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Circle Chains Inscribed in Symmetrical Lenses and Integer Sequences

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Abstract. We derive the conditions for inscribing, inside a symmetrical lens, a chain of tangent circles having the property that the ratio between the largest circle radius and the radius of any other circle of the chain is an integer.

1. Introduction

A lens is the common area between two intersecting circles. If the radii of the circles are equal, the lens is symmetrical. The aim of this paper is to show some connections between the infinite chains of tangent circles that can be inscribed in a symmetrical lens and certain integer sequences. A generic example of circle chain is shown in Figure 1.

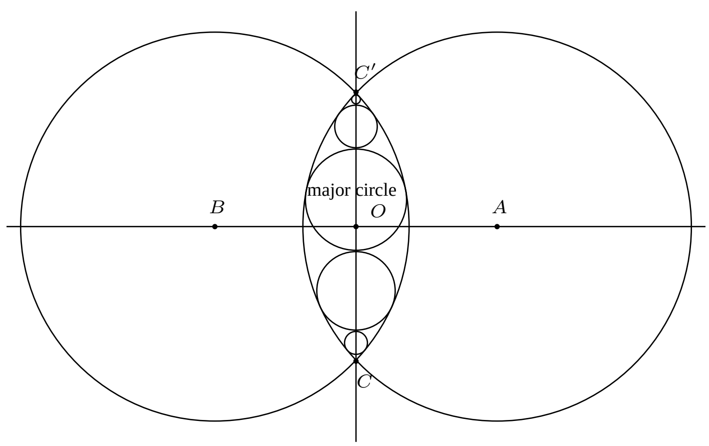

Figure 1. Example of a circle chain inscribed inside a symmetrical lens

To the best of our knowledge, only two papers ([1, 2]) can be found in literature dealing with the problem of an infinite chain of mutually tangent circles inscribed inside a lens. In the paper [2], by using the circular inversion technique, the present author derived, expressions for center coordinates and radius of the circles belonging to the chain. The results are general and they are valid also for a generic lens not necessarily symmetrical. In [1], J. Kocik, by using a different approach, showed that the radii relevant to chains of mutually tangent circles inscribed in a symmetrical lens can be obtained by a non-homogeneous linear recurrence formula.

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2. Some definitions and useful expressions

A circle chain in the symmetrical lens is doubly infinite. We label the circles $\mathcal{C}_k$, with radius r_k, for integers k. It is convenient to define the *major circle* $\mathcal{C}_0$ (see Figure 1) This induces a subdivision of the chain into two sub-chains:
(i) an *up-chain* $\mathcal{C}_{\mathrm{u}k} = \mathcal{C}_k$, $k = 0, 1, 2, \ldots$, starting from the major circle and converging to point C', and
(ii) a *down-chain* $\mathcal{C}_{\mathrm{d}k} = \mathcal{C}_{-k}$, $k = 0, 1, 2, \ldots$, starting from the major circle and converging to point C.

With a Cartesian coordinate system with axes along the line AB and its perpendicular bisector, the circle chain and its up- and down-subchains are completely determined by the ordinate of the center of the major circle (see Figure 2). We shall assume

- $R =$ the common radius of the circles forming the lens,
- $2a$ (with $a < R$) the distance between the centers of the two intersecting circles,
- the two circles intersect at $C(0, -h)$ and $C'(0, h)$, $h = \sqrt{R^2 - a^2}$,
- y_0 is the ordinate of the center K of the major circle in the chain; note that $y_0 \in \left[-\frac{R^2-a^2}{2R}, \frac{R^2-a^2}{2R}\right]$,
- the radius of the major circle is

$$r_0(y_0) = R - \sqrt{a^2 + y_0^2}. \tag{1}$$

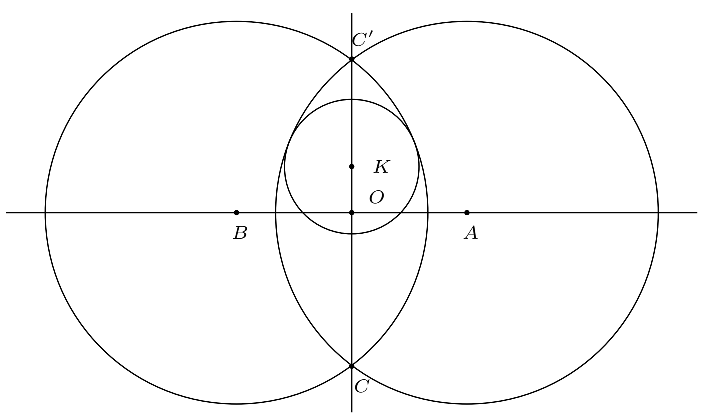

Figure 2. Symmetrical lens with major circle of the chain

For two particular values for y_0, the up- and down-chains are symmetrical.

