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A Characterization of the Rhombus

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Abstract. In this article we discuss a simple characterization of the rhombus by considering the triangles formed by a point and each of the sides of the rhombus.

1. The characteristic property

A rhombus $AB\Gamma\Delta$ has a symmetry center coinciding with the intersection point O of its diagonals, defining four triangles $\{OAB, OB\Gamma, O\Gamma\Delta, O\Delta A\}$, which are congruent. (see Figure 1). One can inversely ask, if there is another kind of quadri-

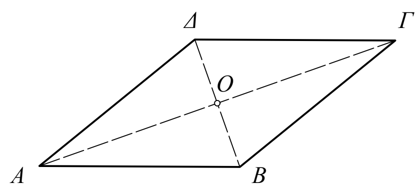

Figure 1. Four congruent triangles

lateral with the same property. The answer is no, and this is the subject of the following theorem.

Theorem 1. *A quadrilateral $AB\Gamma\Delta$ is a rhombus, if and only if, there is a point O in its plane, such that the four triangles*

$$OAB,\ OB\Gamma,\ O\Gamma\Delta,\ O\Delta A \quad \textit{are congruent}. \tag{*}$$

The formulation of the theorem is quite general and refers to arbitrary quadrilaterals, convex, non-convex, self-intersecting, but, nevertheless, non-degenerate, i.e. having no three vertices collinear. The proof is amusing, since it scarcely needs something more in background than pure logic. It is however not totally trivial, its tricky part being the arrangement of the angles around the point O.

The key-idea arises then naturally and consists in the study of the possible configurations of two "adjacent" triangles, which share a common side, like for example $\{OAB, OB\Gamma\}$.

The necessity part of the condition $(^*)$ being trivial, the following sections supply the details of the proof for the sufficiency part. Below we often refer to this condition as the "fundamental assumption".

The figures that follow seem, some times, to deviate from a correct graphical representation of this assumption. This is though necessary and reflects the incompatibility of the assumption with some other additional assumptions made in each case.

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2. Congruent triangles with a common side

The following lemma is trivial, and its proof can be read from Figure 2. In this

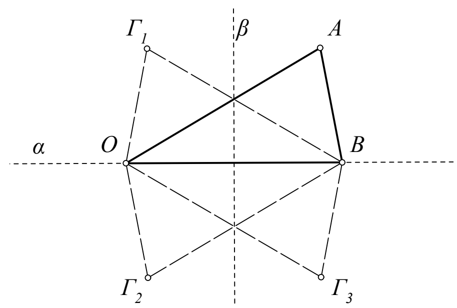

Figure 2. Congruent triangles with a common side

are seen also the symmetry axes of the configuration, which are α : the line OB, and β : the medial line of OB.

Lemma 2. *Given the triangle OAB, there are three other, congruent to it, triangles sharing with it the side OB.*

Returning to the initial problem, we can imagine, that if a point O, satisfying the fundamental assumption $(^*)$ exists, and we fix the triangle OAB, then the vertex Γ of the quadrilateral in question must have one of the positions $\{\Gamma_1, \Gamma_2, \Gamma_3\}$. The proof then, results naturally by repeating this simple construction for each one of the possibilities

$$O\Gamma_1,\ O\Gamma_2,\ O\Gamma_3, \tag{**}$$

and considering analogously the three possible places for the vertex Δ. It turns out, that only $O\Gamma_3$ and one, out of the three, possibilities for it, is compatible with $(^*)$ and leads to the rhombus.

In this section we adopt for Γ the position Γ_1 of the previous figure and examine the three resulting possibilities for the Δ's as shown in the figure 3-I. The first case,

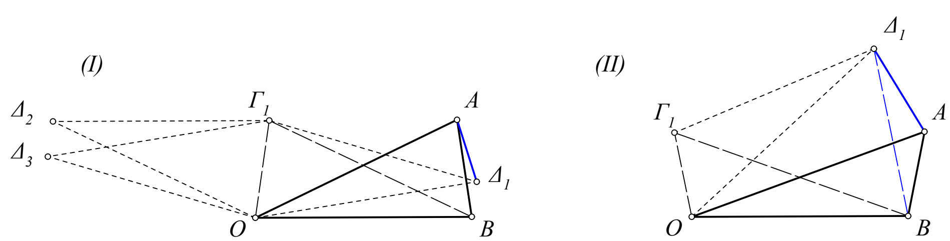

Figure 3. $\{\Delta_1, \Delta_2, \Delta_3\}$ for Γ_1 ... and incompatible $\{\Gamma_1, \Delta_1\}$

seen in this figure, resulting by selecting $\{\Gamma_1, \Delta_1\}$, leads to an incompatibility with $(^*)$.

