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Steiner-Lehmus Type Results Related to The Gergonne Point of a Triangle

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Abstract. A *Gergonne cevian* (abbreviated GC) is a line segment joining the vertex of a triangle with the point of tangency of the triangle's incircle with the opposite side. In this paper, we determine monotonicity results for various segments determined by the intersection of Gergonne cevians and angle bisectors (both internal and external) within a triangle. We first consider the problem, in response to a prior question, of comparing certain segment lengths determined by the intersection of a fixed GC with the external bisectors of the other two angles of a triangle. We then consider the analogous problem wherein one fixes an angle bisector and considers the segment lengths determined by its intersection with the GC's emanating from the other two vertices. Finally, we prove some results for a related question comparing segments determined by the intersection of the GC from $\angle B$ with an angle bisector from $\angle C$ within a triangle ABC to those determined by the intersection of a bisector from $\angle B$ with the GC from $\angle C$.

1. Introduction

Segments within an isosceles triangle that are symmetric with respect to the line of symmetry of the triangle are always congruent. Conversely, one might ask when congruence of some particular pair of symmetric segments within a triangle implies congruence of the corresponding sides. For example, it is readily shown that congruence of altitudes or medians to two sides of a triangle implies congruence of the sides. The comparable problem involving angle bisectors is more difficult and the fact that the same result holds for angle bisectors is known as the Steiner-Lehmus theorem. Many proofs have been given of this result and we refer the reader to [2, 3, 4, 7] and also to the references contained within [10, 11]. See [5] for stronger versions of the theorem and [1] for a version involving extensions of the angle bisector. Results such as these arise frequently as particular cases of more general monotonic behavior. For example, a median, altitude or angle bisector to a shorter side is always longer, and conversely. On the other hand, the congruence of a pair of segments that would be congruent within an isosceles triangle by virtue of their symmetry need not imply that the triangle within which they lie is isosceles. See, e.g., [6] for an example.

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Here, we consider variants of the Steiner-Lehmus result involving the Gergonne point. A *Gergonne cevian* (GC) joins a vertex of a triangle with the point of tangency of the opposite side with the triangle's incircle. The Gergonne cevians of a triangle are concurrent, and their point of concurrency is known as the *Gergonne point* of the triangle (see, e.g., [3, p. 13]). In [9], a Gergonne analogue of the Steiner-Lehmus theorem was proven. That is, if BD and CE are GC's within $\triangle ABC$, then $BD = CE$ implies $AB = AC$. It is also shown in [9] that if the internal angle bisectors of $\angle B$ and $\angle C$ within $\triangle ABC$ intersect the Gergonne cevian AD at E and F, respectively, then $BE = CF$ implies $AB = AC$. The comparable question involving external instead of internal angle bisectors is raised at the conclusion of [9], and here we provide an affirmative answer to this question by establishing a more general monotonicity criterion.

This paper is organized as follows. In the next section, after addressing the problem from [9] described above, we consider the analogous problem wherein the roles of the GC's and angle bisectors are reversed. That is, given an angle bisector, we compare the distances, when traveled along a GC, from the other two vertices to the bisector. We address this problem in the case of both an internal and an external bisector. In the third section, we consider a related question which extends work from [6] wherein we compare segment lengths determined by the intersection of the GC from $\angle B$ with the bisector (either internal or external) from $\angle C$ within $\triangle ABC$ to those determined by the intersection of the comparable bisector from $\angle B$ with the GC from $\angle C$. We prove a more general monotonicity property from which the specific Steiner-Lehmus type result follows in each case. Our arguments primarily make use of establishing certain trigonometric and/or algebraic inequalities.

We will use at times the following results. The first may be obtained by slightly generalizing the argument presented for [9, Theorem 2].

Lemma 1. *Let BD and CE be Gergonne cevians within $\triangle ABC$. If $AB < AC$, then $BD < CE$.*

The following formula (see, e.g., [8, p. 46]) for the sine of a half-angle in terms of the side lengths of a triangle will also be useful.

Lemma 2. *If s denotes the semiperimeter of a triangle ABC, then $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$.*

In the proofs below, we let $a = BC$, $b = AC$ and $c = AB$, with $s = \frac{a+b+c}{2}$.

2. An answer to a previous question and related result

In Figure 1 below, the point of tangency of the incircle of $\triangle ABC$ with side BC is denoted by D. The exterior angle bisectors of $\angle B$ and $\angle C$ meet AD (extended) at points E and F, respectively.

Our first result shows that $BE = CF$ in Figure 1 implies $AB = AC$, answering a question raised in [9].

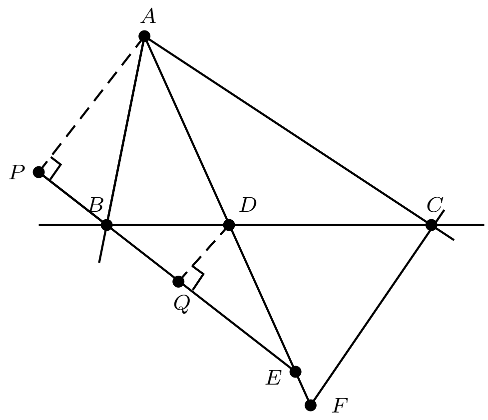

Figure 1. Intersection of Gergonne cevian with exterior angle bisectors of the other two angles.

