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How to Compute a Triangle with Prescribed Lengths of Its Internal Angle Bisectors

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Dedicated to my friend Professor Dr. Ewald Reinhart
on the occasion of his 80th birthday

Abstract. In 1994 P. Mironescu and L. Panaitopol published a non-constructive proof that any three given positive real numbers are the lengths of the internal angle bisectors of a triangle which is unique up to isometries. In the present paper it will be shown that this result can be obtained also by a constructive proof which in addition leads to an efficient method for computing the lengths of the sides of the triangle in question.

1. Preliminaries

Let $\langle A, B, C\rangle$ be a triangle, $a = |\overrightarrow{BC}|$, $b = |\overrightarrow{CA}|$, $c = |\overrightarrow{AB}|$ the lengths of its sides, and $\alpha = \angle(CAB)$, $\beta = \angle(ABC)$, $\gamma = \angle(BCA)$ its angles. The lengths a, b, c are positive real numbers satisfying the triangle inequalities

$$a < b + c,\ b < c + a,\ c < a + b. \tag{1}$$

Conversely, if a, b, c are positive real numbers satisfying the inequalities (1), then there is a triangle $\langle A, B, C\rangle$, unique up to isometries, such that $|\overrightarrow{BC}| = a$, $|\overrightarrow{CA}| = b$, $|\overrightarrow{AB}| = c$. The line through A bisecting α is given by $\left\{A + \lambda(\frac{\overrightarrow{AC}}{b} + \frac{\overrightarrow{AB}}{c}) : \lambda \in \mathbb{R}\right\}$. It cuts the side $\langle B, C\rangle$ of the triangle at

$$X = A + \lambda' \left(\frac{\overrightarrow{AC}}{b} + \frac{\overrightarrow{AB}}{c}\right) = B + \tau' \overrightarrow{BC}, \tag{2}$$

where $\lambda' = \frac{bc}{b+c}$ and $\tau' = \frac{c}{b+c}$ (see Figure 1). The segment $\langle A, X\rangle$ is the so called internal angle bisector (of $\langle A, B, C\rangle$) at A. For the length $t_A = |\overrightarrow{AX}|$ of $\langle A, X\rangle$ the well-known formula

$$t_A^2 = bc\left(1 - \frac{a^2}{(b+c)^2}\right) \tag{3}$$

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can be easily derived from (2). The internal angle bisectors $\langle B, Y\rangle$ and $\langle C, Z\rangle$ at B and C, as well as t_B and t_C are defined similarly.

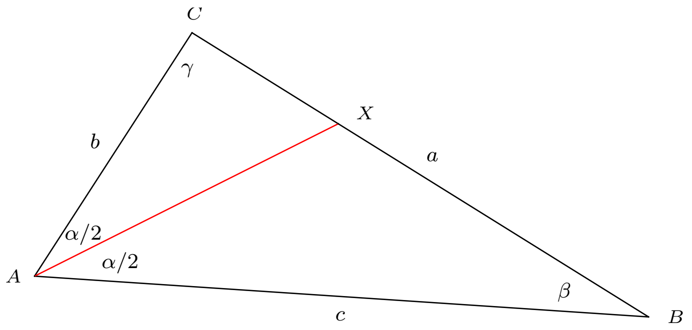

Figure 1. Triangle $\langle A, B, C\rangle$ with internal angle bisector $\langle A, X\rangle$

Whereas it is well-known, that a triangle can be reconstructed up to isometries by compass and ruler from the lengths of its altitudes or of its medians, it is not possible in general to reconstruct it by compass and ruler from the lengths of its internal angle bisectors, as was shown probably first by A. Korselt in 1897 [4] (see also [7]). It was even not clear for a long time, whether a triangle is uniquely defined up to isometries by the lengths of its internal angle bisectors, and if there is a relation between these lengths. These questions were answered by P. Mironescu and L. Panaitopol in 1994 [5]. Their surprising result was the following theorem.

Theorem (Mironescu and Panaitopol)**.** *Given any three positive real numbers m, n, p, then there is always a triangle $\langle A, B, C\rangle$ such that $t_A = m$, $t_B = n$, and $t_C = p$. This triangle is unique up to isometries.*

Their proof however was not constructive and therefore did not show up an efficient way how to compute a triangle with the desired property. Nevertheless, the first part of their proof is also the basis of the computational method to be developed in the present paper. This first part consists of the derivation of a relation of t_A, t_B, and t_C to the positive numbers

$$u = \frac{b+c-a}{2},\ v = \frac{c+a-b}{2},\ w = \frac{a+b-c}{2}, \tag{4}$$

appearing in several formulas for triangles.

