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Similarities on a Sphere

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Abstract. We define dilations and similarities on a sphere with the use of quaternions. As an application, we produce a simple algorithm for the calculation of the angles of a polyhedron.

1. Introduction

The orientation preserving similarities of the usual plane are well known: They are composition of translations, rotations and dilations. We can define as well rotations and dilations on a sphere in a simple manner by the use of quaternions. The big advantage of quaternions is to facilitate the calculation of the composition $R_2 \circ R_1$ of two rotations in the usual 3D-space ([2]), and to avoid singularities like the poles.

We begin by looking for the local similarities on the unit sphere S^2 of the usual 3D-real euclidean space $\mathbb{R}^3$. We say that f is a local similarity of S^2 if there exist neighbourhoods V_Ω, $V_{\Omega'}$ of points Ω, Ω' such that $f : V_\Omega \to V_{\Omega'}$ preserves the distance ratios. Recall that the exponential map $exp_\Omega : m \in T_\Omega \mapsto M \in V_\Omega$ sends points m in a neighbourhood of Ω in the tangent plane T_Ω to points M in the unit sphere. Geometrically, it corresponds to laying off a length equal to Ωm along the geodesic that passes through m in the direction of $\overrightarrow{\Omega m}$. So we can define the local induced map $\tilde{f} : T_\Omega \to T_{\Omega'}$ by $\tilde{f} = exp_{\Omega'}^{-1} \circ f \circ exp_\Omega$. Indeed, it is a local plane similarity because distance ratios and angles from Ω are preserved. If we specialize to positive similarities with a fixed point Ω, we then obtain that $f = exp_\Omega \circ \tilde{f} \circ exp_\Omega^{-1}$ is a local rotation or dilation of center Ω (or their composition), which we want now to describe in terms of quaternions.

2. Local similarities in terms of quaternions

We first recall some basic facts and notations about the space $\mathbb{H}$ of quaternions. We will denote by $\mathbf{q} = a + q$ or $\mathbf{q} = (a, q)$ a quaternion of scalar part (or real part) a and vector part (or imaginary part) q, with $q = bi + cj + dk$, $(b, c, d) \in \mathbb{R}^3$. $\mathbb{H}$ is then an associative, anticommutative real algebra of dimension 4 with respect to the usual addition and the multiplication rule:

$$ii = jj = kk = -1,\ ij = k,\ jk = i,\ ki = j.$$

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We also identify the number q with the vector (b, c, d) of $\mathbb{R}^3$. The conjugate $\mathbf{q}^*$ of $\mathbf{q}$ is defined by $\mathbf{q}^* = a - q$, in such a way that $\mathbf{q}\mathbf{q}^* = \mathbf{q}^*\mathbf{q} = a^2 + b^2 + c^2 + d^2$ is the usual euclidean norm of $\mathbb{R}^4$. This leads us to define the set of unit quaternions $\mathbb{H}(\|1\|)$. In fact, we can introduce a scalar product on $\mathbb{H}$ by $\mathbf{q} \cdot \mathbf{q}' = aa' + bb' + cc' + dd'$ and $\mathbf{q}\mathbf{q}^*$ is nothing but $\mathbf{q} \cdot \mathbf{q} = \|\mathbf{q}\|^2$. On the second hand, each unit quaternion $\mathbf{q}$ can be rewritten as $\cos(\varphi) + \sin(\varphi)u$ where u is a pure imaginary and unit quaternion. Finally, we can represent rotations $R_{\vec{u},\theta}$ with axis $\mathbb{R}\vec{u}$ ($\vec{u}$ unitary) and angle θ by the quaternion $\mathbf{q} = \cos(\theta/2) + \sin(\theta/2)u$ in the following manner: $R_{\vec{u},\theta}(\vec{v}) = \mathbf{q}v\mathbf{q}^*$. Moreover, the mapping $\mathbf{q} \in \mathbb{H}(\|1\|) \mapsto R \in SO(3)$ is a group homomorphism for multiplication and composition, that is the composition of two rotations is represented by the product of two unit quaternions.