Theorem 3. *If $AB < AC$ in Figure 1, then $BE < CF$. In particular, if $BE = CF$, then $AB = AC$.*

Proof. Let P and Q be the feet, respectively, of points A and D on the line $\overleftrightarrow{BE}$. Since $\angle ABP = \angle DBQ = \frac{\pi}{2} - \frac{\angle B}{2}$, we have $AP = c\cos\frac{B}{2}$, $DQ = (s-b)\cos\frac{B}{2}$, $BP = c\sin\frac{B}{2}$ and $PQ = BP + BQ = c\sin\frac{B}{2} + (s-b)\sin\frac{B}{2}$. Let $x = EQ$. Since $\triangle DQE \sim \triangle APE$, we have $\frac{QE}{PE} = \frac{DQ}{AP}$, which implies

$$\frac{x}{x + (s-b+c)\sin\frac{B}{2}} = \frac{s-b}{c}$$

and thus $x = \frac{(s-b+c)(s-b)\sin\frac{B}{2}}{b+c-s}$. We then have

$$BE = EQ + BQ = \frac{s-b+c}{b+c-s}(s-b)\sin\frac{B}{2} + (s-b)\sin\frac{B}{2} = \frac{2c(s-b)\sin\frac{B}{2}}{b+c-s}.$$

By symmetry, we have $CF = \frac{2b(s-c)\sin\frac{C}{2}}{c+b-s}$. Thus, $BE < CF$ if and only if $c(s-b)\sin\frac{B}{2} < b(s-c)\sin\frac{C}{2}$. By Lemma 2, we have $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$ and $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$, and thus the preceding inequality is equivalent to $\sqrt{c(s-b)} < \sqrt{b(s-c)}$, i.e., $c(a-b+c) < b(a+b-c)$. The last inequality holds since $a(c-b) < 0 < b^2 - c^2$, which completes the proof. □

We now consider the analogous problem in which the roles of the angle bisectors and GC's are reversed. In Figure 2 below, BD and CE are GC's in $\triangle ABC$. The exterior and interior bisectors of $\angle A$ meet BD (extended) at the points F and J, respectively, and CE at G and I, respectively.

The first part of the following theorem is an analogue of [9, Theorem 3].

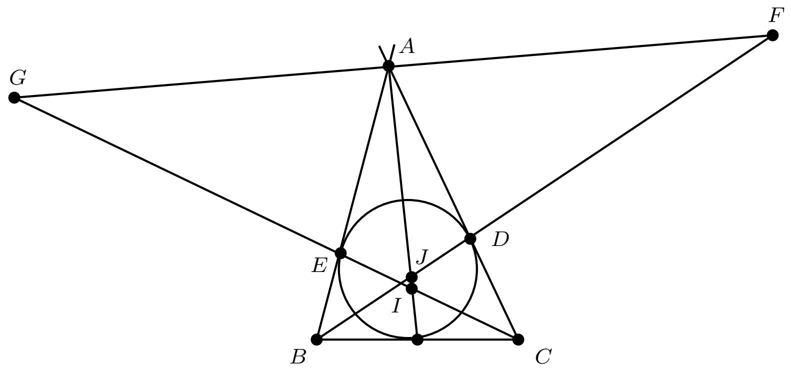

Figure 2. Intersection of interior and exterior angle bisector with the GC's to the two adjacent sides.

Theorem 4. *If $AB < AC$ in Figure 2, then $BJ < CI$ and if $BC \leq AB < AC$, then $CG < BF$.*

Proof. By the angle bisector theorem applied to $\triangle ABD$, we have

$$BJ = \frac{BJ}{BJ + JD} BD = \frac{1}{1 + \frac{AD}{AB}} BD = \frac{c}{c + s - a} BD,$$

and similarly $CI = \frac{b}{b+s-a} CE$. Since $BD < CE$, by Lemma 1, and since $c < b$ and $s > a$ implies $\frac{c}{c+s-a} < \frac{b}{b+s-a}$, it follows that $BJ < CI$.

To prove the second statement, we first find an expression for BF. By the law of cosines in $\triangle ABF$,

$$\begin{aligned} BF^2 &= AF^2 + AB^2 - 2AF \cdot AB \cos(\angle BAF) \\ &= AF^2 + c^2 - 2cAF \cos\left(\frac{A}{2} + \frac{\pi}{2}\right) \\ &= AF^2 + 2cAF \sin\frac{A}{2} + c^2. \end{aligned}$$

By the proof of Theorem 3 above, we have $AF = \frac{2c(s-a)\sin\frac{A}{2}}{s-b}$. Thus,

$$\begin{aligned} BF^2 &= \frac{4c^2(s-a)^2 \sin^2\frac{A}{2}}{(s-b)^2} + \frac{4c^2(s-a)\sin^2\frac{A}{2}}{s-b} + c^2 \\ &= \frac{4c^2(s-a)[(s-a)+(s-b)]\sin^2\frac{A}{2}}{(s-b)^2} + c^2 \\ &= \frac{4c^3(s-a)\sin^2\frac{A}{2}}{(s-b)^2} + c^2 = \frac{4c^2(s-a)(s-c)}{b(s-b)} + c^2, \end{aligned}$$

where we have applied Lemma 2 in the last equality.