- If $y_0 = 0$, we have $r_0 = R - a$, and the major circle is bisected by the x-axis (central symmetry; see Figure 3).
- If $y_0 = \pm\frac{R^2-a^2}{2R}$, we have $r_0 = \frac{R^2-a^2}{2R}$ and two equal major circles (see Figure 4). The up- and down-chains are symmetrical about the x-axis. In order to avoid complication of re-indexing, we exclude this case in the discussion below.

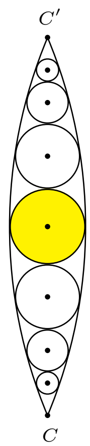

Figure 3

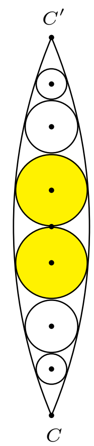

Figure 4

In this paper, we investigate the conditions for which the ratios $\tau_k := \frac{r_0}{r_k}$ ($k = \pm 1, \pm 2, \dots$) between the radii of the major circle and the generic k-th circle are integers for both the up- and down-sub-chains. In other words, we find conditions (provided they exist) for the radius of any generic circle of the chain is a submultiple of the major circle radius.

In [2], we applied inversion techniques (see [4]) and obtained expressions for the radii and the coordinates of the centers of the circles in the up- and down-chains. We choose the circle of inversion with center C and radius $\rho = 2h$, passing through C'. Here is a summary of the main formulas and results.

(a) The two intersecting circles forming the lens are transformed into the two straight lines

$$y = \pm \frac{a}{h} x + h, \tag{2}$$

through C'.

(b) The major circle is transformed into a circle with radius

$$R_0 = \left| \frac{\rho^2}{(y_0 + h)^2 - r_0^2} \right| r_0. \tag{3}$$

(c) The inversive images of the circles of the chain form another chain tangent to two lines given by (2) in (a). Their centers and radii are

$$\begin{aligned} (x'_k,\ y'_k) =&\ \left(0,\ \frac{\omega^{-k} R R_0}{h} + h\right), \\ r'_k =&\ \omega^{-k} R_0, \end{aligned}$$

for $k = 0, \pm 1, \pm 2, \dots$. Here,

$$\omega = \frac{R - h}{R + h}.$$

(d) The centers and radii of the circles in the lens are

$$(x_k,\ y_k) = \left(0,\ s_k\left(\frac{\omega^k R R_0}{h} + 2h\right) - h\right),$$
$$r_k = |s_k|\omega^k R_0,$$

for $k = 0, \pm 1, \pm 2, \ldots$. Here

$$s_k = \left(-\frac{\omega^{2k} R_0^2}{4h^2} + \left(1 + \frac{\omega^k R R_0}{2h^2}\right)^2\right)^{-1}.$$

In Figure 5 an example of a circle chain inside a symmetrical lens is shown with the inversive images of the circles.

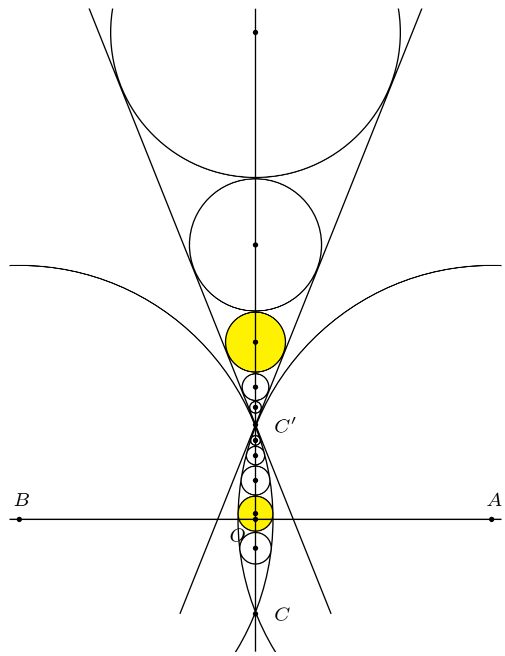

Figure 5

With the aids of the formulas, we have, for $k = 0, \pm 1, \pm 2, \ldots$,

$$r_k = r_0\left((G_k + G_k^{-1})\frac{a(R - \sqrt{a^2 + y_0^2})}{2h^2} + \frac{R(R - \sqrt{a^2 + y_0^2})}{h^2}\right)^{-1}, \qquad (4)$$

where

$$G_k = \frac{\omega^k a(R - \sqrt{a^2 + y_0^2})}{y_0 h - a^2 + R\sqrt{a^2 + y_0^2}}.$$

If k is positive, we have the up-chain while, if k is negative, we have the down-chain.

