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Iterated Harmonic Divisions and the Golden Ratio

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Abstract. On the projective real line we consider sequences of points in which every four consecutive points form two harmonic point pairs. Surprisingly, the asymptotic behavior of one type of these sequences is characterized by the golden ratio. Another type of these sequences is projectively equivalent to a dense set on the unit circle generated by an irrational rotation by nearly 137.5°.

We consider point pairs (A, B) and (C, D) on the projective real line $\mathbb{R} \cup \infty$, which are harmonic conjugate, i.e. their cross ratio $\frac{AC}{AD} \cdot \frac{DB}{CB}$ is -1. Because of the relevance of harmonic division it seems worth to consider also various types of iterated applications of this notion.

Starting with three different points P_0, P_1, P_2 on the projective real line there are three possibilities to define a fourth harmonic point in order to obtain two harmonic conjugate point pairs. Correspondingly, we consider three types of sequences (P_n) defined by a successive harmonic division:

Type 1: P_n is defined as the harmonic conjugate of P_{n-3} with respect to P_{n-1} and P_{n-2} for $n > 2$.

Type 2: P_n is defined as the harmonic conjugate of P_{n-2} with respect to P_{n-1} and P_{n-3} for $n > 2$.

Type 3: P_n is defined as the harmonic conjugate of P_{n-1} with respect to P_{n-2} and P_{n-3} for $n > 2$.

In the following we discuss a few properties of these sequences.

Sequences of type 1. At first we look for sequences of this type of a very simple form. A good candidate are geometric sequences q^n. q satisfies the condition

$$-1 = \frac{(q^n - q^{n-1})(q^{n-2} - q^{n-3})}{(q^n - q^{n-2})(q^{n-1} - q^{n-3})} = \frac{q}{(q+1)^2},$$

i.e. $q^2 + 3q + 1 = 0$ with the solutions $q_1 = \frac{-3+\sqrt{5}}{2} = -\Phi^{-2} = \Phi - 2$ and $q_2 = \frac{-3-\sqrt{5}}{2} = -\Phi^2 = -1 - \Phi$, where $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. For an arbitrary sequence of type 1 we have a projective isomorphism $f : t \to \frac{at+b}{ct+d}$ with

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$(-\Phi^{-2})^n \to P_n$. It follows $\lim_{n\to\infty} P_n = f(0)$. For a finite limit we have $d \neq 0$ and

$$\lim_{n\to\infty} \frac{P_{n+2}P_{n+1}}{P_{n+1}P_n} = \lim_{n\to\infty} \frac{\frac{aq_1^{n+2}+b}{cq_1^{n+2}+d} - \frac{aq_1^{n+1}+b}{cq_1^{n+1}+d}}{\frac{aq_1^{n+1}+b}{cq_1^{n+1}+d} - \frac{aq_1^{n}+b}{cq_1^{n}+d}} = \frac{q_1(cq_1^n + d)}{(cq_1^{n+2}+d)} = -\Phi^{-2},$$

i.e. $P_{n+2}P_{n+1} \approx -\Phi^{-2}\, P_{n+1}P_n$ for large n. For an infinite limit we have $d = 0$ and

$$\lim_{n\to\infty} \frac{P_{n+2}P_{n+1}}{P_{n+1}P_n} = \lim_{n\to\infty} \frac{\frac{aq_1^{n+2}+b}{cq_1^{n+2}} - \frac{aq_1^{n+1}+b}{cq_1^{n+1}}}{\frac{aq_1^{n+1}+b}{cq_1^{n+1}} - \frac{aq_1^{n}+b}{cq_1^{n}}} = \frac{1}{q_1} = -\Phi^2.$$

Therefore, although projective transformations do not preserve ratios, not only cross ratios but also limits of certain ratios are preserved after projective transformations.

We can consider $q_2^n = (-\Phi^2)^n$ as a real subsequence of the complex geometric sequence $(i\Phi)^n$ or as a real subset of the golden spiral $(e^{t(\ln\Phi + i\frac{\pi}{2})})_{t\in\mathbb{R}}$ in the complex plane. The author has not found hints in the literature that four consecutive points of the intersection of the golden spiral with a line through the center form two harmonic point pairs.

Sequences of type 2. By Definition we have $P_{n+4} = P_n$. (P_n) is a 4-periodic sequence of two harmonic conjugate pairs (P_0, P_2) and (P_1, P_3).

Sequences of type 3. Looking for a geometric sequence of this type we require

$$-1 = \frac{(q^n - q^{n-2})(q^{n-3} - q^{n-1})}{(q^n - q^{n-3})(q^{n-2} - q^{n-1})} = \frac{(q+1)^2}{q^2+q+1}.$$

i.e. $q^2 + \frac{3}{2}q + 1 = 0$ with complex solutions $q_\pm = \frac{-3\pm i\sqrt{7}}{4}$. We remark that $|q_\pm^n| = 1$. For an arbitrary sequence of type 3 we have a projective isomorphism from the unit circle to the projective real line with $q_+^n \to P_n$ by a complex Möbius transformation. We obtain q_+^{n+1} by rotating q_+^n through an angle $\arg(q_+)$. q_+ is not a root of unity, because the minimal polynomial $2q^2 + 3q + 2$ is not a cyclotomic polynomial, cf. [1]. Consequently $\arg(q_+)$ is an irrational angle and the points q_+^n lie dense on the unit circle and correspondingly (P_n) is dense on the projective real line.

We remark that even though type 3 sequences seem not to be related to the golden ratio, the rotation angle $\arg(q_+) \approx 138.59°$ is curiously very near to the golden angle $\frac{360°}{\Phi^2} \approx 137.5°$.

Reference

[1] I. Stewart, *Galois Theory*, Chapman and Hall, London, 1973.

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