Noting the comparable expression for CG^2, it follows that $BF > CG$ if and only if

$$\frac{4c^2(s-a)(s-c) + bc^2(s-b)}{b(s-b)} > \frac{4b^2(s-a)(s-b) + b^2c(s-c)}{c(s-c)},$$

which we rewrite as

$$4(s-a)(c^3(s-c)^2-b^3(s-b)^2) > bc(b^2-c^2)(s-b)(s-c). \tag{1}$$

Dividing both sides of (1) by $b-c$ gives

$$4(s-a)f(b,c) > bc(b+c)(s-b)(s-c), \tag{2}$$

where

$$f(b,c) = -(b^2+bc+c^2)s^2 + 2s(b^2+c^2)(b+c) - (b^4+b^3c+b^2c^2+bc^3+c^4).$$

Simplifying the expression for $f(b,c)$ using $s = \frac{a+b+c}{2}$, inequality (2) is equivalent to

$$bc(b^2+bc+c^2)+a(b^2+c^2)(b+c)-\frac{1}{4}(b^2+bc+c^2)(a+b+c)^2 > \frac{bc(b+c)(s-b)(s-c)}{4(s-a)}. \tag{3}$$

Dividing both sides by b^2+bc+c^2, and noting $bc \le \frac{1}{2}(b^2+c^2)$, to show (3), it suffices to show

$$bc + \frac{2}{3}a(b+c) - \frac{1}{4}(a^2+2a(b+c)+(b+c)^2) > \frac{(b+c)(s-b)(s-c)}{12(s-a)}.$$

Let $b+c = 2\ell$ where $\ell > a$, with $b = \ell + y$ and $c = \ell - y$ for some $0 < y < \frac{a}{2}$. Rewriting the last inequality in terms of y and ℓ, we have

$$-\frac{a^2}{4} - y^2 > \frac{\ell\left(\frac{a^2}{4}-y^2\right)}{3(2\ell-a)} - \frac{a\ell}{3}. \tag{4}$$

To prove (4), we consider the cases $a < \ell < \frac{3a}{2}$ or $\ell \ge \frac{3a}{2}$. In the first case, since $c = \ell - y \ge a$, by assumption, it suffices to verify (4) when $y = \ell - a$. That is, we must show

$$-\frac{a^2}{4} + \left(\frac{\ell}{3(2\ell-a)} - 1\right)(\ell-a)^2 > \frac{a^2\ell}{12(2\ell-a)} - \frac{a\ell}{3}, \qquad a < \ell < \frac{3a}{2}. \tag{5}$$

Clearing fractions in (5), rearranging, and dividing by $\ell - a$, we see that (5) is equivalent to $4(3a-5\ell)(\ell-a) > a(3a-8\ell)$ for $a < \ell < \frac{3a}{2}$, which is readily verified. If $\ell \ge \frac{3a}{2}$, then since $0 < y < \frac{a}{2}$, to show (4) in this case, it suffices to show that it holds (possibly with equality) when $y = \frac{a}{2}$, which is seen to be the case for all such ℓ. This implies inequality (4), and thus (3), which completes the proof. $\square$

Remark: The second part of the prior theorem may not hold if b or c less than a is allowed. To show this, let $a = 1$ and $\epsilon := b - c$ where $0 < \epsilon < 1$. If ϵ is fixed, then both sides of (2) are seen to be functions of $s > 1$. Let $\epsilon = .04$. If $b = .54$ and $c = .50$, then $s = 1.02$ and a direct calculation shows that the right side of (2) exceeds the left. If $b = 1.54$ and $c = 1.5$, then $s = 2.02$ and the left side of (2) exceeds the right in this case. By continuity, there exists some $s' \in (1.02, 2.02)$ such that there is equality in (2). Then pick b and c such that $b - c = .04$ and $b + c = 2s' - 1$ which yields a triangle ABC having $BF = CG$ with $AB \ne AC$. Similar examples can be found for other small ϵ. On the other hand, if $AB, AC \geq BC$, then $BF = CG$ implies $AB = AC$, by the previous theorem.

3. Other monotonicity results

In this section, we establish some comparable monotonicity results in which we consider pairs of segment lengths determined by the intersection of angle bisectors with Gergonne cevians. We first consider the case of exterior angle bisectors. In Figure 3 below, BD and CE are GC's in $\triangle ABC$. The exterior bisectors of $\angle C$ and $\angle B$ intersect BD and CE (extended) at F and G, respectively.

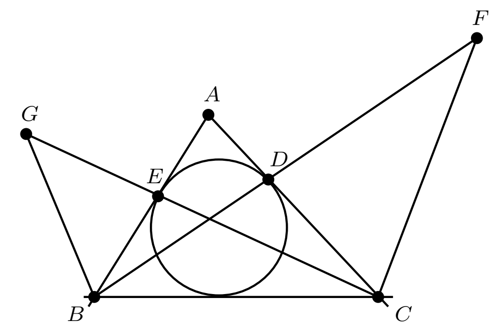

Figure 3. Intersection of GC's with the exterior angle bisectors of the respective angles.

Our next result compares the distances from F and G to the points B and C.

Theorem 5. *If $AB < AC$ in Figure 3, then (i) $BG < CF$ and (ii) $CG < BF$.*

Proof. To show (i), first note that $BG < CF$ if and only if $\frac{(s-b)\sin\frac{B}{2}}{s-c} < \frac{(s-c)\sin\frac{C}{2}}{s-b}$, by the formulas found in the proof of Theorem 3 above. The last inequality, together with the relations $s - b = r\cot\frac{B}{2}$ and $s - c = r\cot\frac{C}{2}$ where r is the inradius of $\triangle ABC$, implies $\cot^2\frac{B}{2}\sin\frac{B}{2} < \cot^2\frac{C}{2}\sin\frac{C}{2}$, i.e., $\cot\frac{B}{2}\cos\frac{B}{2} < \cot\frac{C}{2}\cos\frac{C}{2}$. That $BG < CF$ follows from $\angle B > \angle C$ and the fact that cosine and cotangent are decreasing on $(0, \frac{\pi}{2})$.

