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Distances Among the Feuerbach Points

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Abstract. We find simple formulas for the distances from the Feuerbach points of a triangle to the vertices, and among themselves.

Consider a triangle ABC with the midpoints X, Y, Z of its sides BC, CA, AB respectively, circumcenter O, the incenter I, and the excenters I_a, I_b, I_c. The radii of the circumcircle, incircle, and excircles are denoted by R, r, r_a, r_b, r_c respectively. The nine-point circle is the circle through X, Y, Z; it has center N and radius $\frac{R}{2}$. By the famous Feuerbach theorem, the nine-point circle is tangent to the incircle and each of the excircles. The points of tangency are the Feuerbach points, F_e with the incircle, and F_a, F_b, F_c with the excircles (I_a), (I_b), (I_c) respectively.

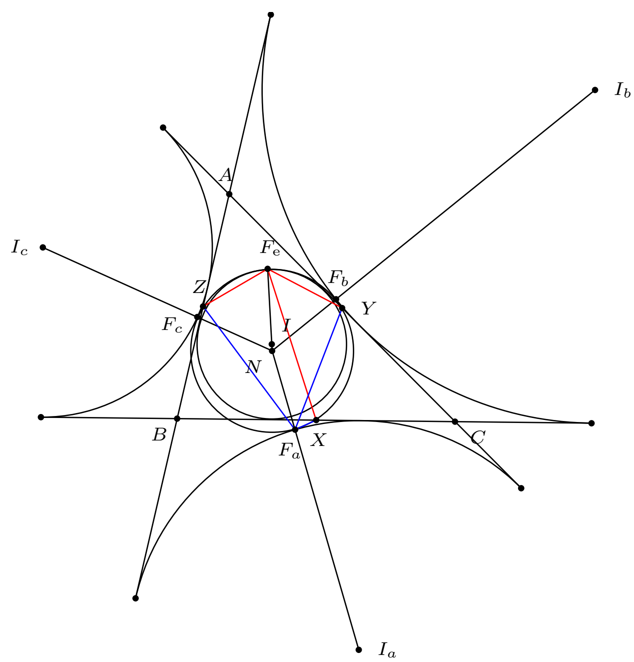

Figure 1

In [2], we have computed the distances from the Feuerbach points to X, Y, Z (see Figure 1). Specifically, if the lengths of the sides BC, CA, AB are a, b, c,

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then

$$F_eX = \frac{|b-c|R}{2\cdot OI}, \quad F_eY = \frac{|c-a|R}{2\cdot OI}, \quad F_eZ = \frac{|a-b|R}{2\cdot OI}; \tag{1}$$
$$F_aX = \frac{|b-c|R}{2\cdot OI_a}, \quad F_aY = \frac{(c+a)R}{2\cdot OI_a}, \quad F_aZ = \frac{(a+b)R}{2\cdot OI_a}. \tag{2}$$

By Euler's formula, $OI^2 = R(R-2r)$ and $OI_a^2 = R(R+2r_a)$, (see [1, Theorems 152, 153]), these are equivalent to the following formulas.

$$F_eX^2 = \frac{(b-c)^2R}{4(R-2r)}, \quad F_eY^2 = \frac{(c-a)^2R}{4(R-2r)}, \quad F_eZ^2 = \frac{(a-b)^2R}{4(R-2r)}; \tag{3}$$
$$F_aX^2 = \frac{(b-c)^2R}{4(R+2r_a)}, \quad F_aY^2 = \frac{(c+a)^2R}{4(R+2r_a)}, \quad F_aZ^2 = \frac{(a+b)^2R}{4(R+2r_a)}. \tag{4}$$

In this note, we find simple formulas analogous to (1), (2) for the distances among the Feuerbach points. We begin with the distances to the vertices (see Figure 2).

Proposition 1. *The distances from the Feuerbach point F_e to the vertices of triangle ABC are given by*

$$AF_e^2 = \frac{(s-a)^2R - rS_A}{R-2r}, \quad BF_e^2 = \frac{(s-b)^2R - rS_B}{R-2r}, \quad CF_e^2 = \frac{(s-c)^2R - rS_C}{R-2r},$$

where s is the semiperimeter of the triangle, and $S_A := \frac{b^2+c^2-a^2}{2}$, $S_B = \frac{c^2+a^2-b^2}{2}$, and $S_C = \frac{a^2+b^2-c^2}{2}$.