We will deduce this relation here for the sake of completeness.

Solving (4) for a, b, c gives

$$a = v + w,\ b = w + u,\ c = u + v. \tag{5}$$

Using the identity $bc = \frac{1}{4}((b+c)^2 - (b-c)^2)$, it is easily verified that

$$t_A^2 + v^2 = \frac{1}{4}\left(b + c + \frac{a(c-b)}{b+c}\right)^2. \tag{6}$$

Since $a(c-b) = v^2 - w^2$, we have

$$b + c + \frac{a(c-b)}{b+c} = \frac{4u^2 + 2v^2 + 4wu + 2wv + 4uv}{b+c} > 0.$$

Hence (6) implies

$$\sqrt{t_A^2 + v^2} = \frac{1}{2}\left(b + c + \frac{a(c-b)}{b+c}\right). \tag{7}$$

Interchanging b and c in this formula, we get

$$\sqrt{t_A^2 + w^2} = \frac{1}{2}\left(b + c - \frac{a(c-b)}{b+c}\right). \tag{8}$$

Adding (7) and (8) results in

$$b + c = \sqrt{t_A^2 + v^2} + \sqrt{t_A^2 + w^2}. \tag{9}$$

Replacing $b + c$ by $w + 2u + v$, (9) can be written as

$$u = \frac{\sqrt{t_A^2 + v^2} - v}{2} + \frac{\sqrt{t_A^2 + w^2} - w}{2}. \tag{10}$$

By symmetry we have also

$$v = \frac{\sqrt{t_B^2 + w^2} - w}{2} + \frac{\sqrt{t_B^2 + u^2} - u}{2}, \tag{11}$$

and

$$w = \frac{\sqrt{t_C^2 + u^2} - u}{2} + \frac{\sqrt{t_C^2 + v^2} - v}{2}. \tag{12}$$

2. The approach of Mironescu and Panaitopol

After the derivation of the relations (10), (11), (12), Mironescu and Panaitopol introduced for every real $q > 0$ the function f_q defined for all $t \in \mathbb{R}$ by

$$f_q(t) = \frac{\sqrt{q^2 + t^2} - t}{2} \in \mathbb{R}, \tag{13}$$

and for every triple (m, n, p) of positive real numbers, the mapping $F_{(m,n,p)}$ defined for all $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ by

$$F_{(m,n,p)}(x) = \begin{pmatrix} f_m(x_2) + f_m(x_3) \\ f_n(x_3) + f_n(x_1) \\ f_p(x_1) + f_p(x_2) \end{pmatrix} \in \mathbb{R}^3. \tag{14}$$

Then the relations (10), (11), (12) are restated as: $(u, v, w)^T$ is a fixed point of $F_{(t_A, t_B, t_C)}$. The main result however is, that for every triple (m, n, p) of positive real numbers, $F = F_{(m,n,p)}$ has a unique fixed point, and that this fixed point is located in the interior of $Q = [0, x_u]$, where $0 = (0,0,0)^T,\ x_u = (m,n,p)^T$, and $[x, y]$ denotes the set of all $z \in \mathbb{R}^3$ satisfying $x \leq z \leq y$ with the usual order relation

$$x = (x_1, x_2, x_3)^T \leq y = (y_1, y_2, y_3)^T \text{ if and only if } x_i \leq y_i,\ i = 1, 2, 3.$$

We rewrite the proof, using also the strict order relation

$$x = (x_1, x_2, x_3)^T < y = (y_1, y_2, y_3)^T \text{ if and only if } x_i < y_i,\ i = 1, 2, 3.$$

Obviously $F(x) > 0$ for all $x \in \mathbb{R}^3$. This implies $x' > 0$ if x' is a fixed point of F. Now, $0 < f_q(t) < \frac{q}{2}$ for all $t > 0$ implies $F(x) < x_u$ if $x > 0$. Hence $0 < x' < x_u$ if x' is a fixed point of F. If $0 \leq x \leq x_u$, then $0 < F(x) \leq x_u$. Hence the continuous mapping F maps the convex compact set Q into Q, implying by Brouwer's fixed point theorem that F has a fixed point x' in Q, for which in fact $0 < x' < x_u$ must hold. From $-\frac{1}{2} < f_q'(t) < 0$ if $t > 0$, the authors concluded $||F(x) - F(y)||_2 < ||x - y||_2$ if $x, y \in Q$ and $x \neq y$, i.e. $F|Q$ is strictly nonexpansive with respect to the euclidean norm $|| \cdot ||_2$. But this shows the uniqueness of x'.

