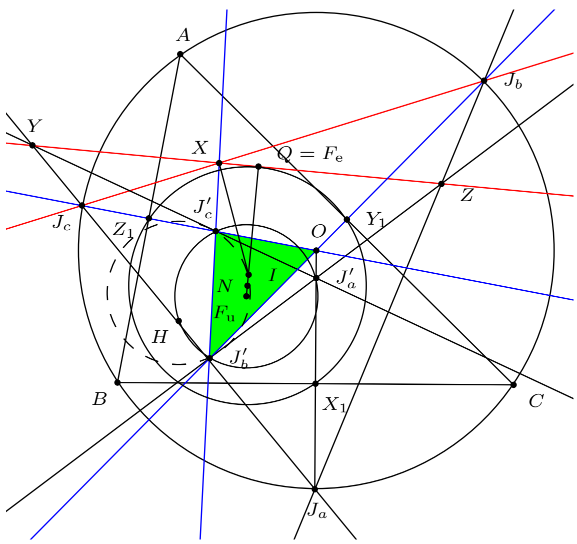

Figure 8

Proof. Applying Menelaus' theorem to triangle $OJ_b'J_c'$ with transversal XJ_bJ_c, we have

$$\frac{J_b'X}{XJ_c'} \cdot \frac{J_c'J_c}{J_cO} \cdot \frac{OJ_b}{J_bJ_b'} = -1 \implies \frac{J_b'X}{XJ_c'} = -\frac{J_cO}{J_c'J_c} \cdot \frac{J_bJ_b'}{OJ_b} = -\frac{J_bJ_b'}{J_cJ_c'}.$$

Therefore,

$$\frac{XJ_b'}{XJ_c'} = \frac{J_bJ_b'}{J_cJ_c'} = \frac{\sin^2 \frac{B}{2}}{\sin^2 \frac{C}{2}} = \left(\frac{\sin \frac{B}{2}}{\sin \frac{C}{2}}\right)^2$$

On the other hand,

$$\frac{IJ_b'}{IJ_c'} = \frac{\sin \frac{B}{2}}{\sin \frac{C}{2}}.$$

Therefore, $\frac{XJ_b'}{XJ_c'} = \left(\frac{IJ_b'}{IJ_c'}\right)^2$. This means that IX is tangent to the circle $(IJ_b'J_c')$, and $IX^2 = XJ_b' \cdot XJ_c'$. Thus, X lies on the radical axis of the Fuhrmann circle and the point-circle I. The same is true for Y and Z. This shows that the perspectrix XYZ is perpendicular to the line joining F_{u} and I, which is the same as the line NI. If XYZ intersects NI at Q,

$$QF_{\mathrm{u}}^2 - QI^2 = (QF_{\mathrm{u}} - QI)(QF_{\mathrm{u}} + QI) = IF_{\mathrm{u}} \cdot 2QN = 4NI \cdot QN = 2(R-2r)QN.$$

It follows that $2(R-2r)QN = \rho^2 = OI^2 = R(R-2r)$, and $QN = \frac{R}{2}$. This means that Q lies on the nine-point circle of triangle ABC, and is the Feuerbach point F_{e}. The line XYZ is the tangent to the nine-point circle at F_{e}. □

Theorem 12. *The triangles $J_aJ_bJ_c$ and $J_a'J_b'J_c'$ are orthologic.*

(a) *The perpendiculars from J_a' to J_bJ_c etc are concurrent at the Nagel point N_{a}.*

(b) *The perpendiculars from J_a to $J_b'J_c'$ etc are concurrent at the superior of the Feuerbach point.*

Proof. (a) The perpendiculars from J_a' to J_bJ_c, J_b' to J_cJ_a, and J_c' to J_aJ_b are the lines

$$\begin{aligned}
(b-c)x + \quad by - \quad cz &= 0,\\
-ax + (c-a)y + \quad cz &= 0,\\
ax - \quad by + (a-b)z &= 0.
\end{aligned}$$

These three lines are concurrent at the point $(x:y:z)$, where

$$\begin{aligned}
x:y:z &= \begin{vmatrix} c-a & c \\ -b & a-b \end{vmatrix} : -\begin{vmatrix} -a & c \\ a & a-b \end{vmatrix} : \begin{vmatrix} -a & c-a \\ a & -b \end{vmatrix}\\
&= a(b+c-a) : a(c+a-b) : a(a+b-c)\\
&= b+c-a : c+a-b : a+b-c.
\end{aligned}$$

This is the Nagel point.

