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An Improved Inequality for the Perimeter of a Quadrilateral

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Abstract. We demonstrate that a new inequality for the perimeter of a quadrilateral is strict even in the nonconvex case.

In [2] the following problem was proposed:

> Let $ABCD$ be a convex quadrilateral. Let E be the midpoint of AC, and let F be the midpoint of BD. Show that $|AB| + |BC| + |CD| + |DA| \geq |AC| + |BD| + 2|EF|$. (Here $|XY|$ denotes the distance from X to Y.)

In words, the perimeter of a convex quadrilateral is at least equal to the sum of the diagonals plus twice the length of the line segment connecting their midpoints. The solution [3] reveals that the statement is true even if the word *convex* is removed. In [5] it was stated without detailed proof that for a convex quadrilateral, the inequality is strict. We show here that the inequality is strict for any nondegenerate quadrilateral (i.e., for which no three vertices are collinear).

Theorem 1. *For any nondegenerate quadrilateral $ABCD$, let E be the midpoint of AC, and let F be the midpoint of BD. Then*

$$|AB| + |BC| + |CD| + |DA| > |AC| + |BD| + 2|EF|. \qquad (1)$$

Proof. The solution (see [3]) to the original problem even with *convex* removed follows immediately by letting A, B, C, $D \in \mathbb{C}$ and $x = A - B$, $y = B - C$, $z = C - D$ in Hlawka's inequality

$$|x| + |y| + |z| + |x + y + z| \geq |x + y| + |y + z| + |z + x|.$$

(Note: $|A - B|$ is equivalent to $|AB|$, and $|C - D + A - B| = 2\left|\frac{A+C}{2} - \frac{B+D}{2}\right| = 2|EF|$.) We now prove strictness. In Proof 1 of [1], multiplying both sides of Hlawka's inequality by $|x| + |y| + |z| + |x + y + z|$ leads to the equivalent form

$$\begin{aligned}
&(|x| + |y| - |x + y|)(|z| - |x + y| + |x + y + z|)\\
&+ (|y| + |z| - |y + z|)(|x| - |y + z| + |x + y + z|)\\
&+ (|z| + |x| - |z + x|)(|y| - |z + x| + |x + y + z|) \geq 0,
\end{aligned}$$

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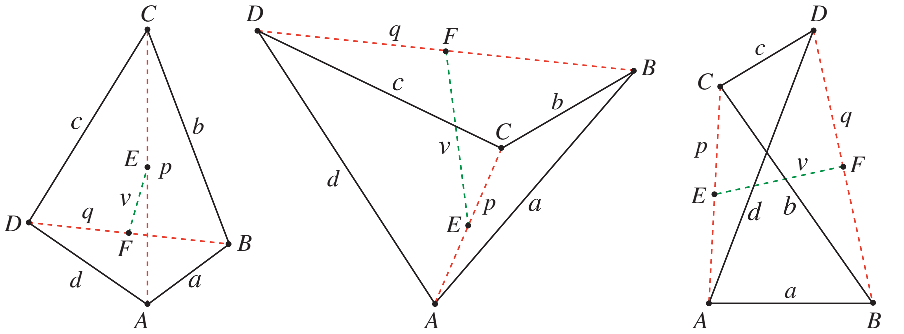

(a) (b) (c)

Figure 1

and since each of the six factors is nonnegative by the triangle inequality, Hlawka's inequality is proved. Inserting the given values,

$$\begin{aligned}&(|A-B|+|B-C|-|A-C|)(|C-D|-|A-C|+|A-D|)\\&+(|B-C|+|C-D|-|B-D|)(|A-B|-|B-D|+|A-D|)\\&+(|C-D|+|A-B|-|C-D+A-B|)\\&\quad\cdot(|B-C|-|C-D+A-B|+|A-D|)\geq 0.\end{aligned}$$

In order for equality to be attained, at least one factor in each of the three pairs of factors above must be equal to 0. Inspection reveals that for the first two pairs of factors, a factor is 0 only if three vertices of the quadrilateral are collinear. Since we are excluding degenerate cases, not just one but in fact the first two products are strictly positive. This proves the strictness in (1). □

There is a geometric interpretation, which may be easier to visualize using more convenient notation. Let $|AB|=a$, $|BC|=b$, $|CD|=c$, $|DA|=d$, $|AC|=p$, $|BD|=q$, $|EF|=v$. Then Theorem 1 can be written as

Theorem 1* *For any nondegenerate quadrilateral with consecutive sides a, b, c, d, diagonals p, q, and v the length of the line segment connecting the midpoints of the diagonals*,

$$a+b+c+d>p+q+2v. \tag{2}$$

The equivalent form of Hlawka's inequality is now

$$(a+b-p)(c-p+d) \tag{3}$$

$$+(b+c-q)(a-q+d) \tag{4}$$

$$+(c+a-2v)(b-2v+d)\geq 0. \tag{5}$$

Whether the quadrilateral is convex as in (a), concave as in (b), or crossing (i.e., nonsimple) as in (c) of Figure 1, the three terms in each factor of (3) and (4) are sides of a triangle and so each of these factors is strictly positive by the triangle inequality.

