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The 19 Congruent Jacobi Triangles

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Abstract. With a given triangle and three angles, Jacobi's theorem involves the construction of a new triangle. An investigation into the case when these two triangles are congruent is presented. It is claimed that there are always nineteen possibilities and that they fall into four distinct classes.

Jacobi Triangles.

With ABC being any triangle, construct the points P, Q, R so that

$$\angle RAB = \angle QAC = \alpha,\ \angle PBC = \angle RBA = \beta \text{ and } \angle QCA = \angle PCB = \gamma.$$

These points form a *Jacobi triangle* for ABC and Jacobi's theorem states that the lines $AP,\ BQ$ and CR are concurrent (at the Jacobi point K), see Figure 1. Proofs of this result are readily available e.g. [1, pp.55-56] and [2]. Let Δ be the area and a, b, c be the lengths of the sides of ABC. With

$$X = 2\Delta(\cot\alpha + \cot A),\ Y = 2\Delta(\cot\beta + \cot B),\ Z = 2\Delta(\cot\gamma + \cot C),$$

the coordinates of the key points are (using areal coordinates based upon ABC)

$$P(-a^2, Z, Y),\ Q(Z, -b^2, X),\ R(Y, X, -c^2),\ K(1/X, 1/Y, 1/Z).$$

This note gives details for the cases when the triangles ABC and PQR are congruent. It is straightforward to write down the necessary conditions for this to occur and numerical experiments suggest that there are always nineteen real solutions (excluding degenerate cases with at least one of α, β, γ being zero). Even for an equilateral triangle, they are distinct. They fall into four distinct groups as listed below. In the figures, the triangle ABC is the one with dashed lines and its circumcircle and circumcenter are shown. The position of K is indicated by a cross.

The properties of Classes 2 and 3 were first noticed from the figures and then verified algebraically with the aid of Maple. Thus informative proofs are not available.

Class 1. This has a unique member given by

$$\alpha = \pi/2 - A,\ \beta = \pi/2 - B,\ \gamma = \pi/2 - C.$$

The points P, Q, R are diametrically opposite A, B, C on the circumcircle of ABC so that PQR corresponds to a rotation of ABC by π. Also K is the center of the circumcircle.

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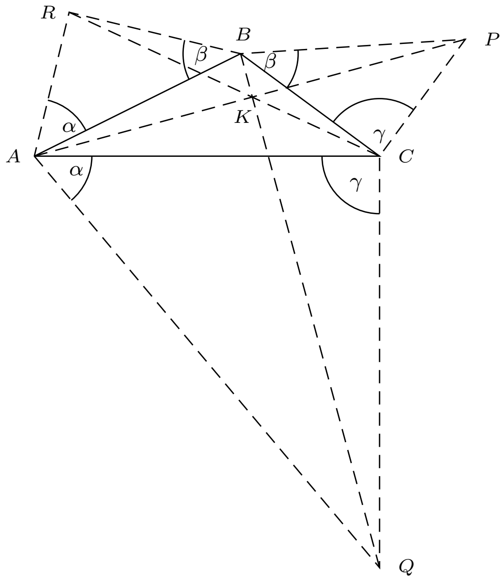

Figure 1. The points P, Q, R are constructed on a base triangle ABC with pairs of angles equal as shown. Jacobi's theorem states that AP, BQ, CR are concurrent and K will be used for the common point.

Class 2. The six members of this group satisfy

$$a^2X + b^2Y + c^2Z = 0$$

which implies that K lies on the circumcircle of ABC. This is illustrated in Figure 2. It will be seen that the six triangles form two groups of three, the orientations within a group being the same. Furthermore they correspond to rotations of $\pm 2\pi/3$

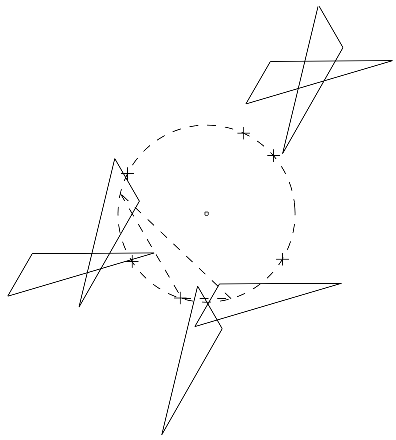

Figure 2. Class 2: The Jacobi points lie on the circumcircle of ABC.