In fact, from the assumed congruence of triangles $\{OAB, O\Delta_1 A\}$, follows the equality of the angles $\widehat{OA\Delta_1} = \widehat{OAB}$ and $\widehat{AOB} = \widehat{AO\Delta_1}$. Hence, either the two triangles coincide, which is not acceptable, or they are symmetric with respect to AO. In the later case line $B\Delta_1$, which is parallel to $O\Gamma_1$, is orthogonal to OA. In addition, from the fundamental assumption, follows the equality of the angles $\widehat{\Delta_1 OA} = \widehat{O\Delta_1\Gamma_1}$. This implies, that the quadrilateral $OA\Delta_1\Gamma_1$ must be an isosceles trapezium with a right angle at O (see Figure 3-II), hence a rectangle. This implies, in turn, that $\{B, A, \Delta_1\}$ are collinear, which is not acceptable.

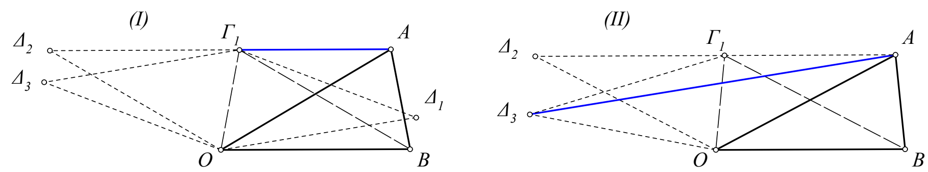

Figure 4. Incompatible $\{\Gamma_1, \Delta_2\}$ ... and incompatible $\{\Gamma_1, \Delta_3\}$

The position Δ_2 for Δ leads directly to impossibility, since in that case, as is easily seen, the points $\{A, \Gamma_1, \Delta_2\}$ are collinear (see Figure 4-I), which is not acceptable.

Finally, the position Δ_3 leads also to incompatibility (see Figure 4-II). In fact, in this case the triangles $\{OAB, O\Delta_3 A, O\Delta_3\Gamma_1\}$, supposed to be congruent, must have also equal the angles opposite to equal sides, and this implies the equality of the angles $\{\widehat{OA\Delta_3}, \widehat{O\Gamma_1\Delta_3}\}$. However, since $\{A, \Gamma_1, \Delta_2\}$ are collinear and $O\Gamma_1\Delta_2\Delta_3$ is a cyclic quadrilateral, this implies that A must coincide with either Γ_1 or Δ_2. The coincidence of A with Γ_1 is not acceptable, and the coincidence of A

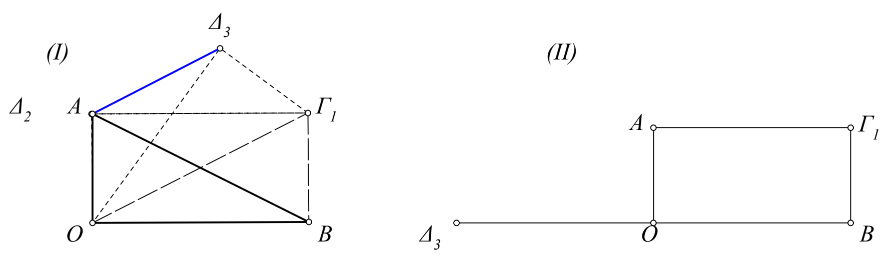

Figure 5. Case with coincident $\{A, \Delta_2\}$

with Δ_2 leads to a parallelogram $OB\Gamma_1 A$ with equal diagonals, hence a rectangle (see Figure 5-I). This implies that $AO\Delta_3$ is a right angle, hence either Δ_3 coincides with B, which is not acceptable, or it is symmetric to B with respect to AO, which is incompatible with $(*)$ (see Figure 5-II).