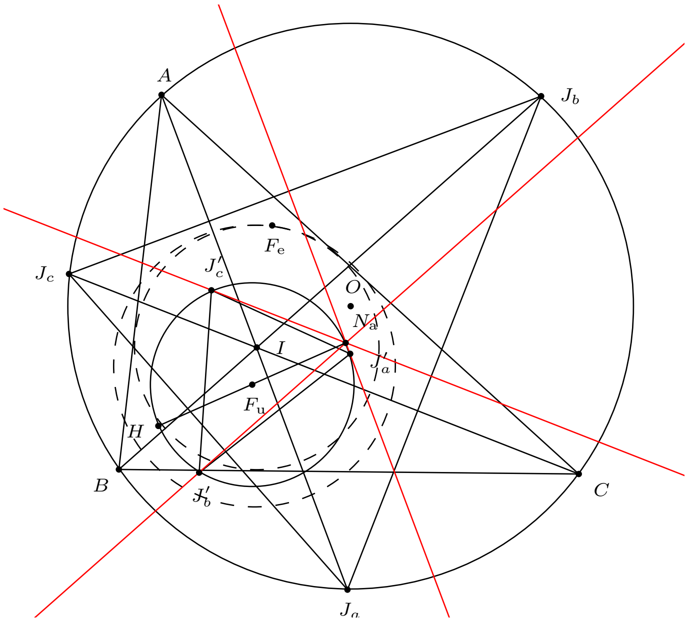

Figure 9

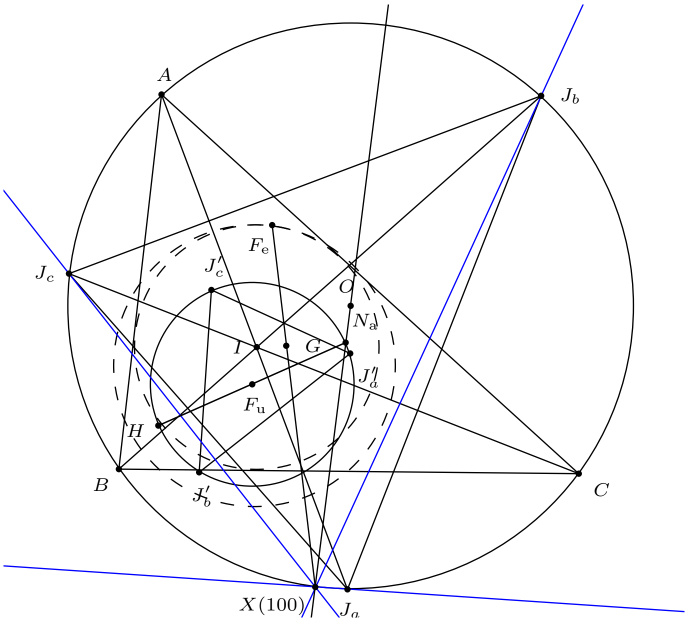

Figure 10

(b) The perpendiculars from J_a to $J'_bJ'_c$, J_b to $J'_cJ'_a$, and J_c to $J'_aJ'_b$ are the lines

$$\begin{aligned}
-(b+c)(b-c)x+ \quad a(c-a)y+ \quad a(a-b)z &= 0,\\
b(b-c)x-(c+a)(c-a)y+ \quad b(a-b)z &= 0,\\
c(b-c)x+ \quad c(c-a)y-(a+b)(a-b)z &= 0.
\end{aligned}$$

These three lines are concurrent at the point $(x : y : z)$, where

$$\begin{aligned}&(b-c)x : (c-a)y : (a-b)z\\ =&\begin{vmatrix}-(c+a) & b\\ c & -(a+b)\end{vmatrix} : -\begin{vmatrix}b & b\\ c & -(a+b)\end{vmatrix} : \begin{vmatrix}b & -(c+a)\\ c & c\end{vmatrix}\\ =&a(a+b+c) : b(a+b+c) : c(a+b+c).\end{aligned}$$

Therefore, $(x : y : z) = \left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$. This is the triangle center $X(100)$ in [3]. It is the superior of the Feuerbach point. $\square$

Remarks. (1) The Nagel point N_{a} is the triangle center $X(74)$ of the Fuhrmann triangle.

(2) The superior of the Feuerbach point is the triangle center $X(100)$. It is $X(74)$ of $J_aJ_bJ_c$.