Next, let us look at spherical dilation. We fix a point $\Omega \in S^2$ and a real positive number λ. The dilation $H_{\Omega,\lambda}$ of center Ω and ratio λ is a map from the spherical cap $V_{\Omega,\pi/\lambda} = \left\{M \in S^2 :\ \angle O\Omega M < \pi/\lambda\right\}$ to S^2. By noting M' the image of M and $\mathscr{C}_{\Omega,M}$ the meridian of origin Ω passing through M, it is defined by: $M' \in \mathscr{C}_{\Omega,M}$ and $\widetilde{\Omega M'} = \lambda \cdot \widetilde{\Omega M}$. This motion may be described by quaternion interpolation ([3]) in the following way. Let $\mathbf{q_0} = q_0$ and $\mathbf{q} = q$ be the unit quaternions (and purely imaginary) representing the vectors $\overrightarrow{O\Omega}$ and $\overrightarrow{OM}$. Then the quaternion (where θ is the angle between q_0 and q)

$$\mathbf{q}'(\lambda) = q'(\lambda) = \frac{q_0 \cdot \sin(1-\lambda)\theta + q \cdot \sin(\lambda\theta)}{\sin(\theta)}$$

is exactly the quaternion representing M'. It is also denoted by $Slerp(\lambda; q_0, q)$ and it is known as "Spherical Linear Interpolation". Let us summarize the previous results.

Theorem 1. *The local positive similarities of the unit sphere S^2 with a fixed point Ω are the rotations $R_{\overrightarrow{O\Omega},\delta}$, the dilations $H_{\Omega,\lambda}$ of center Ω, and their compositions. They can be algebraically represented by quaternions as:*

- $R_{\overrightarrow{O\Omega},\delta} : v \mapsto qvq^*$ *where* $q = \cos\dfrac{\delta}{2} + \sin\dfrac{\delta}{2} \cdot \omega$,
- $H_{\Omega,\lambda} : v \mapsto Slerp(\lambda; \omega, v) = \dfrac{\omega \cdot \sin(1-\lambda)\theta + v \cdot \sin(\lambda\theta)}{\sin(\theta)}$,

where ω is the quaternion representing Ω and $\theta = \angle(\widehat{\omega, v})$.

3. Application to polyhedra

We apply here the above results to compute some angles of a given polyhedral angle. As in our technical report ([4]), we are interested in the calculation of the last three angles of such a polyhedra. That is, let $\mathscr{P}$ be a polyhedral angle of vertex p and degree n, and $F_1, ..., F_n$ its faces. It intersects the unit sphere of center p in a spherical polygon $(q_0, q_1, ..., q_{n-1}, q_n = q_0)$. If we fix the internal angles $\alpha_1 = \angle q_0 p q_1, \ldots, \alpha_{n-1} = \angle q_{n-2} p q_{n-1}$ of the faces $F_1, ..., F_{n-1}$ and the external (dihedral) angles $\delta_1 = \angle(F_1, F_2), \ldots, \delta_{n-2} = \angle(F_{n-2}, F_{n-1})$, then the polyhedron is entirely known up to a motion (see [1] for rigidity-type results). Thus we may compute the last three angles $\alpha_n = \angle q_{n-1} p q_n$, $\delta_{n-1} = \angle(F_{n-1}, F_n)$ and

$\delta_n = \angle(F_n, F_1)$. The angles α_i are measured in $]0, \pi[$ and are equal to the lengths of the geodesic arcs $\widetilde{q_{i-1}q_i}$, whereas the angles δ_i are measured in $]0, 2\pi[$ and are equal to the angles $(\widetilde{q_iq_{i-1}}, \widetilde{q_iq_{i+1}})$.

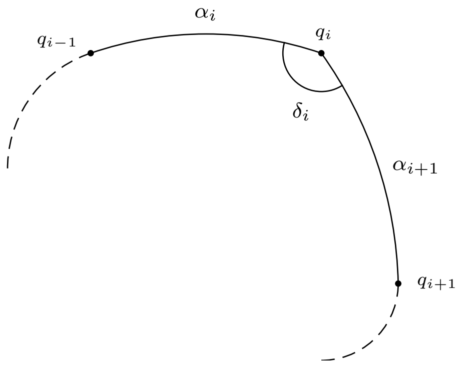

Figure 1. Polygonal view of $\mathscr{P}$ on S^2

If we associate with each point $q_i = (b, c, d)$ the quaternion $\mathbf{q_i} = q_i = bi + cj + dk$, then we have the simple following relationship between q_{i-1}, q_i and q_{i+1}:

$$\begin{aligned} q_{i+1} &= Slerp(\frac{\alpha_{i+1}}{\alpha_i}; q_i, Q_i q_{i-1} Q_i^*) \\ &= \frac{\sin(\alpha_i - \alpha_{i+1}) q_i + \sin(\alpha_{i+1}) Q_i q_{i-1} Q_i^*}{\sin(\alpha_i)} \end{aligned} \tag{1}$$

