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Minimal Proof of a Generalized Droz-Farny Theorem

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Abstract. We give a simple proof of a recent generalization of the Droz-Farny line theorem by using complex numbers.

Theorem. ([2]) (1) *Let H be the orthocenter of a triangle ABC and ℓ, ℓ' two perpendicular lines through H that cut the sidelines BC, CA, and AB at D, E, F and D', E' F', respectively* (*see Figure 1*)*. If one erects directly similar triangular ears on the segments DD', EE', and FF', the ear apices are collinear.*

(2) *The ratio in which one ear apex cuts the segment of the other two is independent of the ear shape.* (*In particular, the ratios for D, E, F and D', E', F' are equal.*)

We present here a very simple proof. Droz-Farny took the midpoints of the segments as ear apices (see [1, 2] for further references).

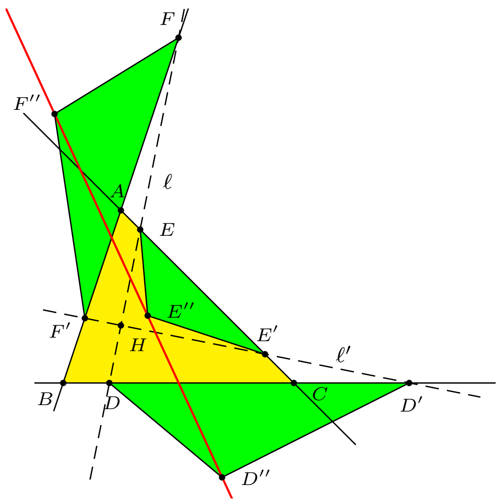

Figure 1

Proof. Without loss of generality, we choose in the complex plane

$$A = ia, \quad B = -b, \text{ and } C = c \text{ with } a, b, c > 0.$$

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The altitude through B being the line $y = \frac{c(x+b)}{a}$, the orthocenter is $H = ih$ for $h = \frac{bc}{a}$. The perpendiculars through H are

$$\ell: \ y = mx + h \quad \text{and} \quad \ell': \ y = -\frac{x}{m} + h, \ m \notin \left\{0, -\frac{a}{c}, \frac{c}{a}, \frac{a}{b}, -\frac{b}{a}\right\}$$

as they have to intersect the sidelines of the triangle, which are

$$BC: \ y = 0, \qquad CA: \ y = -\frac{a}{c}(x - c), \qquad AB: \ y = \frac{a}{b}(x + b).$$

One finds at once

$$\begin{aligned} D &= -\frac{h}{m}, & D' &= hm; \\ E &= \frac{c(a^2 - bc)}{a(cm + a)} + i\frac{c(am + b)}{cm + a}, & E' &= \frac{cm(a^2 - bc)}{a(am - c)} + i\frac{c(bm - a)}{am - c}; \\ F &= \frac{b(a^2 - bc)}{a(bm - a)} + i\frac{b(am - c)}{bm - a}, & F' &= \frac{-bm(a^2 - bc)}{a(am + b)} + i\frac{b(cm + a)}{am + b}. \end{aligned}$$

In order to erect triangular ears directly similar to the triangle with vertices 1, 0, and w on the segments DD', EE', and FF', one has to take the ear apices

$$D'' = wD + (1 - w)D', \qquad E'' = wE + (1 - w)E', \qquad F'' = wF + (1 - w)F'.$$

The apices D'', E'', F'' are collinear as a straightforward computation gives the real quotient

$$\frac{F'' - D''}{E'' - D''} = \frac{\left(m + \frac{a}{c}\right)\left(m - \frac{c}{a}\right)}{\left(m - \frac{a}{b}\right)\left(m + \frac{b}{a}\right)}.$$

E'' divides segment $D''F''$ in a ratio depending on the slopes of ℓ and the sidelines of triangle ABC, but not on the ear shape! □

References

[1] J.-L. Ayme, A synthetic proof of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004) 219–224.
[2] C. Letrouit, On a new generalization of the Droz-Farny line, *Forum Geom.*, 16 (2016) 367–369.

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