Actually a much easier proof of the existence of x' (attributed in [3] to a communication from M. Krein) can be given:

$F(Q)$ is a compact subset of Q. Hence there is an $x' \in F(Q)$ such that

$$||F(x') - x'||_2 \leq ||F(x) - x||_2 \text{ for all } x \in F(Q).$$

If we assume $F(x') \neq x'$, then

$$||F(F(x')) - F(x')||_2 < ||F(x') - x'||_2,$$

which is a contradiction since $x = F(x') \in F(Q)$.

In order to verify that there is a triangle $\langle A', B', C' \rangle$ such that $m = t_{A'}$ $n = t_{B'}$, $p = t_{C'}$, we set $a' = x_2' + x_3'$, $b' = x_3' + x_1'$, $c' = x_1' + x_2'$. Then $a' + b' > c'$, $b' + c' > a'$, $c' + a' > b'$. Hence there is a triangle $\langle A', B', C' \rangle$ such that $a' = |\overrightarrow{B'C'}|$, $b' = |\overrightarrow{C'A'}|$, $c' = |\overrightarrow{A'B'}|$.

Let us show $m = t_{A'}$. The first equation of the identity $x' = F_{(m,n,p)}(x')$ is

$$x_1' = \frac{\sqrt{m^2 + x_2'^2} - x_2'}{2} + \frac{\sqrt{m^2 + x_3'^2} - x_3'}{2}.$$

It is equivalent to

$$b' + c' = 2x_1' + x_2' + x_3' = \sqrt{m^2 + x_2'^2} + \sqrt{m^2 + x_3'^2}.$$

Squaring gives

$$(b' + c')^2 = 2m^2 + x_2'^2 + x_3'^2 + 2\sqrt{(m^2 + x_2'^2)(m^2 + x_3'^2)}.$$

Rearranging and squaring once more results in

$$((b' + c')^2 - 2m^2 - (x_2'^2 + x_3'^2))^2 = 4(m^2 + x_2'^2)(m^2 + x_3'^2).$$

Expanding, cancelling and rearranging gives

$$4(b' + c')^2 m^2 = (b' + c')^4 - 2(b' + c')^2(x_2'^2 + x_3'^2) + (x_2'^2 + x_3'^2)^2 - 4x_2'^2 x_3'^2.$$

Since $x_2' = \frac{c'+a'-b'}{2}$ and $x_3' = \frac{a'+b'-c'}{2}$, we get

$$\begin{aligned}(x_2'^2 + x_3'^2)^2 - 4x_2'^2 x_3'^2 &= (x_3'^2 - x_2'^2)^2 = ((x_3' + x_2')(x_3' - x_2'))^2 = (a'(b'-c'))^2, \\ x_2'^2 + x_3'^2 &= \frac{a'^2 + (b'-c')^2}{2}, \text{ and} \\ 4(b'+c')^2 m^2 &= (b'+c')^4 - (b'+c')^2(a'^2 + (b'-c')^2) + (a'(b'-c'))^2 \\ &= (b'+c')^4 - 4a'^2 b'c' - (b'+c')^2(b'-c')^2.\end{aligned}$$

Consequently,

$$m^2 = \frac{(b'+c')^2}{4} - \frac{a'^2 b'c'}{(b'+c')^2} - \frac{(b'-c')^2}{4} = b'c'\left(1 - \frac{a'^2}{(b'+c')^2}\right),$$

which is equivalent to $m = t_{A'}$. Similarly, or by symmetry, we get $n = t_{B'}$ and $p = t_{C'}$.

The uniqueness of $\langle A', B', C'\rangle$ up to isometries is seen as follows: Assuming that $(m, n, p) = (t_A, t_B, t_C)$ holds also for the triangle $\langle A, B, C\rangle$, then, if u, v, w are defined by (4), $(u, v, w)^T$ is a fixed point of $F_{(m,n,p)}$ as shown in Section 1. Hence $(u, v, w)^T = x'$. But this implies

$$a = x_2' + x_3' = a', \quad b = x_3' + x_1' = b', \quad c = x_1' + x_2' = c'.$$

Hence $\langle A, B, C\rangle$ and $\langle A', B', C'\rangle$ are congruent.