From (4), we define the sequence (τ_k) of the ratios between the major circle radius and the k-th circle radius:

$$\tau_k = \frac{r_0}{r_k} = (G_k + G_k^{-1})\frac{a(R-\sqrt{a^2+y_0^2})}{2h^2} + \frac{R(R-\sqrt{a^2+y_0^2})}{h^2}. \tag{5}$$

By means of algebra, we can show that for $k = 0, \pm 1, \pm 2, \dots,$

$$\tau_k = \alpha\omega^k + \beta\omega^{-k} + \gamma, \tag{6}$$

for

$$\begin{aligned} \alpha &= \frac{a^2(R-\sqrt{a^2+y_0^2})^2}{2h^2(hy_0 - a^2 + R\sqrt{a^2+y_0^2})}, \\ \beta &= \frac{hy_0 - a^2 + R\sqrt{a^2+y_0^2}}{2h^2}, \\ \gamma &= \frac{R(R-\sqrt{a^2+y_0^2})}{h^2}. \end{aligned}$$

It is important, for the following, to point out that in [1], J. Kocik showed how Binet-like formulas can be expressed by means of a recursive relation of the type:

$$\tau_{k+2} = \left(\omega + \frac{1}{\omega}\right)\tau_{k+1} - \tau_k + \gamma\left(2 - \omega - \frac{1}{\omega}\right). \tag{7}$$

3. Conditions for integer sequences

In general, the doubly infinitely sequence (τ_k) is composed of real numbers. Here we want to find the conditions for which τ_k, $k = 0, \pm 1, \pm 2, \dots,$ are integers. In other words, what are the values of $\frac{a}{R}$ and $\frac{y_0}{R}$ to guarantee (τ_k) to be an integer sequence?

First of all, by (5), it is possible to assign two specific values μ and λ (both > 1) to τ_1 and τ_{-1} respectively.

By making the substitutions

$$\begin{cases} X &= \frac{a}{R}, \quad 0 < X < 1, \\ Y &= \frac{y_0}{R}, \quad -\frac{1-X^2}{2} \le Y \le \frac{1-X^2}{2}, \end{cases} \tag{8}$$

we rewrite, for $k = 1$ and $k = -1$, (5) as

$$\begin{cases} \mu &= \left(\dfrac{\frac{1-\sqrt{1-X^2}}{1+\sqrt{1-X^2}}X(1-\sqrt{Y^2+X^2})}{Y\sqrt{1-X^2}-X^2+\sqrt{Y^2+X^2}} + \dfrac{Y\sqrt{1-X^2}-X^2+\sqrt{Y^2+X^2}}{\frac{1-\sqrt{1-X^2}}{1+\sqrt{1-X^2}}X(1-\sqrt{Y^2+X^2})} \right) \dfrac{X(1-\sqrt{Y^2+X^2})}{2(1-X^2)} \\ & \quad + \dfrac{1-\sqrt{Y^2+X^2}}{1-X^2}, \\ \lambda &= \left(\dfrac{\frac{1+\sqrt{1-X^2}}{1-\sqrt{1-X^2}}X(1-\sqrt{Y^2+X^2})}{Y\sqrt{1-X^2}-X^2+\sqrt{Y^2+X^2}} + \dfrac{Y\sqrt{1-X^2}-X^2+\sqrt{Y^2+X^2}}{\frac{1+\sqrt{1-X^2}}{1-\sqrt{1-X^2}}X(1-\sqrt{Y^2+X^2})} \right) \dfrac{X(1-\sqrt{Y^2+X^2})}{2(1-X^2)} \\ & \quad + \dfrac{1-\sqrt{Y^2+X^2}}{1-X^2}. \end{cases} \tag{9a}$$