Proof. We apply the median theorem for the triangles F_eBC, F_eCA, F_eAB. From (3) above, we have

$$F_eX^2 = \frac{BF_e^2 + CF_e^2}{2} - \frac{a^2}{4}, \tag{5}$$
$$F_eY^2 = \frac{CF_e^2 + AF_e^2}{2} - \frac{b^2}{4}, \tag{6}$$
$$F_eZ^2 = \frac{AF_e^2 + BF_e^2}{2} - \frac{c^2}{4}. \tag{7}$$

The combination $-(5)+(6)+(7)$ gives

$$-F_eX^2 + F_eY^2 + F_eZ^2 = AF_e^2 - \frac{b^2+c^2-a^2}{4}.$$

Hence,

$$\begin{aligned} AF_{\rm e}^2 &= -F_{\rm e}X^2 + F_{\rm e}Y^2 + F_{\rm e}Z^2 + \frac{b^2+c^2-a^2}{4} \\ &= \frac{(-(b-c)^2+(c-a)^2+(a-b)^2)R}{4(R-2r)} + \frac{S_A}{2} \\ &= \frac{2(a-b)(a-c)R}{4(R-2r)} + \frac{2(R-2r)S_A}{4(R-2r)} \\ &= \frac{(2(a-b)(a-c)+2S_A)R-4rS_A}{4(R-2r)}. \end{aligned}$$

Since

$$2(a-b)(a-c)+2S_A = 2(a-b)(a-c)+(b^2+c^2-a^2) = (b+c-a)^2 = 4(s-a)^2,$$

we have $AF_{\rm e}^2 = \frac{(s-a)^2R-rS_A}{R-2r}$. The other two expressions follow similarly. □

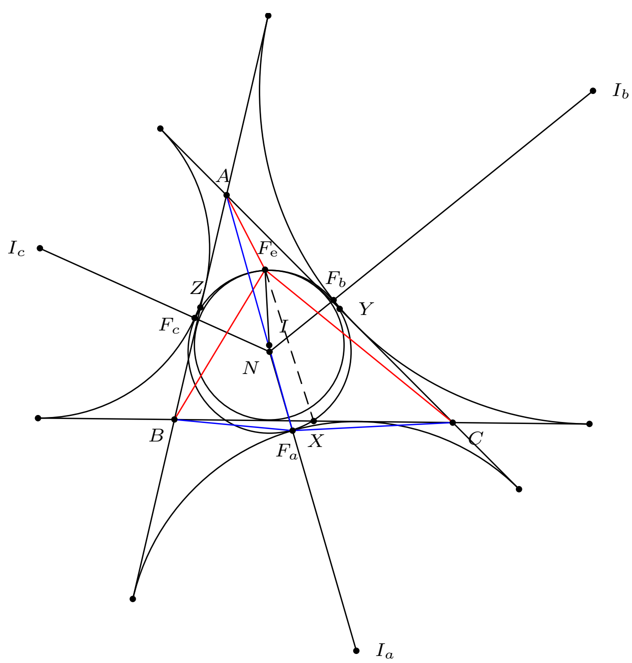

Figure 2

Proposition 2. *The distances from the Feuerbach point F_a to the vertices of triangle ABC are given by*

$$AF_a^2 = \frac{s^2R+r_aS_A}{R+2r_a}, \quad BF_a^2 = \frac{(s-c)^2R+r_aS_B}{R+2r_a}, \quad CF_a^2 = \frac{(s-b)^2R+r_aS_C}{R+2r_a}.$$

Proof. We applying the median theorem to triangles F_aBC, F_aCA, F_aAB. From (4) above, we have

$$F_aX^2 = \frac{BF_a^2 + CF_a^2}{2} - \frac{a^2}{4}, \tag{8}$$

$$F_aY^2 = \frac{CF_a^2 + AF_a^2}{2} - \frac{b^2}{4}, \tag{9}$$

$$F_aZ^2 = \frac{AF_a^2 + BF_a^2}{2} - \frac{c^2}{4}. \tag{10}$$

The combination $-$(8)+(9)+(10) gives

$$-F_aX^2 + F_aY^2 + F_aZ^2 = AF_a^2 - \frac{b^2 + c^2 - a^2}{4}.$$

Hence,

$$\begin{aligned} AF_a^2 &= -F_aX^2 + F_aY^2 + F_aZ^2 + \frac{S_A}{2} \\ &= \frac{(-(b-c)^2 + (c+a)^2 + (a+b)^2)R}{4(R+2r_a)} + \frac{S_A}{2} \\ &= \frac{2(a+b)(a+c)R}{4(R+2r_a)} + \frac{2(R+2r_a)S_A}{4(R+2r_a)} \\ &= \frac{(2(a+b)(a+c) + 2S_A)R + 4r_aS_A}{4(R+2r_a)}. \end{aligned}$$

Since

$$2(a+b)(a+c) + 2S_A = 2(a+b)(a+c) + (b^2 + c^2 - a^2) = (a+b+c)^2 = 4s^2,$$

we have $AF_a^2 = \frac{s^2R + r_aS_A}{R+2r_a}$.