Theorem 13. *The seven circles $IJ_aJ'_a$, $IJ_bJ'_b$, $IJ_cJ'_c$, $AJ'_bJ'_c$, $BJ'_cJ'_a$, $CJ'_aJ'_b$ and the circumcircle (O) have a common point K. Moreover, let J be the intersection point of the ray HF_{e} and the circumcircle (O) and I be the incenter of ABC. Then, K, I, J are collinear.*

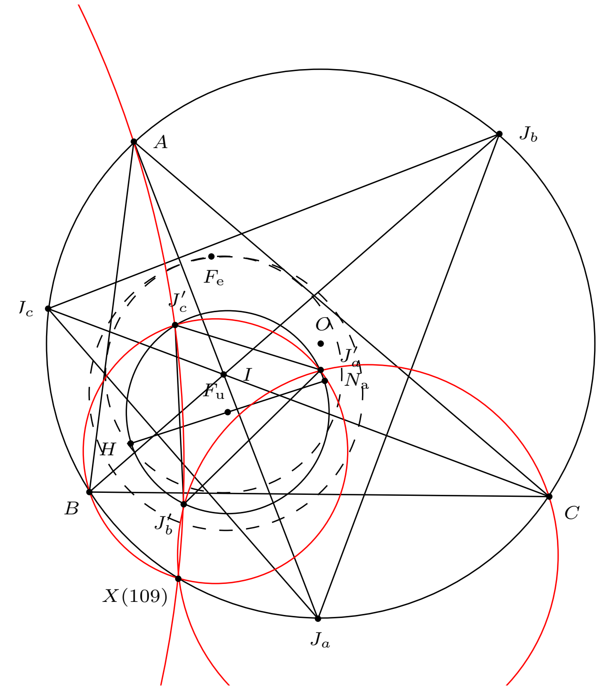

Figure 11

Proof. The equation of the circle $AJ'_bJ'_c$ is

$$(b^2 - bc + c^2 - a^2)(a^2yz + b^2zx + c^2xy) - (x + y + z)f(x, y, z) = 0,$$

where

$$f(x, y, z) = c^2(c - a)(c + a - b)y - b^2(a - b)(a + b - c)z.$$

This contains the point

$$X(109) = \left(\frac{a^2}{(b - c)(b + c - a)} : \frac{a^2}{(b - c)(b + c - a)} : \frac{a^2}{(b - c)(b + c - a)} \right)$$

on the circumcircle since

$$\begin{aligned} &f\left(\frac{a^2}{(b - c)(b + c - a)}, \frac{a^2}{(b - c)(b + c - a)}, \frac{a^2}{(b - c)(b + c - a)} \right) \\ &= c^2b^2 - b^2c^2 = 0. \end{aligned}$$

Therefore, the circle $AJ'_bJ'_c$ contains $X(109)$. Similarly, the circles $BJ'_cJ'_a$ and $CJ'_aJ'_b$ also contain the same point.

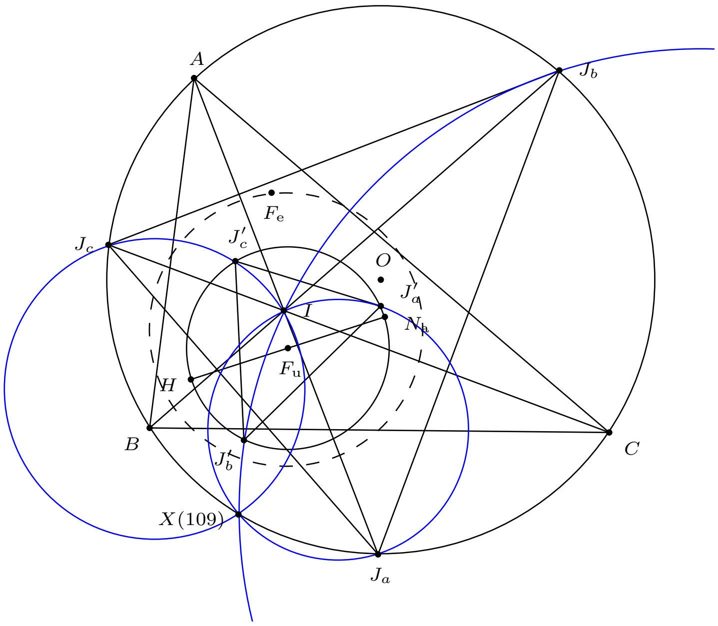

Figure 12

The equation of the circle $IJ_aJ'_b$ is

$$(b - c)(a + b + c)(b + c - a)(a^2yz + b^2zx + c^2xy) - (x + y + z)g(x, y, z) = 0,$$

where

$$g(x, y, z) = bc(b-c)(b+c)(b+c-a)x - a^2c(c-a)(c+a-b)y - a^2b(a-b)(a+b-c)z.$$

This contains the point $X(109)$ with coordinates given above since

$$g\left(\frac{a^2}{(b-c)(b+c-a)},\frac{a^2}{(b-c)(b+c-a)},\frac{a^2}{(b-c)(b+c-a)}\right)$$
$$=a^2bc(b+c)-a^2b^2c-a^2bc^2=0.$$

Similarly, the circles IJ_bJ_b' and IJ_cJ_c' contain the same point.

□

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