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Putting the Icosahedron into the Octahedron

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Abstract. We compute the dimensions of a regular tetragonal pyramid, which allows a cut by a plane along a regular pentagon. In addition, we relate this construction to a simple construction of the icosahedron and make a conjecture on the impossibility to generalize such sections of regular pyramids.

1. Pentagonal sections

The present discussion could be entitled *Organizing calculations with Menelaos*, and originates from a problem from the book of Sharygin [2, p. 147], in which it is required to construct a pentagonal section of a regular pyramid with quadrangular

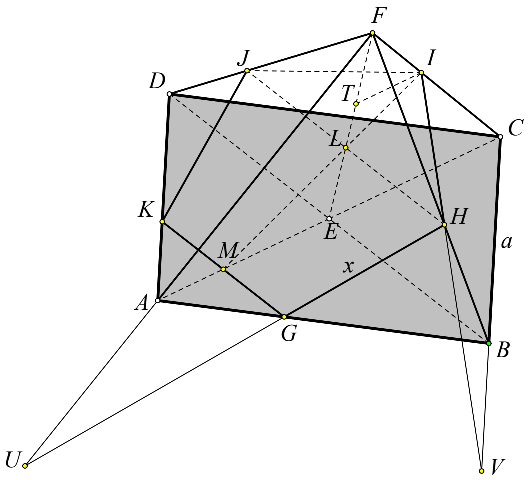

Figure 1. Pentagonal section of quadrangular pyramid

basis. The basis of the pyramid is a square of side-length a and the pyramid is assumed to be regular, i.e. its apex F is located on the orthogonal at the center E of the square at a distance $h = |EF|$ from it (See Figure 1). The exercise asks for the determination of the height h if we know that there is a section $GHIJK$ of the pyramid by a plane which is a regular pentagon. The section is tacitly assumed to be symmetric with respect to the diagonal symmetry plane AFC of the pyramid and defined by a plane through the three points K, G, I. The first two, K and G, lying symmetric with respect to the diagonal AC of the square. The third point I, lying on the edge CF. The, otherwise, excellent book, uses in this case some calculations, that I could not put in a systematic schedule somehow. So, I decided

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to do them in my way, trying to deduce them from some organizing principle. Here are the results of this effort, which, among other things, show that the exercise has some interesting extensions, but also prove that the corresponding calculations, made in the book, were not correct.

The organizing principle for doing the calculations is the theorem of Menelaos ([1, p.153]). We apply this theorem two times. The first on the triangle FBC, using the secant line VHI. This leads to the determination of the side x of the pentagon in terms of the side a of the square basis. Then we apply the theorem on the triangle ABF, using the secant UGH. This leads to the determination of the edge's length, $d = |AF|$, in terms of a and x and completes the proof of the following theorem.

Theorem 1. *A regular pyramid on a square basis of side a can be intersected by a plane along a regular pentagon lying symmetric with respect to one of its planes of symmetry if and only if it has equilateral faces i.e. the pyramid is a half regular octahedron.*

Before to delve into the details, let us fix the notation used in the figure shown. Points M and L are respectively the middles of KG and JH. Point L is located on the symmetry axis EF of the pyramid. Point T is the orthogonal projection of I on this axis. Finally, points U and V are respectively the intersections of line-pairs (AF, GH) and (BC, HI).

2. Menelaos once

The first application of the theorem of Menelaos to FBC with secant VHI leads to the equation

$$\frac{VB}{VC} \cdot \frac{IC}{IF} \cdot \frac{HF}{HB} = 1. \tag{1}$$

The three quotients entering the equation can be calculated easily:

$$\frac{VB}{VC} = \frac{ME}{MC} = \frac{ME}{ME+EC} = \frac{1}{1+\frac{EC}{ME}} = \frac{1}{1+\frac{a/\sqrt{2}}{a/\sqrt{2}-x/2}} = \frac{\sqrt{2}a - x}{2\sqrt{2}a - x}, \tag{2}$$

$$\frac{IC}{IF} = \frac{TE}{TF} = \frac{FE-FT}{TF} = \frac{FE}{TF} + 1 = 1 - \frac{EC}{TI} = 1 - \frac{EC}{ME}\frac{ME}{TI}. \tag{3}$$