On the other hand, the combination (8)$-$(9)+(10) gives

$$F_aX^2 - F_aY^2 + F_aZ^2 = BF_a^2 - \frac{c^2 + a^2 - b^2}{4}.$$

Therefore,

$$\begin{aligned} BF_a^2 &= F_aX^2 - F_aY^2 + F_aZ^2 + \frac{S_B}{2} \\ &= \frac{((b-c)^2 - (c+a)^2 + (a+b)^2)R}{4(R+2r_a)} + \frac{S_B}{2} \\ &= \frac{(2(a+b)(b-c) + 2S_B)R + 4r_aS_B}{4(R+2r_a)}. \end{aligned}$$

Since

$$2(a+b)(b-c) + 2S_B = 2(a+b)(b-c) + (c^2 + a^2 - b^2) = (a+b-c)^2 = 4(s-c)^2,$$

we have

$$BF_a^2 = \frac{(s-c)^2R + r_aS_B}{R+2r_a}.$$

The proof of the expression for CF_a^2 is similar. $\square$

Now we compute the distances among the Feuerbach points.

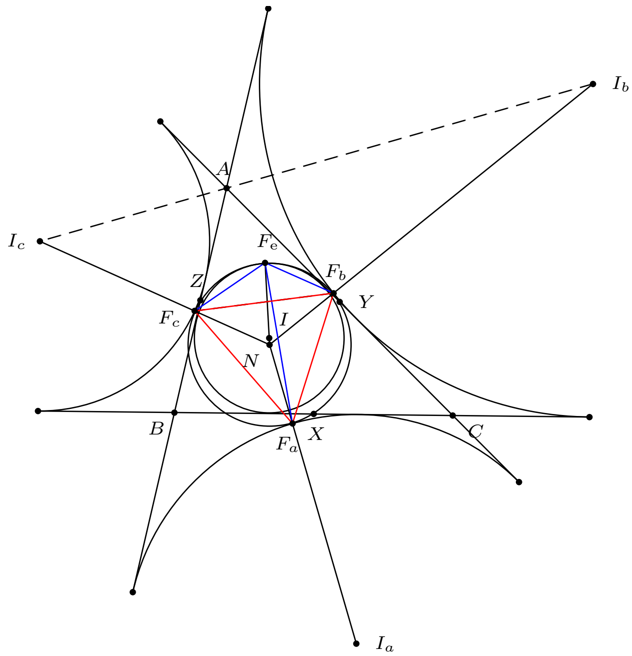

Figure 3

Theorem 3. $F_bF_c = \frac{(b+c)R^2}{OI_b \cdot OI_c}$, $F_cF_a = \frac{(c+a)R^2}{OI_c \cdot OI_a}$, $F_aF_b = \frac{(a+b)R^2}{OI_a \cdot OI_b}$.

Proof. It is enough to prove the first formula. Triangle NF_bF_c is isosceles with $NF_b = NF_c = \frac{R}{2}$; we have

$$F_bF_c^2 = \frac{R^2}{2}(1 - \cos F_bNF_c).$$

Applying the law of cosines to triangles NI_bI_c, noting that $I_bI_c = 4R\cos\frac{A}{2}$, we have

$$\begin{aligned}
&\left(4R\cos\frac{A}{2}\right)^2 \\
=\ & NI_b^2 + NI_c^2 - 2 \cdot NI_b \cdot NI_c \cos I_bNI_c \\
=\ & \left(\frac{R}{2}+r_b\right)^2 + \left(\frac{R}{2}+r_c\right)^2 - 2\left(\frac{R}{2}+r_b\right)\left(\frac{R}{2}+r_c\right)\cos I_bNI_c \\
=\ & \left[\left(\frac{R}{2}+r_b\right) - \left(\frac{R}{2}+r_c\right)\right]^2 + 2\left(\frac{R}{2}+r_b\right)\left(\frac{R}{2}+r_c\right)(1-\cos F_bNF_c) \\
=\ & (r_b - r_c)^2 + \frac{1}{2}(R+2r_b)(R+2r_c)(1-\cos F_bNF_c).
\end{aligned}$$

Therefore,

$$F_bF_c^2 == \frac{R^4\left(\left(4R\cos\frac{A}{2}\right)^2-(r_b-r_c)^2\right)}{OI_b^2\cdot OI_c^2}.$$