For (ii), note first note that by the law of cosines in $\triangle BCF$ and the formula for CF, we have that BF^2 equals

$$\begin{aligned} & CF^2 + BC^2 - 2CF \cdot BC\cos\left(\frac{C}{2} + \frac{\pi}{2}\right) \\ = & \frac{4a^2(s-c)^2\sin^2\frac{C}{2}}{(s-b)^2} + \frac{4a^2(s-c)\sin^2\frac{C}{2}}{s-b} + a^2 \\ = & \frac{4a^2(s-c)[(s-c)+(s-b)]\sin^2\frac{C}{2}}{(s-b)^2} + a^2 \\ = & \frac{4a^3(s-c)\sin^2\frac{C}{2}}{(s-b)^2} + a^2. \end{aligned}$$

Thus $CG < BF$ if and only if

$$\frac{(s-b)\sin^2\frac{B}{2}}{(s-c)^2} < \frac{(s-c)\sin^2\frac{C}{2}}{(s-b)^2},$$

which implies $\cot^3\frac{B}{2}\sin^2\frac{B}{2} < \cot^3\frac{C}{2}\sin^2\frac{C}{2}$. The last inequality is true since $\cot\frac{B}{2}\cos^2\frac{B}{2} < \cot\frac{C}{2}\cos^2\frac{C}{2}$ as $\pi > \angle B > \angle C > 0$, which completes the proof. □

Before proving our next result, we will need the following trigonometric inequality.

Lemma 6. *If $\angle B > \angle C$ in $\triangle ABC$, then*

$$\frac{\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}} < \frac{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{C}{2} + \sin B + \sin(B+C)}{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{B}{2} + \sin C + \sin(B+C)} < \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}. \tag{6}$$

Proof. To show the right-hand inequality in (6), note first that

$$\sin\frac{C}{2}(\sin B+\sin(B+C)) = 2\sin\frac{C}{2}\sin\left(B+\frac{C}{2}\right)\cos\frac{C}{2} = \sin C\sin\left(B+\frac{C}{2}\right)$$

and $\sin\frac{B}{2}(\sin C + \sin(B+C)) = \sin B\sin\left(\frac{B}{2}+C\right)$. Also, we have

$$\begin{aligned}\sin\frac{C}{2}\tan\frac{C}{2} - \sin\frac{B}{2}\tan\frac{B}{2} &= \frac{1-\cos^2\frac{C}{2}}{\cos\frac{C}{2}} - \frac{1-\cos^2\frac{B}{2}}{\cos\frac{B}{2}}\\ &= \frac{\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)\left(1+\cos\frac{B}{2}\cos\frac{C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}.\end{aligned}$$

Subtracting, to prove the right inequality in (6), we must show

$$\begin{aligned}&\sin C\sin\left(B+\frac{C}{2}\right) - \sin B\sin\left(\frac{B}{2}+C\right)\\ &\quad+\frac{2\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)\left(1+\cos\frac{B}{2}\cos\frac{C}{2}\right)\sin^2\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}} < 0.\end{aligned} \tag{7}$$

Since $\angle B > \angle C$ and cosine is decreasing, the third term on the left-hand side of (7) is negative. Thus, to show (7), it is enough to show $\sin C\sin\left(B+\frac{C}{2}\right) < \sin B\sin\left(\frac{B}{2}+C\right)$.

Applying the product-to-difference cosine formula to both sides of this last inequality, and rearranging, gives

$$\cos\left(\frac{C}{2}-B\right) - \cos\left(\frac{B}{2}-C\right) < \cos\left(\frac{3C}{2}+B\right) - \cos\left(\frac{3B}{2}+C\right).$$

By the difference-to-product sine formula, this preceding inequality may be rewritten as

$$-\sin\left(\frac{3(B-C)}{4}\right)\sin\left(\frac{B+C}{4}\right) < \sin\left(\frac{5(B+C)}{4}\right)\sin\left(\frac{B-C}{4}\right). \tag{8}$$

To show (8), first note that since $\angle B > \angle C$, the left side of (8) is negative whereas the right side is non-negative if $B+C \leq \frac{4\pi}{5}$. On the other hand, if $B+C > \frac{4\pi}{5}$, then $-\sin\left(\frac{5(B+C)}{4}\right) = \sin\left(\frac{5(B+C)}{4} - \pi\right) < \sin\left(\frac{B+C}{4}\right)$ since $\angle B + \angle C < \pi$ and sine is increasing in the first quadrant. Furthermore, $\sin\left(\frac{B-C}{4}\right) < \sin\left(\frac{3(B-C)}{4}\right)$ since $0 < \frac{B-C}{4} < \frac{3(B-C)}{4} < \pi - \left(\frac{B-C}{4}\right)$, whence (8) follows in this case by multiplication. This implies inequality (7), which completes the proof of the right inequality in (6).