In the last expression we can replace

$$\frac{EC}{ME} = \frac{a/\sqrt{2}}{a/\sqrt{2}-x/2}, \quad \frac{ME}{TI} = \frac{ML}{LI} = \phi, \tag{4}$$

where $\phi = \frac{\sqrt{5}+1}{2}$ the golden section. Introducing this into equation (3) we obtain

$$\frac{IC}{IF} = \frac{\sqrt{2}a(1-\phi) - x}{\sqrt{2}a - x}. \tag{5}$$

Finally the third quotient entering (1) evaluates to

$$\frac{HF}{HB} = \frac{HF}{FB-FH} = \frac{1}{\frac{FB}{HF}+1} = \frac{1}{1-\frac{\sqrt{2}a}{\phi x}} = \frac{\phi x}{\phi x - \sqrt{2}a}. \tag{6}$$

Using equations (2), (5) and (6), equation (1) transforms to

$$\frac{\sqrt{2}a - x}{2\sqrt{2}a - x} \cdot \frac{\sqrt{2}a(1-\phi) - x}{\sqrt{2}a - x} \cdot \frac{\phi x}{\phi x - \sqrt{2}a} = 1, \tag{7}$$

which, by taking into account the equation satisfied by $\phi: \quad \phi^2 - \phi - 1 = 0$, and simplifying reduces to

$$x = \frac{\sqrt{2}a}{\phi + 1}. \tag{8}$$

3. Menelaos twice

Before to start with the second application of Menelaos theorem, let us compute the side $y = |UA|$ of the triangle UAG lying on the plane of the face ABF. The side UG has the length ϕx of the diagonal JH of the regular pentagon. Also $|AG| = x/\sqrt{2}$ and the $\cos(\omega)$, where $\omega = \widehat{UAG}$, is

$$\cos(\omega) = -\cos(\pi - \omega) = -\frac{a}{2d}.$$

Thus, the cosine theorem applied to side UG of triangle UAG leads to equation

$$\begin{aligned} (\phi x)^2 &= y^2 + \frac{x^2}{2} + 2y\frac{x}{\sqrt{2}}\frac{a}{2d} \\ \Leftrightarrow y^2 &+ \frac{ax}{\sqrt{2}d}y + (x^2/2 - (\phi x)^2) = 0. \end{aligned} \tag{9}$$

Now, coming to the Menelaos equation for triangle ABF and the secant UGH we have:

$$\frac{UA}{UF} \cdot \frac{HF}{HB} \cdot \frac{GB}{GA} = 1. \tag{10}$$

Using equation (6) for the middle factor and computing the last one we get for their product

$$\frac{GB}{GA} = \frac{AB - AG}{GA} = \frac{AB}{GA} + 1 = 1 - \frac{a}{(x/\sqrt{2})} = \frac{x - \sqrt{2}a}{x}, \tag{11}$$

$$\frac{HF}{HB} \cdot \frac{GB}{GA} = \frac{\phi x}{\phi x - \sqrt{2}a} \cdot \frac{x - \sqrt{2}a}{x} = \frac{\phi(x - \sqrt{2}a)}{\phi x - \sqrt{2}a}. \tag{12}$$

On the other side, the first factor is

$$\frac{UA}{UF} = \frac{UA}{UA + AF} = \frac{1}{1 + \frac{AF}{UA}} = \frac{1}{1 + \frac{d}{y}}. \tag{13}$$

Combining equations (10), (12), (13) and using (8) for x we come at

$$1 + \frac{d}{y} = \frac{\phi(x - \sqrt{2}a)}{\phi x - \sqrt{2}a} = \frac{\phi(\frac{\sqrt{2}a}{\phi+1} - \sqrt{2}a)}{\phi\frac{\sqrt{2}a}{\phi+1} - \sqrt{2}a} = \phi^2 = \phi + 1,$$

which implies

$$d = y \cdot \phi. \tag{14}$$

This, taking into account (8) and (9), implies, after a short calculation

$$d^2 = a^2 \qquad \Leftrightarrow \qquad d = a. \tag{15}$$

This implies that the tetragonal pyramid has equilateral faces, i.e. it is a half-octahedron and its height is

$$h = \sqrt{d^2 - \frac{a^2}{2}} = \frac{a}{\sqrt{2}}. \tag{16}$$

This completes the proof of the first half of the assertion of the theorem, the other half being obvious, since the steps of the proof can be reversed, if we assume that the pyramid is a half regular octahedron.