of ABC. No simple formula for the values of X, Y, Z has been found. However, for an isosceles triangle ($b = a$) solutions are

$$X, Y = c^2u \pm [(a^2 + 2a^2u - c^2u)(3a^2 - 2a^2u - c^2u)]^{1/2}, \ Z = -2a^2u$$

where

$$(-16a^4 - 16a^2c^2 + 8c^4)u^3 + (8a^4 - 4a^2c^2 - 4c^4)u^2 + (20a^4 - 4a^2c^2 + 2c^4)u + (6a^4 - 3a^2c^2) = 0.$$

Class 3. The six members of this group satisfy

$$X + Y + Z = (a^2 + b^2 + c^2)/2 \Rightarrow \cot\alpha + \cot\beta + \cot\gamma = 0. \tag{1}$$

However, they form two disparate groups, each with three members.

Class 3A. Here the circumcircles of PQR and ABC touch, see Figure 3. The orientation of the Jacobi triangles involves a reflection as well as a rotation. The analytic solution for this case also appears to be unwieldy but when $b = a$ it may be written as

$$X, Y = \frac{-c^2 \pm 3c\sqrt{c^2 + 8a^2}}{4}, \ Z = a^2 + c^2$$

and, as the third root,

$$X = Y = 2a^2, \ Z = c^2/2 - 3a^2.$$

The coordinates of the three points of contact are

$$(c \pm \sqrt{c^2 + 8a^2}, \ c \mp \sqrt{c^2 + 8a^2}, \ 4c) \ \text{ and } \ (2a^2, 2a^2, -c^2).$$

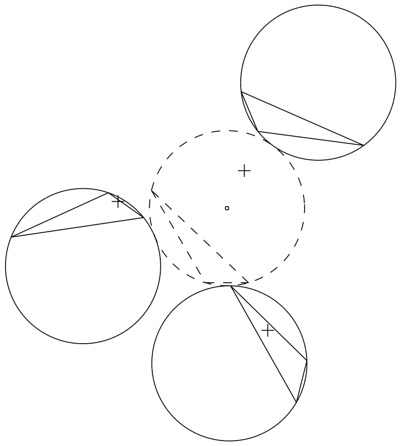

Figure 3. Class 3A: The circumcircle of PQR touches that of ABC.

Class 3B. The members of this sub-class satisfy the additional condition

$$X \cot \alpha = Y \cot \beta = Z \cot \gamma.$$

This condition together with (1) implies that

$$1/X + 1/Y + 1/Z = 0$$

and so K is at infinity. Indeed PQR has the same orientation as ABC, (see Figure 4).

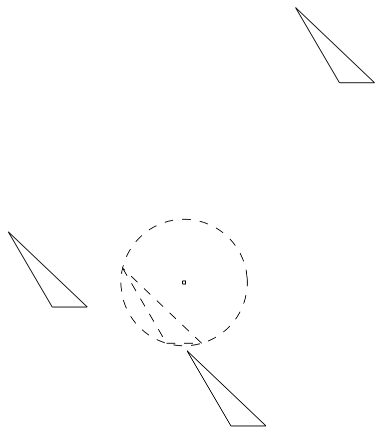

Figure 4. Class 3B: The orientations of the Jacobi triangles are the same as that of ABC. The Jacobi points are at infinity.