3. Existence, uniqueness and computability of x' by Banach's fixed point theorem.

In the sequel we write x_l instead of

$$F(x_u) = (f_m(n) + f_m(p),\ f_n(p) + f_n(m),\ f_p(m) + f_p(n))^T.$$

A first suggestion for approximating x' was made by G. Dinca and J. Mawhin [1]. They considered the perturbed mappings $\lambda F|Q, 0 < \lambda < 1$, which are contractions of Q into Q. By Banach's fixed point theorem there is a unique fixed point x_λ of $\lambda F|Q$ which can be approximated (theoretically) with arbitrary precision by the iteration $x_0 \in Q,\ x_k = \lambda F(x_{k-1}), k = 1, 2, \dots$. For λ close to 1, x_λ is used as an approximation of x'.

Actually, a Theorem of M. Edelstein [2] (see also [6, 12.3.6]) shows that for any $x_0 \in Q$ the sequence $\{x_k\}$ defined by $x_{k+1} = F(x_k), k = 0, 1, \dots$, converges to x'. In both cases, however, error estimates are not given.

In the following it will be demonstrated how Banach's fixed point theorem can be applied to a restriction of F to a suitable subset of Q.

We introduce the absolute value $|x| = (|x_1|, |x_2|, |x_3|)^T$ and the 1-norm $||x||_1 = |x_1| + |x_2| + |x_3|$ for all $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$. Since $f_q'(t) = \frac{1}{2}\left(\frac{t}{\sqrt{q^2+t^2}} - 1\right) < 0$ for all $t \in \mathbb{R}$, f_q is strictly decreasing. But this implies that F is order-reversing, i.e. $F(y) \le F(x)$ if $x \le y$. An important consequence of this property is the following

Proposition 1. *If F maps $[x, y]$ into itself, then it also maps $[F(y), F(x)]$ into itself.*

Proof. If $F(y) \leq z \leq F(x)$ then $F(F(x)) \leq F(z) \leq F(F(y))$. Since $x \leq F(y)$ implies $F(F(y)) \leq F(x)$, and $F(x) \leq y$ implies $F(y) \leq F(F(x))$, we have $F(y) \leq F(F(x)) \leq F(z) \leq F(F(y)) \leq F(x)$. $\square$

A consequence of Proposition 1 is: Since F maps $[0, x_u]$ into itself, it maps also $[F(x_u), F(0)] = [x_l, x_u]$ into itself. If x' is a fixed point of F, we have seen already that $0 < x' < x_u$ must hold. $F(0) \geq F(x') = x' \geq F(x_u)$ shows that x' must be contained even in $[x_l, x_u]$.

Next we use the fact that we have for all $t_0 > 0$, $0 < -f_q'(t_0) < 1/2$, and $|f_q(t_2) - f_q(t_1)| \leq -f_q'(t_0)|t_2 - t_1|$ if $t_1, t_2 \geq t_0$. This implies for all $x, y \geq x_l$ (> 0), $|F(y) - F(x)| \leq P|y - x|$, where

$$P = \begin{pmatrix} 0 & -f_m'(x_{l2}) & -f_m'(x_{l3}) \\ -f_n'(x_{l1}) & 0 & -f_n'(x_{l3}) \\ -f_p'(x_{l1}) & -f_p'(x_{l2}) & 0 \end{pmatrix}. \tag{15}$$

Consequently we have $||F(y) - F(x)||_1 \leq \lambda ||y - x||_1$ with

$$\lambda = \max\{-f_n'(x_{l1}) - f_p'(x_{l1}), -f_m'(x_{l2}) - f_p'(x_{l2}), -f_m'(x_{l3}) - f_n'(x_{l3})\}$$

for all $x, y \geq x_l$. $\lambda < 1$ since $-f_q'(t) < \frac{1}{2}$ if $t > 0$.

Now we can conclude from Banach's fixed point theorem: There is exactly one fixed point x' of F, $x' \in [x_l, x_u]$, and starting the iteration

$$x_{k+1} = F(x_k),\ k = 0, 1, \ldots,$$

with any $x_0 \in [x_l, x_u]$, then $||x' - x_{k+1}||_1$ is bounded above by $\frac{\lambda^{k+1}}{1-\lambda}||x_1 - x_0||_1$ (a priori error estimate), and by $\frac{\lambda}{1-\lambda}||x_{k+1} - x_k||_1$ (a posteriori error estimate).