4. Relation to Icosahedron

Using (8) in formula (5) and calculating similarly the other ratios we see that the golden ratio appears in all side divisions:

$$\frac{CI}{IF} = \frac{BG}{GA} = \frac{FJ}{JD} = \phi. \tag{17}$$

Drawing also the line orthogonal to the plane of the pentagon at its center O and taking its intersection N with the pyramid's edge AF, we see easily that $\frac{AN}{NF} = \phi$ and that the pentagonal pyramid $NGHIJK$ is the pentagonal gasket of the pla-

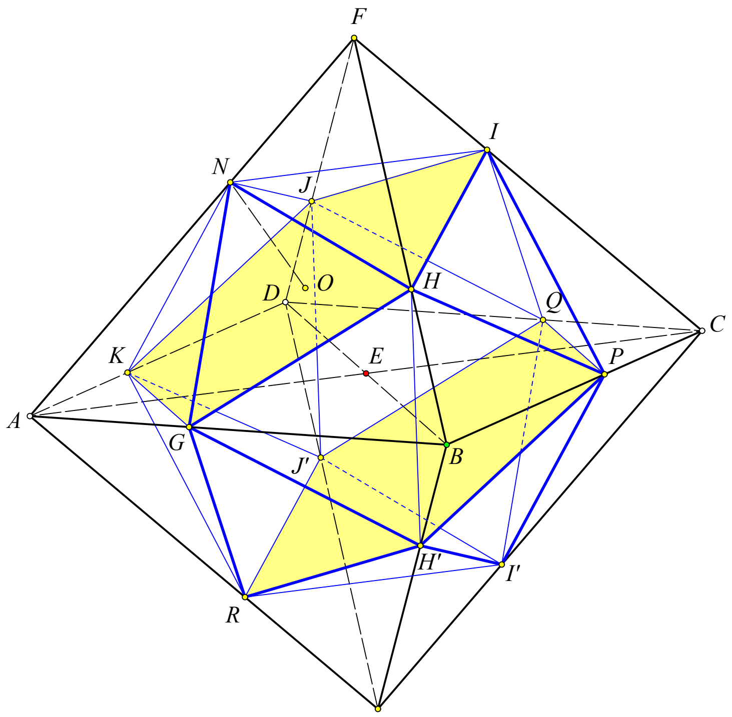

Figure 2. Icosahedron inscribed in the octahedron

tonic regular icosahedron (See Figure 2). The symmetric pentagon $KGH'I'J'$ with respect to the plane of the basic square produces analogously another pentagonal gasket, and the two gaskets can be easily completed to an icosahedron. This can be done by taking the symmetrics of these two pentagons with respect to the symmetry plane BFD of the pyramid. The vertices of the four, thus defined, pentagons determine the vertices of an icosahedron. This shows that the initial exercise relates to the inclusion of the icosahedron into the octahedron, defined by the square $ABCD$. The described procedure gives also a very simple method to construct the icosahedron. It suffices to construct the octahedron, which is simple, and divide its edges in the golden ratio using a certain rule suggested by the figure. The twelve resulting division points are the vertices of an icosahedron. Eight out

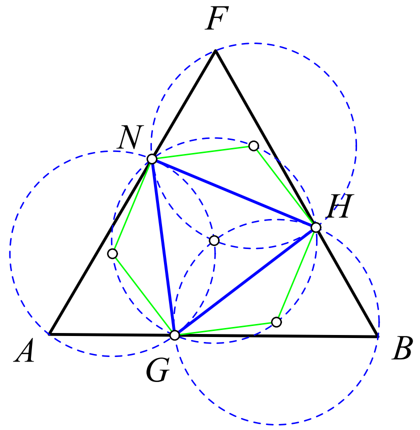

Figure 3. Face in face

of the twenty faces of the icosahedron are inscribed in corresponding faces of the octahedron (See Figure 3), the other twelve sitting entirely inside the octahedron.

5. Generalization

The original problem could be generalized to arbitrary regular pyramids. For triangular basis the answer is easy to supply. The only regular triangular pyramid admiting a square section is the regular tetrahedron, the section being the square of the middles of the segments joining the endpoints of two opposite lying edges (See Figure 4). This, assuming that one side HE of the square is parallel to the edge BC, we have that also the opposite side of the square FG is parallel to BC and since $\frac{AE}{AB} = \frac{HE}{BC} = \frac{FG}{BC} = \frac{DF}{DB}$, we have that EF and AD must be parallel. Since analogously $\frac{BE}{BA} = \frac{AE}{AB}$ we deduce that E and analogously F are the middles of the respective edges AB and BD. Thus, $x = |AE| = |EH| = |EF| = |EB| = |BF|$. This implies that the faces of the pyramid are equilateral triangles and shows the claim.

