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On the Existence of a Triangle with Prescribed Bisector Lengths

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Abstract. We suggest a geometric visualization of the process of constructing a triangle with prescribed bisectors that makes the existence of such a triangle geometrically evident.

1. Introduction

Since antiquity, in construction problems of elementary geometry main attention has been paid to possibility or impossibility of such constructions with only ruler and compass, and problems of constructing regular n-gons, trisection of angle or squaring of a circle has played an important role in the development of mathematics. Existence and number of solutions of these construction problems when one is not confined to using ruler and compass only is often evident and as a rule is not discussed in much detail. However, situation changes when one turns to other problems such as, for example, problems of constructing triangles with prescribed its three elements. In particular, a triangle with given lengths of its sides can be constructed if the 'triangle inequalities' are fulfilled and then the triangle is unique up to an isometry. In a similar way, the problems of constructing a triangle with given lengths of its medians or altitudes can be considered and these constructions with ruler and compass are possible under conditions that certain inequalities are fulfilled. Interestingly enough, the problem of constructing a triangle with prescribed lengths of its bisectors is very different: it was shown that such a construction with only ruler and compass is impossible (see, e.g., [1]). Then the question arises: Given three lengths l_a, l_b, l_c, does there exist a triangle with its angle bisectors equal to these lengths? Positive answer to this question was given by Mironescu and Panaitopol in [2] by analytical method with the use of the Brower fixed point theorem. More elementary analytical proof was suggested by Zhukov and Akulich in [3]. Being absolutely strict, these proofs lack, in our opinion, geometric clearness. In this note we suggest such a geometric visualization of the process of constructing a triangle with prescribed bisectors that makes the possibility of such a construction practically evident. Thus, our aim in this note is to present an elementary visual proof of the Mironescu and Panaitopol theorem that reads:

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Theorem 1. *For any given segments with lengths l_a, l_b, l_c there exists a triangle whose bisectors are equal to these prescribed lengths.*

Informal discussion. Before going to the strict proof of this theorem, we give an informal description of the idea of our proof in which the process of constructing a triangle with prescribed bisector lengths is reduced to two transformations. Let, for definiteness, the prescribed lengths of bisectors satisfy inequalities $l_c < l_a < l_b$. We start from an equilateral triangle with three equal bisector lengths equal to l_b. The first transformation is the decreasing of the angle B in such a way that the triangle becomes isosceles with decreasing equal sides $a = c$ and constant bisector l_b. It is evident that bisectors l_a and l_c decrease also remaining all the time equal to each other. As a result of this first transformation we can reach such an isosceles triangle that its two changing bisectors become equal to the prescribed value l_a. Further decrease of the angle B would lead to decrease of l_a, therefore we combine change of B with rotation of the side b around the intersection point of l_b and b in such a way that l_a remains fixed (as well as l_b) and l_c decreases reaching finally the prescribed value. Possibility of such transformation follows from geometrically evident observation that if angles A and B decrease under condition that l_a and l_b remain constant, then the third angle C increases whereas its bisector l_c decreases. In the limit $\angle A \to 0$, $\angle B \to 0$ we have $l_c \to 0$ and hence l_c can reach any prescribed value $l_c < l_a$, l_b. This visualization of the construction makes intuitively plausible that a triangle with prescribed bisector lengths $l_c < l_a < l_b$ does exist.

2. Preliminary lemmas

Now we turn to formal realization of the above idea. As a preliminary step, we shall prove several Lemmas about angles and sides of a triangle, and its bisectors.

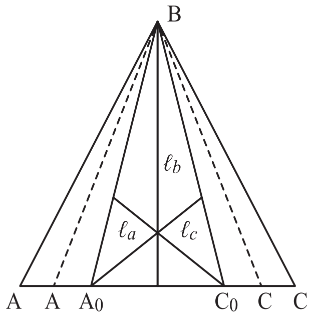

Figure 1. Transformation of equilateral triangle ABC with bisectors $l_a = l_b = l_c = l_1$ to isosceles triangle A_0BC_0 with bisectors $l_a = l_c = l_2$ and $l_b = l_1$ by decreasing the angle $\angle B$.

The above informal description of the idea of the proof actually introduces the elements of the triangle ABC as functions of the angle $\angle B$ under conditions that lengths of one or two bisectors are kept constant. Therefore it is convenient to distinguish the changing variables l_a, l_b, l_c equal to the bisectors lengths of our varying triangle from their prescribed fixed values. To this end, from now on we denote the fixed prescribed values as $l_3 < l_2 < l_1$.