Componentwise error estimates can be obtained from the inequalities

$$|x' - x_{k+1}| \leq (I - P)^{-1} P^{k+1} |x_1 - x_0|$$

and

$$|x' - x_{k+1}| \leq (I - P)^{-1} P |x_{k+1} - x_k|,$$

where I denotes the 3×3 unit matrix (see e.g. [6, 13.1.2]).

The evaluation of the considered error estimates can be avoided however if $x_0 = x_u$ is chosen. In this case the subsequence $\{x_{2k}\}$ of $\{x_k\}$ converges monotonically decreasing to x', and the subsequence $\{x_{2k+1}\}$ converges monotonically increasing to x'. This can be deduced by applying Proposition 1 iteratively: By Proposition 1 we get $[x_1, x_0] \supseteq [x_1, x_2] \supseteq [x_3, x_2]$, and by induction $[x_{2k-1}, x_{2(k-1)}] \supseteq [x_{2k-1}, x_{2k}] \supseteq [x_{2k+1}, x_{2k}], k = 2, 3, \ldots$. Since then $x_{2k-1} \leq x' \leq x_{2k}, k = 1, 2, \ldots$, it is possible to get lower and upper bounds of x' which are as close to x' as one wants.

Let us illustrate these results with the numerical example

$$(m, n, p)^T = (17, 12, 19.36)^T = x_u = x_0.$$

The following data were delivered by a simple MATLAB program using the standard output rounding:

$$x_1 = F(x_0) = x_l = (7.6066, 3.6130, 9.7709)^T,$$

($0 < x_l < x_0$ for the exact value of x_l),

$$\begin{aligned} x_{15} &= (10.4995, 5.0363, 13.2461)^T, \\ x_{16} &= (10.4996, 5.0364, 13.2462)^T, \end{aligned}$$

($x_{15} \le x' \le x_{16}$ for the exact values of x_{15} and x_{16}),

$$P = \begin{pmatrix} 0 & 0.3961 & 0.2508 \\ 0.2323 & 0 & 0.1843 \\ 0.3172 & 0.4083 & 0 \end{pmatrix},$$

$\lambda = 0.8043$, and 9.7853×10^{-4} as the resulting upper bound for $||x' - x_{16}||_1$, $10^{-3} \times (0.1109, 0.0852, 0.1206)^T$ as the resulting upper bound of $|x' - x_{16}|$, and 3.1664×10^{-4} as the resulting upper bound of $|||x' - x_{16}|||_1$.

From the approximation x_{16} of x', $a = 18.8825$, $b = 23.7458$, $c = 15.5360$ were derived. The computation of t_A, t_B, t_C from a, b, c resulted in

$$t_A = 17.0000, \quad t_B = 12.0000, \quad t_C = 19.3600,$$

as it should be.

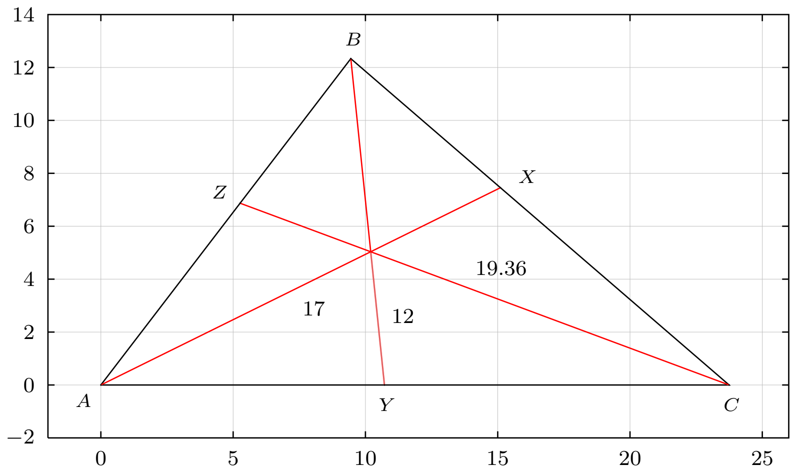

Figure 2. Triangle $\langle A, B, C\rangle$ with prescribed $(t_A, t_B, t_C) = (17, 12, 19.36)$

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