Since $s = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, $s\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right) = 4R\cos\frac{A}{2}\sin\left(\frac{B}{2}-\frac{C}{2}\right)$. From this,

$$F_bF_c^2 = \frac{16R^6\cos^2\frac{A}{2}\left(1-\sin^2\left(\frac{B}{2}-\frac{C}{2}\right)\right)}{OI_b^2\cdot OI_c^2} = \frac{16R^6\cos^2\frac{A}{2}\cos^2\left(\frac{B}{2}-\frac{C}{2}\right)}{OI_b^2\cdot OI_c^2},$$

and

$$\begin{aligned} F_bF_c &= \frac{4R^3\cos\frac{A}{2}\cos\left(\frac{B}{2}-\frac{C}{2}\right)}{OI_b\cdot OI_c} = \frac{2R^3\left(\cos\left(\frac{A}{2}-\frac{B}{2}+\frac{C}{2}\right)+\cos\left(\frac{A}{2}+\frac{B}{2}-\frac{C}{2}\right)\right)}{OI_b\cdot OI_c} \\ &= \frac{2R^3\left(\cos\left(\frac{\pi}{2}-B\right)+\cos\left(\frac{\pi}{2}-C\right)\right)}{OI_b\cdot OI_c} = \frac{2R^3(\sin B+\sin C)}{OI_b\cdot OI_c} = \frac{(b+c)R^2}{OI_b\cdot OI_c}. \end{aligned}$$

□

Theorem 4. $F_eF_a = \frac{|b-c|R^2}{OI\cdot OI_a}$, $F_eF_b = \frac{|c-a|R^2}{OI\cdot OI_b}$, $F_eF_c = \frac{|a-b|R^2}{OI\cdot OI_c}$.

Proof. Again, it is enough to prove the first formula. Triangle NF_eF_a is isosceles with $NF_e = NF_a = \frac{R}{2}$ (see Figure 3); we have

$$F_eF_a^2 = \frac{R^2}{2}(1-\cos F_eNF_a).$$

Applying the law of cosines to triangle NII_a, we have, noting that $II_a = 4R\sin\frac{A}{2}$,

$$\begin{aligned} &\left(4R\sin\frac{A}{2}\right)^2 \\ =\ & NI^2+NI_a^2-2NI\cdot NI_a\cos INI_a \\ =\ & \left(\frac{R}{2}-r\right)^2+\left(\frac{R}{2}+r_a\right)^2-2\left(\frac{R}{2}-r\right)\left(\frac{R}{2}+r_a\right)\cos INI_a \\ =\ & \left[\left(\frac{R}{2}-r\right)-\left(\frac{R}{2}+r_a\right)\right]^2+2\left(\frac{R}{2}-r\right)\left(\frac{R}{2}+r_a\right)(1-\cos F_eNF_a) \\ =\ & (r+r_a)^2+\frac{1}{2}(R-2r)(R+2r_a)(1-\cos F_eNF_a). \end{aligned}$$

Therefore,

$$F_eF_a^2 = \frac{R^2\left(\left(4R\sin\frac{A}{2}\right)^2-\left((b+c)\tan\frac{A}{2}\right)^2\right)}{(R-2r)(R+2r_a)} = \frac{R^4\left(\left(4R\sin\frac{A}{2}\right)^2-\left((b+c)\tan\frac{A}{2}\right)^2\right)}{OI^2\cdot OI_a^2}.$$

Since $(b+c)\tan\frac{A}{2} = 2R(\sin B+\sin C)\tan\frac{A}{2} = 4R\sin\frac{B+C}{2}\cos\frac{B-C}{2}\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B-C}{2}$,

$$F_eF_a^2 = \frac{R^4\left(\left(4R\sin\frac{A}{2}\right)^2-\left(4R\sin\frac{A}{2}\cos\frac{B-C}{2}\right)^2\right)}{OI^2\cdot OI_a^2} = \frac{16R^6\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}}{OI^2\cdot OI_a^2},$$

and

$$F_eF_a = \left|\frac{4R^3 \sin\frac{A}{2}\sin\frac{B-C}{2}}{OI\cdot OI_a}\right| = \left|\frac{2R^3\left(\cos\left(\frac{A}{2}-\frac{B}{2}+\frac{C}{2}\right)-\cos\left(\frac{A}{2}+\frac{B}{2}-\frac{C}{2}\right)\right)}{OI\cdot OI_a}\right|$$

$$= \left|\frac{2R^3\left(\cos\left(\frac{\pi}{2}-B\right)-\cos\left(\frac{\pi}{2}-C\right)\right)}{OI\cdot OI_a}\right|$$

$$= \left|\frac{2R^3(\sin B-\sin C)}{OI\cdot OI_a}\right| = \frac{|b-c|R^2}{OI\cdot OI_a}.$$

□

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