We shall start with formulation of an obvious Lemma 2 that describes the variation of elements of an isosceles triangle in the first type of our transformations.

Lemma 2. *In an isosceles triangle ABC with equal sides $AB = CB$ and fixed length of the bisector $l_b = l_1$ (see Figure 1) the angles $\angle A = \angle C = (180° - \angle B)/2$ are decreasing functions of $\angle B$ and the sides $AB = CB$ as well as bisectors $l_a = l_c$ are increasing functions of $\angle B$. When $\angle B$ changes from $60°$ to zero, the bisectors l_a, l_c change from $l_b = l_1$ to zero.*

In the second type of our transformations the length of two bisectors l_b and l_a are kept constant and we are interested in dependence of the angles $\angle A$, $\angle C$ and the length of the bisector l_c on the angle $\angle B$.

Lemma 3. *Let in a triangle ABC the bisector l_a (of the angle $\angle A$) be equal to $l_a = l_2$, the bisector l_b (of angle $\angle B$) be equal to $l_b = l_1$, and the angles $\angle A$, $\angle B$, $\angle C$ satisfy the condition $\angle B < \angle A < \angle C$ (see Figure 2). Then the following inequalities hold*

$$\frac{1}{2}l_2 < b < a < c < l_1 + l_2. \tag{1}$$

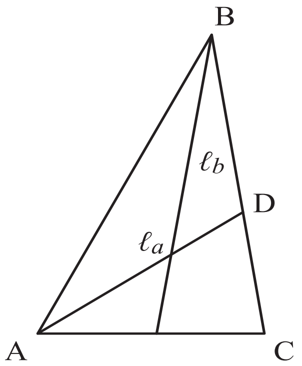

Figure 2. Sketch of triangle $\angle B < \angle A < \angle C$ and $l_a = l_2 < l_b = l_1$.

Proof. Since $\angle ADC = \angle BAD + \angle ABD$, we have $\angle ADC > \angle BAD = \angle CAD$ and, consequently, $CD < AC$. Therefore, we have $AD < AC + CD < 2AC$, hence $AC = b > \frac{1}{2}AD = \frac{1}{2}l_2$. At last, $AB = c$ is less than the sum of the two other sides of a triangle formed by AB and pieces of the bisectors $l_b = l_1$ and $l_a = l_2$. □

Now we consider a continuous set of triangles ABC parameterized by their smallest angle $\angle B$, so that $\angle B < \angle A < \angle C$, and we assume that in all these triangles the bisectors l_a and l_b have the same values. Thus, the angles $\angle A$ and $\angle C$ become some functions of the angle $\angle B$: if we change the angle $\angle B$, then

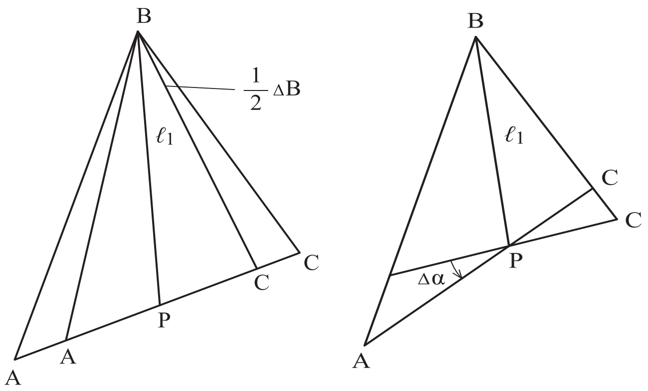

Figure 3. Transformation of the isosceles triangle with bisectors $l_a = l_c = l_2$ and $l_b = l_1$ to the final triangle with bisectors with $l_b = l_1$, $l_a = l_2$ and $l_c = l_3$.

the angles $\angle A$ and $\angle C$ will also change whereas the bisectors l_a and l_b remain constant. At first we shall prove the following simple property of these functions.