These conditions also imply that the circumcircles of AQR, PBR and PQC all touch that of ABC and this is shown in Figure 5. For this case the analytic solution for the general case is reasonably concise; the three values of X satisfy the equation

$$6X^3 + (a^2 - 5b^2 - 5c^2)X^2 + (b^4 + c^4 - a^2b^2 - a^2c^2)X + b^2c^2(a^2 + b^2 + c^2) = 0.$$

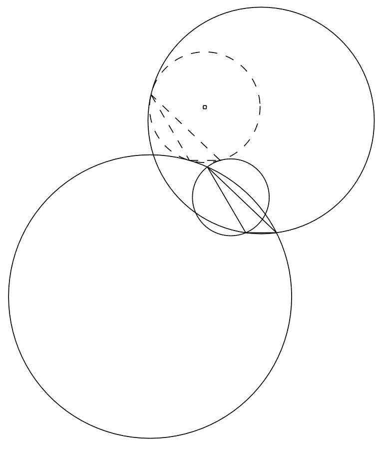

Figure 5. Class 3B: This shows that the circumcircles of AQR, PBR, PQC touch that of ABC.

Class 4. The six members of this group satisfy both

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$$

and

$$\frac{\sin(A+2\alpha)}{\sin A} = \frac{\sin(B+2\beta)}{\sin B} = \frac{\sin(C+2\gamma)}{\sin C}.$$

These conditions were encountered in [2] and provided the instigation of this note. They correspond to the reciprocal triangles that occur when K is at infinity so that

$$1/X + 1/Y + 1/Z = 0.$$

There are three pairs of solutions one of which is as follows:

$$\begin{aligned}
\cot 2\alpha &= \frac{1}{2\sin A\cos\frac{1}{3}(B-C)} - \cot A, \\
\cot 2\beta &= \frac{-1}{2\sin B\cos\frac{1}{3}(B+2C)} - \cot B, \\
\cot 2\gamma &= \frac{-1}{2\sin C\cos\frac{1}{3}(2B+C)} - \cot C
\end{aligned}$$

and as subsidiary relations,

$$A = -2\alpha + \beta + \gamma + \pi,\ B = \alpha - 2\beta + \gamma,\ C = \alpha + \beta - 2\gamma.$$

These determine two sets of values for α, β, γ and the two triangles have the same orientation, see Figure 6.

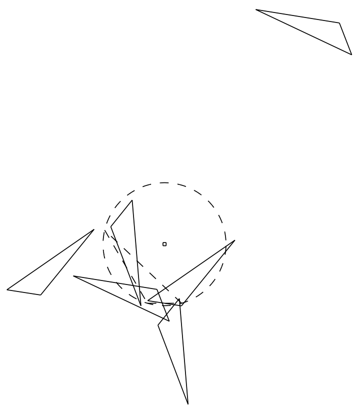

Figure 6. Class 4: There are now three similarly oriented pairs, rotated by $2\pi/3$ from one another. The Jacobi points are at infinity.

Furthermore, the three orientations (i.e. one for each pair) are rotated by $2\pi/3$ with respect to each other. The circumcircles of AQR, PBR, PQC, PBC, AQC, ABR all have the same radius, (see Figure 7), the value being the same for each pair.

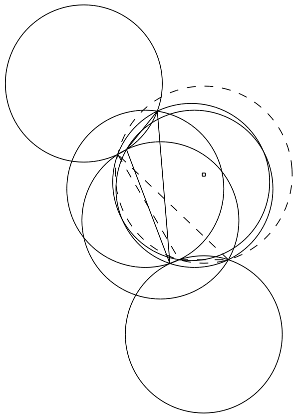

Figure 7. Class 4: To show that the circumradii of AQR, PBR, PQC, PBC, AQC, ABR are equal.

It may be shown that these solutions satisfy not only

$$\frac{1}{\cot A+\cot\alpha}+\frac{1}{\cot B+\cot\beta}+\frac{1}{\cot C+\cot\gamma}=0$$

(which is equivalent to $1/X+1/Y+1/Z=0$) but also

$$\frac{1}{\cot A+\cot 2\alpha}+\frac{1}{\cot B+\cot 2\beta}+\frac{1}{\cot C+\cot 2\gamma}=0.$$

References

[1] P. Baptist, *Die Entwicklung der Neueren Dreiecksgeometrie*, Wissenschaftsverlag, Mannheim/ Leipzig/Wein/Zurich, 1992.

[2] G. T. Vickers, Reciprocal Jacobi triangles and the McCay cubic, *Forum Geom.*, 15 (2015) 179–183.

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