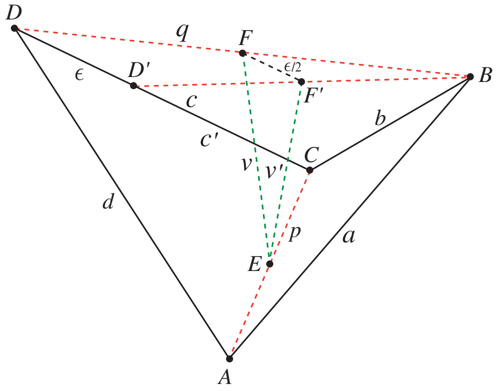

Figure 2

We now show that with but one exception, the product in (5) is strictly positive as well.

Theorem 2. *For any nondegenerate quadrilateral with consecutive sides a, b, c, d, let v be the length of the line segment connecting the midpoints of the diagonals. Then $a + c > 2v$ and $b + d > 2v$, except for a crossing quadrilateral with a pair of parallel sides, in which case the inequality involving the two parallel sides becomes an equality.*

Proof. We know from the proof of Theorem 1 that $a + c \geq 2v$ and $b + d \geq 2v$, so all that remains is to ascertain when equality holds. For $v = 0$ these inequalities obviously are strict, so assume $v > 0$ and $a + c = 2v$. Reverting to the earlier notation used for the third product pair in the proof of Theorem 1, $|C - D| + |A - B| = |C - D + A - B|$ implies that $C - D = \lambda(A - B)$, where $\lambda > 0$, which implies that the quadrilateral is crossing and $AB \parallel CD$, as claimed. Conversely, for any crossing quadrilateral with parallel sides AB and CD, since EF is the midsegment of trapezoid $ABDC$, $|AB| + |CD| = 2|EF|$, that is, $a + c = 2v$. Since only one pair of sides can be crossing, the other inequality remains strict. Relabeling the vertices gives $b + d = 2v$ as the other possibility.

This result also can be proved geometrically, as follows. For a concave quadrilateral $ABCD$ as in Figure 2, choose any point D' on CD not an endpoint, let $c' = |CD'|$ and $\epsilon = |DD'| = c - c'$, and let F' be the midpoint of BD'. Because FF' is the midline of triangle BDD' parallel to DD', $|FF'| = \frac{\epsilon}{2}$. Let $|EF'| = v'$. Then from the triangle inequality applied to $EF'F$, $v' + \frac{\epsilon}{2} > v$ and so $2v' + \epsilon > 2v$. We know from the proof of Theorem 1 that for quadrilateral $ABCD'$, $a + c' \geq 2v'$. Then $a + c = a + c' + \epsilon \geq 2v' + \epsilon > 2v$. An identical argument shows the same for $b + d$. This demonstrates that equality is impossible. Note that this proof fails for either a convex or a crossing quadrilateral since, unlike in the concave case, E, F, and F' may be collinear, but we can restore the proof for these quadrilaterals by observing that E, F, and F' are collinear if and only if the quadrilateral $ABCD$ or $ABDC$ (each for suitably labeled vertices) is a trapezoid (use the fact that $FF' \parallel DD'$). In the first case, the strict inequality still holds, since the sum of the lengths of the parallel sides equals twice the length of the midsegment (which properly contains EF). In the second case, $ABCD$ is a crossing quadrilateral with a pair of parallel sides, and then EF is the midsegment of $ABDC$, hence the exception. Provided neither of these is the case, we can proceed as in the above proof. For an alternative geometric construction in the convex case (and for which the trapezoid presents no real difficulty), Figure 1 in [5] shows all at once that the three terms in every factor of (3), (4), and (5) are sides of a triangle, and so both factors in (5) are strictly positive as well. □

The inequality (2), being true for all (nondegenerate) quadrilaterals, may have application to four-bar linkages in mechanics. See [4] for an introduction and references.

References

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