To prove the left inequality in (6), first note that if $\frac{a}{b} > \frac{c}{d}$ with $\frac{c}{d} < 1$, then $\frac{a+x}{b+x} > \frac{c}{d}$ if all variables represent positive quantities. Thus it suffices to show

$$\frac{\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}} < \frac{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{C}{2} + \sin B}{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{B}{2} + \sin C}. \tag{9}$$

By the sine and cosine double-angle formulas, inequality (9) may be rewritten as

$$\begin{aligned}(\sin B - \sin C)\sin^2\left(\frac{B+C}{2}\right) &< \sin B\cos^2\frac{C}{2} - \sin C\cos^2\frac{B}{2} \\ &= \sin B\left(\frac{1+\cos C}{2}\right) - \sin C\left(\frac{1+\cos B}{2}\right) \\ &= \frac{1}{2}(\sin B - \sin C) + \frac{1}{2}\sin(B-C).\end{aligned}$$

Since $\cos 2x = 1 - 2\sin^2 x$, the last inequality is equivalent to

$$-(\sin B - \sin C)\cos(B+C) < \sin(B-C),$$

i.e.,

$$-2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)\cos(B+C) < 2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B-C}{2}\right). \tag{10}$$

Since $\angle B > \angle C$, inequality (10) is seen to hold as $0 < \cos\left(\frac{B+C}{2}\right) < \cos\left(\frac{B-C}{2}\right)$ with $-\cos(B+C) < 1$, which completes the proof. $\square$

We will also need the following well-known collinearity result known as the Theorem of Menelaus (see, e.g., [3, p. 66]).

Lemma 7. *If ABC is a triangle and D is on an extension of AB, E is on side BC, and F is on side AC, then the three points D, E, and F are collinear if and only if*

$$\left(\frac{AD}{DB}\right)\left(\frac{BE}{EC}\right)\left(\frac{CF}{FA}\right) = -1. \tag{11}$$

Here, the measure of a segment in one direction is considered to be the opposite of its measure in the other. In practice, when using the "only if" direction of this result, one often takes the absolute value of both sides of (11) and no longer considers segments as signed.

We now consider the comparable version of the prior problem wherein exterior are replaced by interior angle bisectors. In Figure 4 below, the interior angle bisectors BH and CJ intersect Gergonne cevians CE and BD at Q and P, respectively.

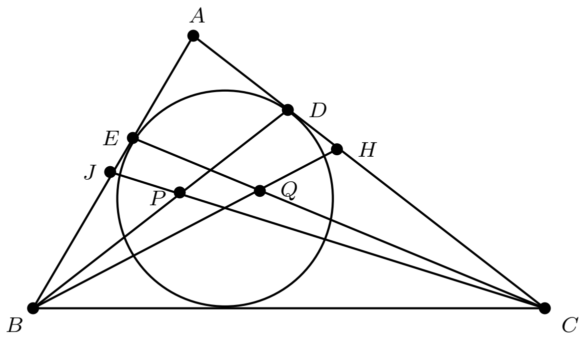

Figure 4. Intersection of GC's with the interior angle bisectors of the respective angles.

In our next result, we compare the internal segments of BH determined by Q with those of CJ determined by P.

Theorem 8. *If $AB < AC$ in Figure 4, then $BQ < CP$. If $\angle A \geq \frac{\pi}{3}$ in $\triangle ABC$, then $AB < AC$ implies $JP < HQ$.*

Proof. By Lemma 7 applied to $\triangle AJC$ with transversal $\overleftrightarrow{BPD}$, we have $\frac{AB}{BJ} \cdot \frac{JP}{PC} \cdot \frac{CD}{DA} = 1$. Note that CJ bisecting $\angle C$ implies $\frac{AB}{BJ} = \frac{AJ+BJ}{BJ} = \frac{b}{a} + 1$. Also, $CD = s - c = r \cot \frac{C}{2}$ and $DA = s - a = r \cot \frac{A}{2} = r \tan\left(\frac{B+C}{2}\right)$. It follows that

$$\frac{JP}{PC} = \frac{BJ}{AB} \cdot \frac{DA}{CD} = \frac{\tan \frac{C}{2} \tan\left(\frac{B+C}{2}\right)}{\frac{b}{a} + 1} = \frac{\tan \frac{C}{2} \tan\left(\frac{B+C}{2}\right) \sin(B + C)}{\sin B + \sin(B + C)},$$

where we have used the fact $\frac{b}{a} = \frac{\sin B}{\sin(B+C)}$ in the last equality. By a formula for the length of the angle bisector, we have $CJ = \frac{2ab \cos \frac{C}{2}}{a+b} = \frac{2a \sin B \cos \frac{C}{2}}{\sin B + \sin(B+C)}$ and thus

$$\begin{aligned} CP &= \left(\frac{CP}{CP + JP}\right) CJ \\ &= \frac{\sin B + \sin(B + C)}{\sin B + \sin(B + C) + \tan \frac{C}{2} \tan\left(\frac{B+C}{2}\right) \sin(B + C)} \cdot \frac{2a \sin B \cos \frac{C}{2}}{\sin B + \sin(B + C)} \\ &= \frac{a \sin B \sin C}{\sin \frac{C}{2} \left(\sin B + \sin(B + C) + \tan \frac{C}{2} \tan\left(\frac{B+C}{2}\right) \sin(B + C)\right)}, \end{aligned}$$

where we have used the identity $\sin 2x = 2 \sin x \cos x$ in the last equality. Noting the comparable expression for BQ, the inequality $BQ < CP$ follows from the right inequality in Lemma 6.