Lemma 4. *If there exists such a transformation of a triangle ABC that the conditions $\angle B < \angle A < \angle C$ are satisfied, the bisectors l_a and l_b are kept constant, and $\angle B \to 0$, then $\angle A \to 0$ and $\angle C \to 180°$.*

Proof. From identities $\sin\angle A/a = \sin\angle B/b = \sin\angle C/c$ and inequalities (1) we find that in the limit $\angle B \to 0°$ and, consequently, $\sin\angle B \to 0$, we have $\sin\angle A \to 0°$, $\sin\angle C \to 0°$. These limiting values of $\angle A$ and $\angle B$ can be realized in two cases

$$\angle A \to 0°, \qquad \angle C \to 180°;$$
$$\angle A \to 180°, \qquad \angle C \to 0°.$$

However in the second case we must have $l_a \to 0$ and that contradicts to the conditions of the Lemma, so this case must be excluded. We note also that inequalities (1) exclude such "exotic" situations as $\angle B \to 0°$, $b \to 0$ or $\angle B \to 0°$, $a \to \infty$, $c \to \infty$ that admit any limiting values of angles A and B provided $\angle A + \angle C \to 180°$ and $\angle A < \angle C$. This completes the proof of Lemma 4. □

Now we concretize the transformation of the second type by combining the decrease of the angle $\angle B$ with rotation of the side AC around the point P at which the bisector l_b and the side AC meet. Let the angle $\angle B$ be decreased by the value ΔB (see Figure 3, left) and the side AC be rotated counterclockwise around point P by the angle $\Delta\alpha$ (Figure 3, right). Then after such a transformation we have $\angle B < \angle C$ since $l_a < l_b$ and also we have $\angle A < \angle C$ since $\Delta A = \frac{1}{2}\Delta B - \Delta\alpha$ and $\Delta C = \frac{1}{2}\Delta B + \Delta\alpha$. Thus, we have proved the following

Lemma 5. *In the defined above transformation we always have $\angle B < \angle A < \angle C$.*

This means that the conditions of Lemma 4 are always fulfilled for our concrete realization of the transformation.

Now we shall study the behavior of the bisector l_c under action of our transformation.

Lemma 6. *When angle $\angle B$ decreases from its value B_0, corresponding to the isosceles triangle with $l_a = l_c = l_2$, to zero, and the bisectors $l_b = l_1$, $l_a = l_2$ are kept constant, then the length of the bisector l_c is a continuous function of $\angle B$ taking all values from the interval $[0, l_2]$.*

Proof. From inspection of Figure 3 (left) it is clear that decrease of the angle $\angle B$ leads to the decrease of the sides AB and AC and to the increase of the angle $\angle A$. Consequently, from the known equation

$$l_a = \frac{2AB \cdot AC}{AB + AC} \cos \frac{\angle A}{2} = \frac{2}{\frac{1}{AB} + \frac{1}{AC}} \cos \frac{\angle A}{2} \tag{2}$$

we find that the length l_a also decreases with decrease of the angle $\angle B$, when the angles $\angle A$ and $\angle C$ are kept constant. To prevent such a decrease of l_a with decrease of $\angle B$ by the value ΔB, we combine it with rotation of the side AB around the point P by the angle $\Delta\alpha$, where $\alpha = \angle APB$. Considered separately, such a rotation increases the sides AB and AC monotonously to infinitely large values, decreasing at the same time the angle $\angle A$ to infinitesimally small values. Then, according to Eq. (2), the length l_a can be increased by such a rotation as we wish and, consequently, we can compensate the decrease of l_a after diminishing of $\angle B$ by the rotation of AC and make the bisector of the angle $\angle A$ equal to $l_a = l_2$. This construction shows that each value of the angle $\angle B$ corresponds to a definite value of the angle $\angle A$ at which $l_a = l_2$ and the function $\angle A = \angle A(\angle B)$ is continuous. The function $l_c(\angle B)$ is also continuous and in the described above transformation the conditions $\angle B < \angle A < \angle C$, $l_a = l_2 = \text{const}$, $l_b = l_1 = \text{const}$ of Lemma 4 are fulfilled and when $\angle B$ tends to zero we have $\angle A \to 0$ and $\angle C \to 180^\circ$, consequently $l_c \to 0$ as $\angle B$ tends to zero. Thus, we arrive at the conclusion that with decrease of $\angle B$ from B_0 to zero the bisector l_c as a continuous function of $\angle B$ takes all values from the interval $[0, l_2]$. □

Remark. As we shall see later, the function is not only continuous but also single-valued and, hence, monotonous.

3. Proof of Theorem 1

Now we can prove Theorem 1 about existence of a triangle with prescribed lengths of its bisectors.