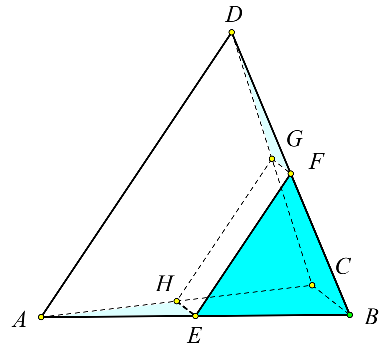

Figure 4. Square section of regular tetrahedron

At this point one is tempted to seek regular $(n+1)-$ sided polygon sections for regular pyramids with basis having $n > 4$ sides. The method used in the preceding paragraphs can be transferred in this more general question and used in order to determine the special (isosceles) triangles, which play the role of the faces of a pyramid allowing a regular polygonal section with $n+1$ sides. Next section, however shows that there are not hexagonal regular sections in a regular pentagonal pyramid and suggests the conjecture that there are no similarly constructible sections for $n > 4$.

6. Impossibility of hexagonal regular sections

In this section, working a bit more general than the case requires, we assume that the basis of the regular pyramid is a regular (odd) n-sided polygon and ask for the particular one which allows a plane-cut along an $(n+1)$-sided regular polygon. In the following discussion we think and calculate for general (odd) n, though the figure we refer to corresponds to $n = 5$ and ultimately it is for this figure that we

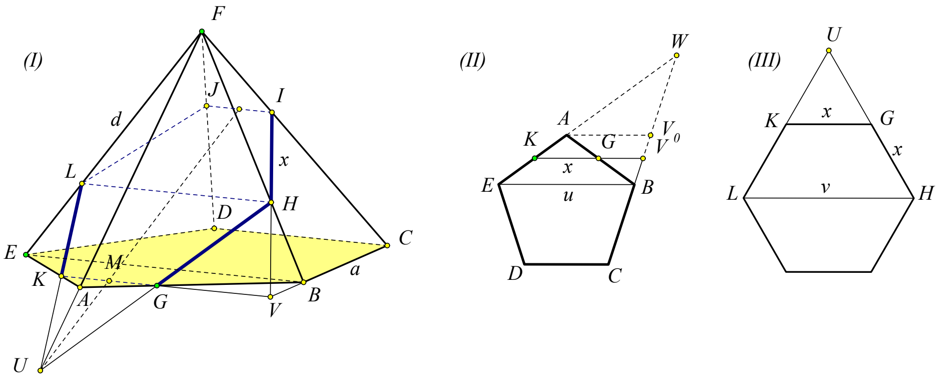

Figure 5. Pyramids with odd sided regular polygon basis

apply the results of the calculation. Our aim is to prove the following theorem (See Figure 5).

Theorem 2. *There is no regular pentagonal pyramid $FABCDE$ admitting a section symmetric with respect to one symmetry plane of the pyramid, which, in addition is a regular hexagon.*

For the proof we apply Menelaos theorem on the face BFC with secant VHI leading to equation:

$$\frac{VC}{VB}\cdot\frac{IF}{IC}\cdot\frac{HB}{HF}=1.$$

The first ratio entering the equation is again easily computed

$$\frac{VC}{VB}=\frac{VB+BC}{VB}=1+\frac{BC}{VB}. \tag{18}$$

By assumption the basis is a regular n-sided polygon, and, as suggested by the figure 5-(II), the length of VB can be computed in terms of the side $x=|KG|$ of the $(n+1)-$gonal section and $a=|AB|$ the side of the basis. In fact, introducing the half-angle ω of the n-sided basis and using the shortcut $s=\sin(\omega)$, we have

$$\begin{aligned}
\omega &= \frac{n-2}{2n}\pi, \qquad |AG|=\frac{x}{2s},\\
|GB| &= a-|AG|=\frac{2as-x}{2s},\\
|VB| &= \frac{|GB|}{|AB|}|BV_0|=\frac{2as-x}{2as}|BV_0|.
\end{aligned} \tag{19}$$

Referring to the figure 5-(II), the segment $u=|EB|=2a\sin(\omega)$ is the smallest diagonal of the $n-$gon and BV_0 satisfies

$$\begin{aligned}
\frac{BV_0}{BW} &= \frac{EA}{EW}=\frac{a}{a+\frac{a/2}{\cos(\pi-2\omega)}}=\frac{2\cos(2\omega)}{2\cos(2\omega)-1}\\
\Rightarrow\ |BV_0| &= \frac{EA}{EW}|BW|=\frac{2\cos(2\omega)}{2\cos(2\omega)-1}\cdot\frac{a/2}{\cos(\pi-2\omega)},\\
|BV_0| &= \frac{a}{1-2\cos(2\omega)}=\frac{a}{4s^2-1}.
\end{aligned}$$

which, by (18) and (19) gives

$$\begin{aligned}
|VB| &= \frac{2as-x}{2as}\cdot\frac{a}{4s^2-1}=\frac{2as-x}{2s(4s^2-1)}\\
\Rightarrow \frac{VC}{VB} &= 1+\frac{a}{|VB|}=\frac{8as^3-x}{2as-x}.
\end{aligned} \tag{20}$$