The discussion made in this section shows that the position $\Gamma = \Gamma_1$ leads, in all cases, to incompatibilities with the fundamental assumption. The next sections handle analogously the cases for Γ_2 and Γ_3.

3. $\varGamma_2$: three incompatibilities

Taking $\varGamma$ to be at the position $\varGamma_2$ of the initial Figure 2, the possible positions for $\varDelta$ under the assumption $(^*)$ are seen in Figure 6. The cases of $\{\varDelta_1, \varDelta_2\}$ are

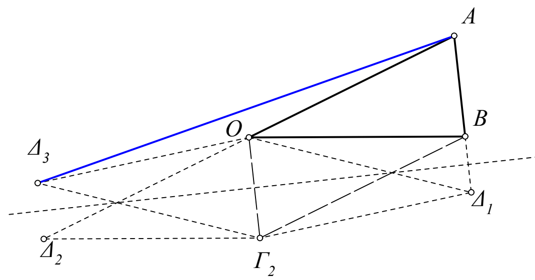

Figure 6. All $AB\varGamma_2\varDelta_i$ incompatible

excluded immediately, since in the first $\{A, B, \varDelta_1\}$ are collinear and in the second $\{A, O, \varDelta_2\}$ are collinear and the triangle $AO\varDelta_2$ is degenerate. The third case $\varDelta = \varDelta_3$ is also excluded, since then, under the fundamental assumption, the angles $\widehat{\varDelta_3 AO} = \widehat{OAB}$, and the congruency of triangles $\{\varDelta_3 AO, BAO\}$ imply that A lies on the medial line of the segment $B\varDelta_3$, which is line $O\varGamma_2$. Hence $\{O, \varGamma_2, A\}$ must be collinear, which is not possible, since $\{AB, O\varGamma_2\}$ are parallel in this case.

4. $\varGamma_3$: one compatibility

The last possibility is to choose for $\varGamma$ the place $\varGamma_3$ in Figure-2. Then the possible places for the vertex $\varDelta$ are shown in Figure 7-I. The first case with $\varDelta = \varDelta_1$

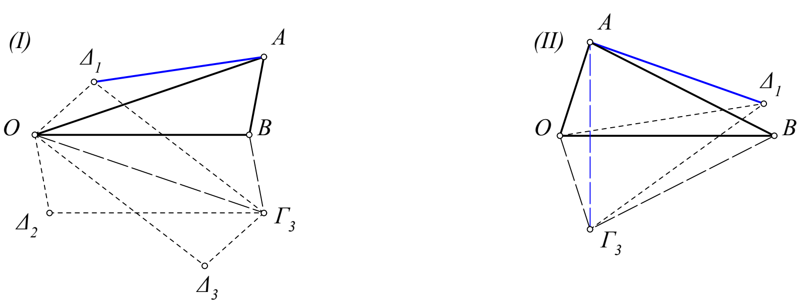

Figure 7. Case $AB\varGamma_3\varDelta_1$ incompatible

leads to incompatibility. In fact, in this case the assumed congruent triangles $\{OA\varDelta_1, O\varDelta_1\varGamma_3\}$ imply the equality of angles $\widehat{O\varDelta_1 A} = \widehat{O\varDelta_1\varGamma_3}$. This implies that A is either coincident with $\varGamma_3$, which is incompatible, or it is the symmetric of $\varGamma_3$ with respect to $O\varDelta_1$. In the later case $\varDelta_1$ must lie on the medial line of $A\varGamma_3$ which is OB. Since it lies, in this case, also on the parallel $B\varDelta_1$ to $O\varGamma_1$, this is impossible.