One can verify that the only solution of system (9a) satisfying the constraints in (8) is

$$\begin{cases} X = \frac{2}{\sqrt{(\lambda+1)(\mu+1)}}, \\ Y = \frac{1}{\lambda+1} - \frac{1}{\mu+1}. \end{cases} \tag{10a}$$

Similarly, one can impose from (5) $\tau_1 = \lambda$ and $\tau_{-1} = \mu$ and obtain

$$\begin{cases} \lambda &= \left(\frac{\frac{1-\sqrt{1-X^2}}{1+\sqrt{1-X^2}} X(1-\sqrt{Y^2+X^2})}{Y\sqrt{1-X^2}-X^2+\sqrt{Y^2+X^2}} + \frac{Y\sqrt{1-X^2}-X^2+\sqrt{Y^2+X^2}}{\frac{1-\sqrt{1-X^2}}{1+\sqrt{1-X^2}} X(1-\sqrt{Y^2+X^2})} \right) \frac{X(1-\sqrt{Y^2+X^2})}{2(1-X^2)} \\ & \quad + \frac{1-\sqrt{Y^2+X^2}}{1-X^2}, \\ \mu &= \left(\frac{\frac{1+\sqrt{1-X^2}}{1-\sqrt{1-X^2}} X(1-\sqrt{Y^2+X^2})}{Y\sqrt{1-X^2}-X^2+\sqrt{Y^2+X^2}} + \frac{Y\sqrt{1-X^2}-X^2+\sqrt{Y^2+X^2}}{\frac{1+\sqrt{1-X^2}}{1-\sqrt{1-X^2}} X(1-\sqrt{Y^2+X^2})} \right) \frac{X(1-\sqrt{Y^2+X^2})}{2(1-X^2)} \\ & \quad + \frac{1-\sqrt{Y^2+X^2}}{1-X^2}, \end{cases} \tag{9b}$$

having the unique solution

$$\begin{cases} X = \frac{2}{\sqrt{(\lambda+1)(\mu+1)}}, \\ Y = -\left(\frac{1}{\lambda+1} - \frac{1}{\mu+1} \right). \end{cases} \tag{10b}$$

Hence, the solutions (10a)-(10b) are the conditions relevant to the half width $R-a$ of the lens and to the ordinate y_0 of the center of the major circle in order to have the ratios $\tau_1 = \mu$ and $\tau_{-1} = \lambda$ or $\tau_1 = \lambda$ and $\tau_{-1} = \mu$ respectively. It is useful to notice that, by applying the Pythagorean theorem to the right triangle OAC in Figure 2, one has from (10a):

$$AC = \frac{1}{\lambda+1} + \frac{1}{\mu+1}.$$

In particular, we are interested to the case where $\mu = n-1$ and $\lambda = m-1$ for integers $m, n \geq 2$. With reference to Figure 2, we state the following property:

For a symmetrical lens with a given ratio $\frac{a}{R}$, the condition of a circle chain with integer ratios τ_k is

$$\begin{cases} AK &= \left(\frac{1}{m} + \frac{1}{n}\right) R, \\ OK &= \left|\frac{1}{m} - \frac{1}{n}\right| R, \end{cases}$$

where m, n are integers ≥ 2 and $(m,n) \neq (2,2)$.

From (10a), we have

$$\frac{y_0}{R} = \frac{1}{m} - \frac{1}{n}, \tag{11}$$

$$\frac{a}{R} = \frac{2}{\sqrt{mn}}. \tag{12}$$

By splitting up (6) into two sequences for the up- and down-chains respectively one has, for $k = 0, 1, 2, \ldots,$

$$\tau_{\mathrm{u}k} = \alpha\omega^k + \beta\omega^{-k} + \gamma,$$

$$\tau_{\mathrm{d}k} = \beta\omega^k + \alpha\omega^{-k} + \gamma,$$

Moreover, by taking (11) and (12) into account, we have

$$\begin{aligned} \omega =& \left(\frac{\sqrt{mn}-\sqrt{mn-4}}{2}\right)^2, \\ \alpha =& \frac{(m-n)\sqrt{mn-4}+(m+n-4)\sqrt{mn}}{2\sqrt{mn}(mn-4)}, \\ \beta =& \frac{-(m-n)\sqrt{mn-4}+(m+n-4)\sqrt{mn}}{2\sqrt{mn}(mn-4)}, \\ \gamma =& \frac{mn-m-n}{mn-4}. \end{aligned}$$