For the second statement, first note that by the formula for the ratio $\frac{JP}{PC}$ found above, we have

$$\begin{aligned}
JP &= \left(\frac{JP}{CP+JP}\right) CJ \\
&= \left(\frac{\tan\frac{C}{2}\tan\left(\frac{B+C}{2}\right)\sin(B+C)}{\sin B+\sin(B+C)+\tan\frac{C}{2}\tan\left(\frac{B+C}{2}\right)\sin(B+C)}\right)\left(\frac{2a\sin B\cos\frac{C}{2}}{\sin B+\sin(B+C)}\right) \\
&= \frac{a\sin B\sin C\tan\left(\frac{B+C}{2}\right)\sin(B+C)}{\cos\frac{C}{2}(\sin B+\sin(B+C))\left(\sin B+\sin(B+C)+\tan\frac{C}{2}\tan\left(\frac{B+C}{2}\right)\sin(B+C)\right)},
\end{aligned}$$

where we have used $\sin C = 2\tan\frac{C}{2}\cos^2\frac{C}{2}$. Noting the comparable expression for HQ, it follows that $JP < HQ$ if and only if

$$\begin{aligned}
&\frac{\cos\frac{B}{2}}{(\sin B+\sin(B+C))\left(\sin B+\sin(B+C)+2\tan\frac{C}{2}\sin^2\left(\frac{B+C}{2}\right)\right)} \\
&< \frac{\cos\frac{C}{2}}{(\sin C+\sin(B+C))\left(\sin C+\sin(B+C)+2\tan\frac{B}{2}\sin^2\left(\frac{B+C}{2}\right)\right)}. \qquad (12)
\end{aligned}$$

Since $\sin x+\sin(x+y) = 2\sin\left(x+\frac{y}{2}\right)\cos\frac{y}{2}$, inequality (12) may be rewritten as

$$\frac{\sin\left(\frac{B}{2}+C\right)\cos^2\frac{B}{2}}{\sin\left(B+\frac{C}{2}\right)\cos^2\frac{C}{2}} < \frac{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{C}{2}+\sin B+\sin(B+C)}{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{B}{2}+\sin C+\sin(B+C)}. \qquad (13)$$

Since $\angle B > \angle C$ and $\angle A \geq \frac{\pi}{3}$, we have $0 < \frac{B}{2}+C < B+\frac{C}{2} \leq \pi-\left(\frac{B}{2}+C\right)$ and therefore $0 < \sin\left(\frac{B}{2}+C\right) \leq \sin\left(B+\frac{C}{2}\right)$. Thus inequality (13) follows from the left inequality in Lemma 6, which completes the proof. □

Remark: The second part of the prior theorem may not hold if $\angle B+\angle C > \frac{2\pi}{3}$. For example, if $\angle B = \frac{\pi}{2}$ and $\angle C = \frac{22\pi}{45}$, then the left-hand side of (13) is seen to be strictly larger than the right. If $\angle B = \frac{\pi}{2}$ and $\angle C \leq \frac{\pi}{6}$, then we know from the previous theorem that the right side of (13) is larger. Thus, by continuity, there exists a triangle ABC in which $JP = HQ$ with $AB \neq AC$. On the other hand, if $\angle A \geq \frac{\pi}{3}$, then $JP = HQ$ implies $AB = AC$.

Our final two results compare the lengths of the internal segments within the GC's that are determined by the points P and Q in Figure 4 above. At this point, we will assume without loss of generality that $a = BC = 1$ and thus $s = \frac{b+c+1}{2}$.

Theorem 9. *If $AB < AC$ in Figure 4, then $BP < CQ$.*

Proof. We first find an expression for BP. By the angle bisector theorem, we have

$$BP = \frac{BP}{BP+DP}BD = \frac{BC}{BC+DC}BD = \frac{1}{1+s-c}BD,$$

and thus by the law of cosines in $\triangle BCD$ and $\triangle ABC$,

$$\begin{aligned} BP^2 &= \frac{1}{(1+s-c)^2}((s-c)^2 + 1 - 2(s-c)\cos C) \\ &= \frac{1}{b(1+s-c)^2}(b(s-c)^2 + b - (s-c)(1+b^2-c^2)). \end{aligned}$$

By the comparable expression for CQ^2, it follows that $BP < CQ$ if and only if

$$\begin{aligned} & c(1+s-b)^2(b(s-c)^2 + b - (s-c)(1+b^2-c^2)) \\ & < b(1+s-c)^2(c(s-b)^2 + c - (s-b)(1+c^2-b^2)). \end{aligned} \tag{14}$$

To show (14), we first rewrite it as

$$\begin{aligned} & bc((s-c)^2 - (s-b)^2 + 2(b-c)(s-b)(s-c)) \\ & + bc(2(c-b) + (s-b)^2 - (s-c)^2) \\ & + b(1+s-c)^2(s-b) - c(1+s-b)^2(s-c) \\ & + (c^2-b^2)(c(1+s-b)^2(s-c) + b(1+s-c)^2(s-b)) < 0. \end{aligned} \tag{15}$$

Note that $(s-c)^2 - (s-b)^2 = (2s-b-c)(b-c) = b-c$ and

$b(1+s-c)^2(s-b) - c(1+s-b)^2(s-c) = (c-b)(b+c-s) + (b-c)(s-b)(s-c)(s+2)$.