with $Q_i = \cos(\frac{\delta_i}{2}) + \sin(\frac{\delta_i}{2}) q_i$. For this calculation, we also need the value of θ which is here $\arccos q_i \cdot (Q_i q_{i-1} Q_i^*)$ (because $Slerp$ is independent of the sign of θ). For the algorithm, we fix the values $q_0 = i, q_1 = \cos(\alpha_1) i + \sin(\alpha_1) j$, and we compute q_{n-1} step by step by using the equation (1) for $i = 1$ to $n - 2$.

Now we can compute $\alpha_n = \arccos(\mathbf{q_{n-1}} . \mathbf{q_n}) = \arccos \dfrac{q_{n-1} q_0^* + q_0 q_{n-1}^*}{2}$.

For the last two dihedral angles, we will use the following lemma.

Lemma 2. *Let u, v be two unit vectorial quaternions, $q = \cos(\delta/2) + \sin(\delta/2) u$ and $Q = \cos(\delta) + \sin(\delta) u$. Then $Q = \dfrac{(u \cdot v) u \wedge v + (qvq^*)((u \cdot v) u - v)}{1 - (u \cdot v)^2}$.*

Proof. Recall that if $q = \cos(\delta/2) + \sin(\delta/2) u$ is a unit quaternion and v a vectorial quaternion, then

$$qvq^* = \cos(\delta) v + (1 - \cos(\delta))(u.v) u + \sin(\delta) . u \wedge v \tag{2}$$

and that the product of two vectorial quaternions q_1 and q_2 is

$$q_1 q_2 = -(q_1 . q_2) + q_1 \wedge q_2.$$

So, if v is unitary, then

$$(qvq^*) v = -\cos(\delta) + (1 - \cos(\delta))(u \cdot v)(-(u \cdot v) + u \wedge v) + \sin(\delta)(u \wedge v) \wedge v.$$

By the vector triple product formula $(u \wedge v) \wedge w = (u.w)v - (v.w)u$, this leads to

$$\begin{aligned}(qvq^*)v = & -(\cos(\delta) + \sin(\delta)u) - (u \cdot v)^2(1 - \cos(\delta)) + \sin(\delta)(u \cdot v)v \\ & + (1 - \cos(\delta))(u \cdot v)u \wedge v.\end{aligned}$$

We have similarly

$$(qvq^*)u = -(u\cdot v)(\cos(\delta)+\sin(\delta)u)-(u\cdot v)(1-\cos(\delta))+\sin(\delta)v-\cos(\delta)u\wedge v.$$

Thus, $(qvq^*)(v - (u \cdot v)u) = Q((u \cdot v)^2 - 1) + (u.v)u \wedge v$ and this proves the lemma. □

Now, formula (1) tells us that

$$Q_i q_{i-1} Q_i^* = \frac{\sin(\alpha_i)q_{i+1} - \sin(\alpha_i - \alpha_{i+1})q_i}{\sin(\alpha_{i+1})}. \tag{3}$$

Thus, we have by Lemma 2

$$\cos(\delta_i) + \sin(\delta_i)q_i = \frac{\cos(\alpha_{i+1})(\cos(\alpha_i) + q_i q_{i-1}) + q_{i+1}(\cos(\alpha_i)q_i - q_{i-1})}{\sin(\alpha_{i+1})\sin(\alpha_i)}$$

giving us the values of δ_{n-1} and δ_n by identifying the real and imaginary parts.

Theorem 3. *The dihedral angles δ_{n-1} and δ_n are given by*

$$\begin{aligned}&\cos(\delta_{n-1}) + \sin(\delta_{n-1})q_{n-1} \\ &= \frac{\cos(\alpha_n)(\cos(\alpha_{n-1}) + q_{n-1}q_{n-2}) + q_0(\cos(\alpha_{n-1})q_{n-1} - q_{n-2})}{\sin(\alpha_n)\sin(\alpha_{n-1})}, \\ &\cos(\delta_n) + \sin(\delta_n)q_0 \\ &= \frac{\cos(\alpha_1)(\cos(\alpha_n) + q_0 q_{n-1}) + q_1(\cos(\alpha_n)q_0 - q_{n-1})}{\sin(\alpha_1)\sin(\alpha_n)}.\end{aligned}$$

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