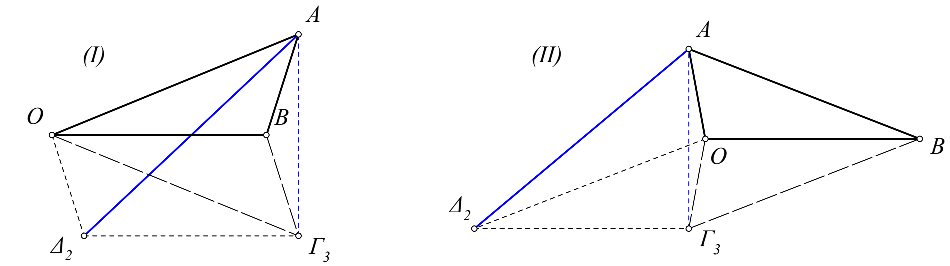

Figure 8. Case $AB\Gamma_3\Delta_2$ incompatible

In the case of $\Delta = \Delta_2$ of Figure 8-I, the triangles $\{OA\Delta_2, O\Delta_2\Gamma_3\}$ assumed congruent, would imply the equality of angles $\widehat{O\Delta_2\Gamma_3} = \widehat{O\Delta_2 A}$. This would imply, either coincidence of $\{A, \Gamma_3\}$, which is not acceptable, or coincidence of A with the symmetric of Γ_3 with respect to $O\Delta_2$. In this case it is readily seen, that, under the fundamental assumption, Δ_2 lies on the medial line of $A\Gamma_3$ and points $\{\Delta_2, O, B\}$ must be collinear (see Figure 8-II), which is not possible, since $\{\Delta_2\Gamma_3, OB\}$ are parallel in this case.

The only acceptable configuration results for $\Delta = \Delta_3$, which is seen in Figure 9-I. The requirement of congruency of triangles $\{OAB, OA\Delta_3\}$ implies that point

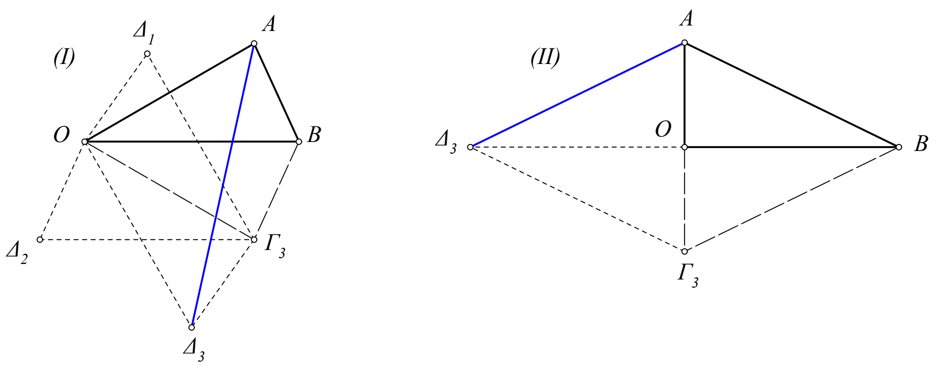

Figure 9. The only compatible case $AB\Gamma_3\Delta_3$

Δ_3, either coincides with B, which is not acceptable, or it is the symmetric of B with respect to AO. Since Δ_3 is also the symmetric of B with respect to $O\Gamma_3$, the two symmetrics coincide when $\{A, O, \Gamma_3\}$ are collinear and OB is orthogonal to AO (see Figure 9-II). This last case produces the rhombus $AB\Gamma_3\Delta_3$ and completes the proof of the sufficiency part of the theorem.

Remark. It is trivial to see that, for triangles, the only one species with an analogous to the previous property, i.e. triangles $AB\Gamma$, for which there is a point O in their plane, such that the triangles $\{OAB, OB\Gamma, O\Gamma A\}$ are congruent, are the equilaterals. For general polygons, however, I don't know alternative characterizations for the analogously defined category. Certainly, one can construct examples by gluing together copies of the same triangle, or rotating a triangle about a ver-

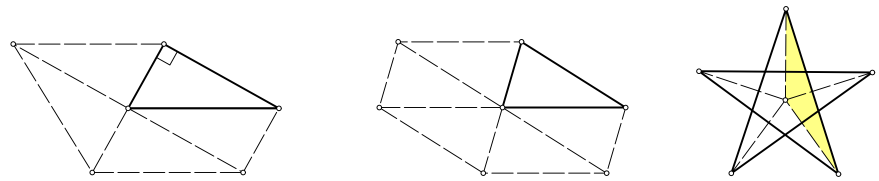

Figure 10. Polygons with the analogous property

tex, as seen in Figure 10. The knowledge, though, of a simple geometric property, giving another aspect of this class of polygons, seems to be missing.

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