Let three segments be given whose lengths l_1, l_2, l_3 satisfy the conditions $l_3 < l_2 < l_1$. Let us consider an equilateral triangle ABC with bisectors equal to $l_a = l_b = l_c = l_1$. We fix the position and the length of the bisector l_b and decrease the angle $\angle B$ in a way shown in Figure 1 down to the moment when we get an isosceles triangle A_0BC_0 with bisectors $l_a = l_c = l_2$ and $l_b = l_1$. Existence of such a triangle follows from Lemma 2: since the bisector lengths l_a and l_c ($l_a = l_c$) are continuous functions of the angle $\angle B$ and they decrease with $\angle B$ to zero value in the limit $\angle B \to 0$, then they take all values in the interval $[0, l_1]$ including the value $l_a = l_c = l_2$ that belongs to this interval due to inequality $0 < l_2 < l_1$. The corresponding value of the angle $\angle B$ will be denoted as $\angle B_0$.

Then we continue transformation of the triangle ABC but now keeping the values of the bisectors l_a and l_b constant and decreasing the length l_c of the third bisector by further decrease of the angle $\angle B$ with simultaneously rotation of the side AC counterclockwise around the point P as is shown in Figure 3. As was shown in Lemma 6, during such an evolution the function $l_c(\angle B)$ behaves as a continuous function taking all values from the interval $[0, l_2]$ including the value $l_c = l_3$ thanks to inequality $0 < l_3 < l_2$. Hence by means of our transformation we arrive at such a triangle whose bisectors have the prescribed lengths. Thus the main Theorem 1 is proved.

We conclude with a few remarks.

Remarks. (1) In [3] an elementary proof is given that if a triangle with prescribed lengths does exist then it is unique up to isometry. Combining this theorem with our proof of existence of such a triangle, we conclude that in our transformation a single value of each parameter $\angle A$, $\angle C$ and l_c corresponds to a given value of $\angle B$; otherwise there would exist several such triangles. Consequently, these continuous functions of the angle $\angle B$ must be monotonous: with decrease of $\angle B$ the angle $\angle A$ and the bisector l_c monotonously decrease whereas the angle $\angle C$ monotonously increases.

(2) The above proof shows that the angles of the final triangle satisfy the inequalities $0° < \angle A < \angle A_0$, $0° < \angle B < \angle B_0$, $\angle C_0 < \angle C < 180°$, where $\angle A_0 = \angle C_0 = (180° - \angle B_0)/2$. These inequalities can be made stronger if one introduces an angle β such that $\angle C = 180° - 2\beta$. Let $\beta = \arcsin(l_3/(l_1 + l_3))$, then we obtain the following estimate for l_c:

$$l_c = \frac{2ab}{a+b}\cos\frac{\angle C}{2} = \frac{2ab}{a+b}\cdot\frac{l_3}{l_1+l_3} < \frac{2a^2}{2a}\cdot\frac{l_3}{l_1+l_3} = l_3\cdot\frac{a}{l_1+l_3} < l_3. \quad (3)$$

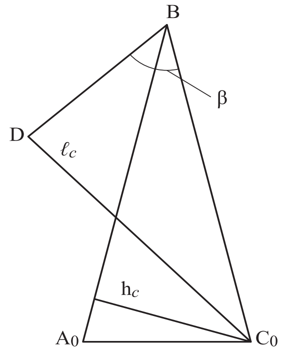

Figure 4. Estimates of the angles $\angle B$ and $\angle A$.

Since the angle $\angle C$ increases monotonously with decrease of $\angle B$ ($\Delta C = \frac{1}{2}\Delta B + \Delta\alpha$) and the length l_c monotonously decreases, then we find from Eq. (3) that $\angle C < 180° - 2\beta$.

For estimation of the angle $\angle A$ we prolong the bisector line l_c up to the point D with $BD \perp CD$; see Figure 4. Then $\angle DBC = 90^\circ - \frac{1}{2}\angle C = \beta$ and we get $\angle B < \beta$. Indeed, if we assume that during our transformation at some moment we reach $\angle B > \beta$, then by continuity argument there exists a triangle $A'BC'$ such that $\angle B = \beta$. Consequently, $\angle B + \frac{1}{2}\angle C = \angle B + (90^\circ - \beta) = 90^\circ$, that is $AB \perp l_c$ and the triangle $A'BC'$ must be isosceles, $AC = BC$, what is impossible because of $l_a \neq l_b$. Hence, taking into account that $\angle A + \angle B + \angle C = 180^\circ$, we arrive at inequality $\angle A > \beta$. Thus, with account of monotonicity of our functions, we conclude that the angles of the triangle ABC satisfy the inequalities $\beta < \angle A < \angle A_0$, $0^\circ < \angle B < \angle B_0$, $\angle C_0 < \angle C < 180^\circ - 2\beta$.

(3) At last we notice that the suggested here method gives actually the algorithm for numerical calculation of the elements of the triangle with prescribed values of its bisectors.

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