Dividing both sides of (15) by $b-c>0$, we then have

$$\begin{aligned} & bc(1+2(s-b)(s-c)) - 3bc + s - b - c + (s+2)(s-b)(s-c) \\ & - (b+c)(c(1+s-b)^2(s-c) + b(1+s-c)^2(s-b)) \\ = {} & 2bc((s-b)(s-c)-1) + 1 - s + (s+2)(s-b)(s-c) \\ & - (b+c)(c(1+s-b)^2(s-c) + b(1+s-c)^2(s-b)) < 0. \end{aligned} \tag{16}$$

To show (16), since $s-b, s-c < 1$ and $s > 1$, it is enough to show

$$(s+2)(s-b)(s-c) - (b+c)(c(1+s-b)^2(s-c) + b(1+s-c)^2(s-b)) < 0. \tag{17}$$

Observe that

$$\begin{aligned} & c(1+s-b)^2(s-c) + b(1+s-c)^2(s-b) \\ & > c(s-c) + b(s-b) + 2(b+c)(s-b)(s-c) \\ & > c(s-c)(s-b) + b(s-b)(s-c) + 2(b+c)(s-b)(s-c) \\ & = 3(b+c)(s-b)(s-c). \end{aligned}$$

For (17), it then suffices to show

$$(s+2)(s-b)(s-c) - 3(b+c)^2(s-b)(s-c) < 0,$$

i.e.,

$$(s+2-3(2s-1)^2)(s-b)(s-c) = (s-1)(1-12s)(s-b)(s-c) < 0,$$

which is true since $s > 1$. This establishes inequality (14) and completes the proof. $\square$

A similar result applies when comparing the segments DP and EQ in the case of an acute triangle.

Theorem 10. *If $\triangle ABC$ is acute and $AB < AC$ in Figure 4, then $EQ < DP$.*

Proof. First note that $DP^2 > EQ^2$ if and only if

$$\begin{aligned}&\frac{(s-c)^2}{b(1+s-c)^2}(b(s-c)^2+b-(s-c)(1+b^2-c^2))\\ &>\frac{(s-b)^2}{c(1+s-b)^2}(c(s-b)^2+c-(s-b)(1+c^2-b^2)),\end{aligned}$$

which may be rewritten as

$$\begin{aligned}&bc[(s-c)^4-(s-b)^4+2(s-b)(s-c)((s-c)^3-(s-b)^3)\\ &+(s-b)^2(s-c)^2((s-c)^2-(s-b)^2)]\\ &+bc[(s-c)^2-(s-b)^2+2(s-b)(s-c)((s-c)-(s-b))]\\ &+b(1+s-c)^2(s-b)^3-c(1+s-b)^2(s-c)^3\\ &+(c^2-b^2)(c(1+s-b)^2(s-c)^3+b(1+s-c)^2(s-b)^3)\\ &>0. \qquad (18)\end{aligned}$$

One may verify

$$b(s-b)^3-c(s-c)^3=s^3(b-c)+3s^2(c^2-b^2)+3s(b^3-c^3)+c^4-b^4,$$

$$b(s-b)^3(s-c)-c(s-b)(s-c)^3=(s-b)(s-c)(s^2(b-c)+2s(c^2-b^2)+b^3-c^3),$$

and

$$b(s-b)^3(s-c)^2-c(s-b)^2(s-c)^3=(s-b)^2(s-c)^2(s(b-c)+c^2-b^2).$$

Dividing both sides of (18) by $b-c$, and noting $b+c=2s-1$, we then have

$$\begin{aligned}&bc(2+2(s-b)(s-c)-(s-b)^2(s-c)^2)+s^3-3s^2(2s-1)+3s((2s-1)^2-bc)\\ &-(2s-1)((2s-1)^2-bc)+2(s-b)(s-c)(s^2-2s(2s-1)+(2s-1)^2-bc)\\ &+(s-b)^2(s-c)^2(1-s)-(b+c)(c(1+s-b)^2(s-c)^3+b(1-s-c)^2(s-b)^3)\\ &>0,\end{aligned}$$

i.e.,

$$\begin{aligned}&bc(s-(s-b)^2(s-c)^2)-s^3+(3+2(s-b)(s-c))s^2-(3+4(s-b)(s-c)\\ &+(s-b)^2(s-c)^2)s+(1+(s-b)(s-c))2-(b+c)(c(1+s-b)^2(s-c)^3\\ &+b(1+s-c)^2(s-b)^3)>0. \qquad (19)\end{aligned}$$

Let $b=\frac{2s-1}{2}+x$ and $c=\frac{2s-1}{2}-x$ for some $0<x<\frac{1}{2}$. Then $bc=s^2-s+\frac{1}{4}-x^2$, with $s-b=\frac{1}{2}-x$ and $s-c=\frac{1}{2}+x$. Substituting this into (19), and simplifying, gives

$$
\begin{aligned}
&(2+2\gamma-\gamma^2)s^2-3(1+\gamma)s-3\gamma^3+(1+\gamma)^2\\
&-(2s-1)\left[\left(s-\frac{1}{2}-x\right)\left(\frac{3}{2}-x\right)^2\left(\frac{1}{2}+x\right)^3+\left(s-\frac{1}{2}+x\right)\left(\frac{3}{2}+x\right)^2\left(\frac{1}{2}-x\right)^3\right]\\
&>0,
\end{aligned}
$$

where $\gamma = \frac{1}{4} - x^2$.