The second ratio entering the Menelaos theorem is

$$\frac{IF}{IC}=\frac{IF}{FC-FI}=\frac{1}{\frac{FC}{IF}+1}=\frac{1}{1-\frac{a}{x}}=\frac{x}{x-a}. \tag{21}$$

Finally the third ratio is

$$\frac{HB}{HF}=\frac{FB-FH}{HF}=\frac{FB}{HF}+1=1-\frac{EB}{LH}=1-\frac{u}{v}. \tag{22}$$

The length $v = |LH|$ is the smallest diagonal of the $(n+1)-$ sided cut of the pyramid and can be expressed in terms of x:

$$|KU| = \frac{x}{2\cos(\psi)}, \quad \frac{v}{x} = \frac{x + |KU|}{x} = \frac{2\cos(\psi)+1}{2\cos(\psi)} \Rightarrow v = x\frac{2c+1}{2c},$$

where we have set $\psi = \widehat{GKU}$ for the complement of the angle of the $(n+1)-$ sided regular polygon and used the shortcut $c = \cos(\psi)$, the relevant elements being displayed in figure 5-(III). Replacing this in (22) we get

$$\frac{HB}{HF} = \frac{x(2c+1) - 4asc}{x(2c+1)}. \tag{23}$$

By substitution into the Menelaos condition we obtain the equation

$$\frac{8as^3 - x}{2as - x} \cdot \frac{x}{x-a} \cdot \frac{x(2c+1) - 4asc}{x(2c+1)} = 1.$$

$$\Leftrightarrow \ \frac{x}{2s} = a\frac{16cs^3 - 2c - 1}{16cs^3 + 8s^3 - 2s - 2c - 1}. \tag{24}$$

If there were a hexagonal section, then the quantity $\frac{x}{2s} = |AG|$, should have the value on the right of the last equation. This, in the case $n = 5$, for which $s = \phi/2, c = 1/2$, evaluates to

$$a\frac{\phi^2 - 2}{2\phi^2 - \phi - 2} = a\frac{2\phi - 1}{3\phi} \approx 0.46065 \cdot a. \tag{25}$$

But this value leads to a contradiction visible in figure 6. In fact, the left figure

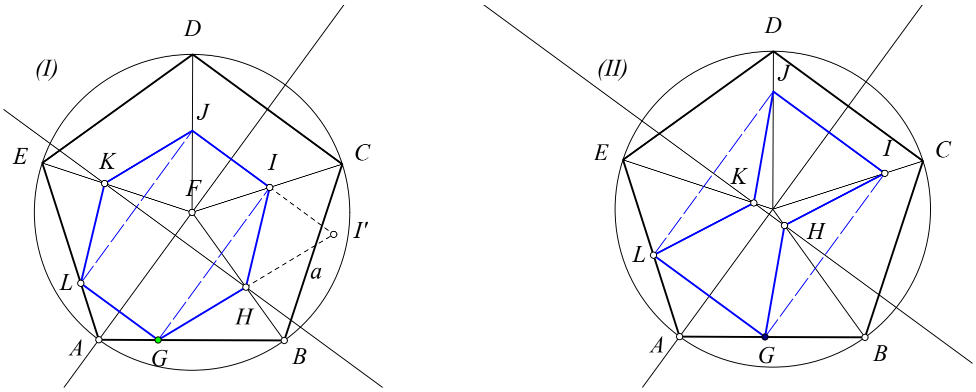

Figure 6. Impossible configuration

6-(I) shows how should look the vertical projection of the regular hexagon on the plane of the basis of the pyramid, if such an hexagon was existing. The right part (II) of the figure, however, shows how the projection should look, for the value $|AG| \approx 0.46065 \cdot a$ according to equation (25). In fact, opposite sides LG, IJ of the projected hexagon are equal segments and $LGIJ$ is a rectangle. The other two vertices project at points K, H, which coincide, respectively, with the intersections of the medial line of GI with the radii FE, FB of the circumcircle of the pentagon.

Thus, the shape of the projected hexagon is completely determined by the length of $|AG|$. However for the value obtained above, it is easy to show that the location of the intersection points K, H of the medial line of GI with the radii FE, FB results to a non-convex hexagon (See Figure 6-II), which is impossible.

References

[1] N. A. Court, *College Geometry*, Dover Publications Inc., New York, 1980.
[2] I. F. Sharygin, *Problems in Solid Geometry*, Mir, Moscow, 1986.

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