With a little algebra, we can show that

$$\begin{aligned} \omega+\frac{1}{\omega} =& \ mn-2, \\ \gamma\left(2-\omega-\frac{1}{\omega}\right) =& \ m+n-mn. \end{aligned}$$

Furthermore,

$$\tau_{\mathrm{u}0}=\tau_{\mathrm{d}0}=\alpha+\beta+\gamma=1.$$

Note also that

$$\begin{cases} \tau_{\mathrm{u}1}=n-1, \\ \tau_{\mathrm{d}1}=m-1. \end{cases}$$

Now, let us focus on the up-chain. $\tau_{\mathrm{u}0}$, $\tau_{\mathrm{u}1}$, $\gamma\left(2-\omega-\frac{1}{\omega}\right)$, and $\omega+\frac{1}{\omega}$ being integers, $\tau_{\mathrm{u}2}$ is also an integer from (7). It follows that $\tau_{\mathrm{u}k}$ is an integer for all $k>0$.

The same reasoning applies to the down-chain. Therefore, the sequence $\{\tau_k\}$ consists entirely of integers.

To conclude, *given a symmetrical lens of ratio* $\frac{a}{R}$*, if a pair* (m,n) *of integers exists so that relation* (12) *is satisfied, then by choosing the ordinate* y_0 *to satisfy* (11)*, it is possible to inscribe inside the lens a circle chain generating two integer sequences* $(\tau_{\mathrm{u}k})$ *and* $(\tau_{\mathrm{d}k})$.

Conversely, relations (12) and (11) can be used to create an inscribed chain starting from an arbitrary pair of integers (m,n) provided that $(m,n)\neq(2,2)$.

4. Symmetrical chains

Some interesting particular cases are represented by the symmetrical chains . As mentioned in Section 2, depending on the ordinate y_0 of the major circle center, one can have two different kinds of symmetrical chains that, consequently, generate identical sequences $\{\tau_{\mathrm{u}k}\}$ and $\{\tau_{\mathrm{d}k}\}$:

- the case with $m=n\geq 3$, central symmetry;
- the case with (m,n) with $m=2$ and $n\geq 3$, or $m\geq 3$ and $n=2$, bicentral symmetry.

A certain number of these sequences can be found in OEIS (*The On Line Encyclopedia of Integer Sequences* [3]). In Table I, some of them are listed.

Table I: Some sequences listed in OEIS related to
circle chains in a symmetric lens

(m,n)	Sequence in OEIS
$(3,3)$	$A064170$
$(4,4)$	$A011922$
$(6,6)$	$A076218$
$(2,3),\ (3,2)$	$A101265$
$(2,4),\ (4,2)$	$A011900$
$(2,5),\ (5,2)$	$A182432$
$(2,6),\ (6,2)$	$A054318$
$(2,7),\ (7,2)$	$A253621$
$(2,8),\ (8,2)$	$A156712$
$(2,10),\ (10,2)$	$A246641$
$(2,12),\ (12,2)$	$A254782$
$(2,14),\ (14,2)$	$A253458$
$(2,16),\ (16,2)$	$A253447$
$(2,22),\ (22,2)$	$A054318$
$(2,32),\ (32,2)$	$A131751$

We conclude with two examples of circle chains generated by sequences listed in OEIS.

Example 1. Circle chain with central symmetry derived from sequence

$$A064170: \qquad 1,\ 2,\ 10,\ 65,\ 442,\ 3026,\ \ldots$$

From the second term we have $\tau_{-1} = \tau_1 = 2$. This yields $m = n = 3$ and finally from (11) and (12), one has

$$\frac{y_0}{R} = 0, \qquad \frac{a}{R} = \frac{2}{3}.$$

Example 2. Circle chain with bicentral symmetry derived from sequence

$$A101265: \qquad 1,\ 2,\ 6,\ 21,\ 77,\ 286,\ \ldots$$

Due to the fact that we are considering a chain with bicentral symmetry, we have $r_0 = |y_0|$. From (1) one can write:

$$y_0 = R - \sqrt{a^2 + y_0^2}.$$

By considering the up-chain (with $y_0 > 0$) and by substituting (11) and (12) in the previous formula, we obtain $m = 2$. Moreover, from the second term of the sequence, we have that $\tau_1 = 2$. This yields $n = 3$ and finally from (11) and (12) one has:

$$\frac{y_0}{R} = \frac{1}{6}, \qquad \frac{a}{R} = \frac{2}{\sqrt{6}}.$$

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