Since $\triangle ABC$ is acute, we have $b^2 + c^2 > 1$ and $c^2 + 1 > b^2$. Translating these conditions in terms of x and s, to complete the proof, we must show $f(x, s) > 0$, for x and s satisfying

$$\frac{1}{2}+\sqrt{\frac{1}{2}-x^2}<s<\frac{1}{2}+\frac{1}{4x},\quad 0<x<\frac{1}{2},$$

where

$$
\begin{aligned}
&f(x,s)\\
&=\left(\frac{39}{16}-\frac{3}{2}x^2-x^4\right)s^2-\left(\frac{15}{4}-3x^2\right)s-\left(\frac{1}{4}-x^2\right)^3+\left(\frac{5}{4}-x^2\right)^2\\
&-(2s-1)\left[\left(s-\frac{1}{2}-x\right)\left(\frac{3}{2}-x\right)^2\left(\frac{1}{2}+x\right)^3+\left(s-\frac{1}{2}+x\right)\left(\frac{3}{2}+x\right)^2\left(\frac{1}{2}-x\right)^3\right].
\end{aligned}
$$

In order to do so, we seek to minimize $f(x, s)$ over its domain C. We consider restrictions on C as follows. Given $0 < \delta < \frac{1}{2}$ (δ to be determined), let $D = D_\delta$ denote the closure of the set of points (x, s) in C such that $\delta < x < \frac{1}{2}$. By *Maple*, there exist no points (x, s) in $\mathbb{R}^2$ such that $\frac{df}{dx}(x, s) = \frac{df}{ds}(x, s) = 0$ with $0 < x < \frac{1}{2}$. This implies that the minimum value of f on the compact set D must occur on its boundary. Let $\alpha(x) = \frac{1}{2} + \frac{1}{4x}$ and $\beta(x) = \frac{1}{2} + \sqrt{\frac{1}{2} - x^2}$. First observe that

$$f(x,\alpha(x))=\frac{(2x+1)(2x-1)^3(16x^4+32x^3+32x^2-21)}{256x^2}.$$

Since the factor $16x^4 + 32x^3 + 32x^2 - 21$ is negative for $0 < x < \frac{1}{2}$, it is seen that $f(x, \alpha(x)) > 0$ for $\delta < x < \frac{1}{2}$. By *Maple*, the derivative of the function $g(x) = f(x, \beta(x))$ has no zeros on the interval $(0, \frac{1}{2})$ with $g(0) \approx .009$ and $g\left(\frac{1}{2}\right) = 0$, which implies $g(x) > 0$ on the interval.

We now consider the values of f on the third piece of the boundary of D, namely, along the vertical line $x = \delta$. Note first that

$$f(0,s)=\frac{21}{16}s^2-\frac{21}{8}s+\frac{81}{64}>0,\qquad s\geq\frac{1+\sqrt{2}}{2}.$$

If $h(s) = f(x, s)$ for a given x, then the position of the vertex of the graph of the parabolic curve $h(s)$ is seen to be a continuous function of the parameter x. Note that the coefficient of s^2 in $f(x,s)$ is given by

$$\frac{39}{16}-\frac{3}{2}x^2-x^4-2\left(\frac{3}{2}-x\right)^2\left(\frac{1}{2}+x\right)^3-2\left(\frac{3}{2}+x\right)^2\left(\frac{1}{2}-x\right)^3,$$

which can be shown to be positive for $0<x<\frac{1}{2}$, and thus the vertex of $h(s)$ always corresponds to a minimum. By continuity, we may then select $\delta^*>0$ such that for all $0<\delta<\delta^*$, we have $f(\delta,s)>0$ for all s such that $\beta(\delta)\le s\le\alpha(\delta)$. If $0<\delta<\delta^*$, it follows from the preceding that $f(x,s)\ge 0$ on D_δ, with equality occurring only at the point $(x,s)=\left(\frac{1}{2},1\right)$. In particular, we have $f(x,s)>0$ for all interior points of D_δ. Upon allowing δ to approach zero, it follows that $f(x,s)$ is positive for all points in C, which completes the proof. □

Remark: The preceding result need not hold if $\triangle ABC$ is obtuse. For example, $f(x,s)<0$, i.e., $DP<EQ$, if $x=.1$ and $s=1.1$, which corresponds to $\triangle ABC$ having side lengths $AB=.5$, $AC=.7$ and $BC=1$. Note that if $\triangle ABC$ is acute, then Theorem 10 implies 9, though not conversely, by Lemma 1.

References

[1] S. Abu-Saymeh, M. Hajja, and H. A. ShahAli, Another variation on the Steiner-Lehmus theme, *Forum Geom.*, 8 (2008) 131–140.
[2] H. S. M. Coxeter, *Introduction to Geometry*, John Wiley & Sons, Inc., 1961.
[3] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Random House, Inc., 1967.
[4] M. Hajja, A short trigonometric proof of the Steiner-Lehmus theorem, *Forum Geom.*, 8 (2008) 39–42.
[5] M. Hajja, Stronger forms of the Steiner-Lehmus theorem, *Forum Geom.*, 8 (2008) 157–161.
[6] T. Mansour and M. Shattuck, Some monotonicity results related to the Fermat point of a triangle, *Forum Geom.*, 16 (2016) 355–366.
[7] R. Oláh-Gál, and J. Sándor, On trigonometric proofs of the Steiner-Lehmus theorem, *Forum Geom.*, 9 (2009) 155–160.
[8] E. Rapaport, *Hungarian Problem Book I*, Random House, Inc., 1963.
[9] K. R. S. Sastry, A Gergonne analogue of the Steiner-Lehmus theorem, *Forum Geom.*, 5 (2005) 191–195.
[10] L. Sauvé, The Steiner-Lehmus theorem, *Crux Math.*, 2 (1976) 19–24.
[11] C. W. Trigg, A bibliography of the Steiner-Lehmus theorem, *Crux Math.*, 2 (1976) 191–193.

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