Forum Geometricorum
Volume 17 (2017) 119–121.

FORUM GEOM
ISSN 1534-1178

Another Purely Synthetic Proof of Lemoine's Theorem

Nguyen Tien Dung

Abstract. In this article, we give another purely synthetic proof the Lemoine's theorem that the symmedian point of a triangle is the unique point which is the centroid of its own pedal triangle.

Tran Quang Hung [6] proposed a new proof using similar triangles and cyclic quadrilaterals of Lemoine's theorem on the symmedian point of a triangle.

Lemoine's Theorem ([6]). *Given a triangle ABC, a point P is the centroid of its own pedal triangle with reference to ABC if and only if P is the symmedian point of the triangle ABC.*

In this article, we shall give another synthetic proof of the theorem, also using similar triangles and cyclic quadrilaterals.

Lemma 1. *Let $ABCD$ be a cyclic quadrilateral. The diagonal AC is a symmedian of triangle ABD if and only if CA is a symmedian of triangle CBD.*

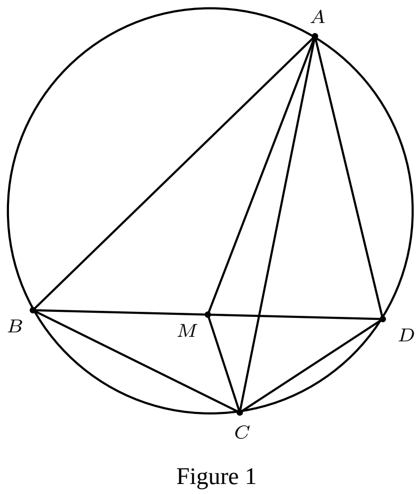

Figure 1

Proof. We only need to prove that if AC is a symmedian of the triangle ABD, then CA is a symmedian of triangle CBD. Let M be the midpoint of the segment BC. As $\angle BAM = \angle CAD$ and $\angle ABM = \angle ACD$, the triangles ABM and

Publication Date: April 24, 2017. Communicating Editor: Paul Yiu.

ACD are similar. It follows that $\dfrac{AB}{AC} = \dfrac{BM}{CD}$, and $AB \cdot CD = AC \cdot BM$. Since $AB \cdot CD + BC \cdot DA = AC \cdot BD$ by Ptolemy's theorem, we have $BC \cdot DA = AC \cdot DM = AC \cdot BM$. Notice that $\dfrac{AC}{BC} = \dfrac{AD}{BM}$ and $\angle CAD = \angle CBM$. The triangles ACD and BCM are similar. Hence, $\angle ACD = \angle BCM$, and CA is a symmedian of triangle CBD. □

Proof of Lemoine's Theorem. Denote by D, E and F the orthogonal projections of P onto BC, CA and AB respectively. The line AP intersects the circumcircle of triangle ABC again at Q. Let M be the midpoint of BC and let L be the reflection of F through P.

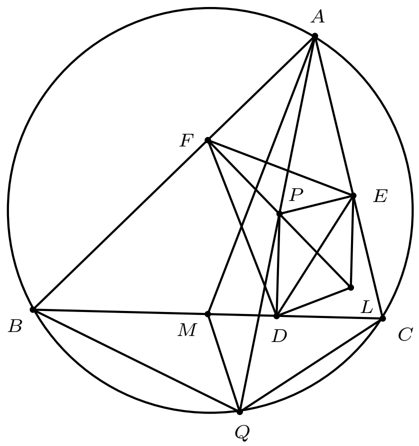

Figure 2

We can see that the quadrilaterals $AEPF$, $BFPD$ and $CDPE$ are cyclic. As $\angle PFE = \angle PAE = \angle QBC$ and $\angle PEF = \angle PAF = \angle BCQ$, the triangles PFE and QBC are similar. It follows that $FE \cdot BQ = FP \cdot BC = FL \cdot BM$. Hence, triangles EFL and MBQ are also similar.

(a) If P is the symmedian point of triangle ABC, AQ is a symmedian of the triangle. According to Lemma 1, QA is the symmedian of the triangle QBC. As the triangles EFL and MBQ are similar, $\angle ELF = \angle MQB = \angle AQC = \angle ABC = \angle DPL$; so $DP \parallel EL$. Similarly, $EP \parallel DL$. The quadrilateral $DPEL$ is a parallelogram. It follows that PL bisects DE. Therefore, FP is a median of the triangle DEF. Similarly, EP is also a median of the same triangle. Hence, P is the centroid of the triangle DEF.

(b) If P is the centroid of triangle DEF, the quadrilateral $DPEL$ is a parallelogram. Since triangles EFL and MBQ are similar, $\angle BQM = \angle FLE = \angle DPL = \angle ABC = \angle AQC$. It follows that QA is a symmedian of triangle QBC. By Lemma 1, AP is a symmedian of triangle ABC. Similarly, BP is also a symmedian of the same triangle ABC. It follows that P is the symmedian point of triangle ABC.

References

[1] J.-P. Ehrmann, Hyacinthos message 95, January 8, 2000.

[2] D. Grinberg, New proof of the symmedian point to be the centroid of its pedal triangle, and the converse, available at
`http://www.cip.ifi.lmu.de/∼grinberg/geometry2.html`.

[3] R. Honsberger, *Episodes of 19th and 20th Century Euclidean Geometry*, Math. Assoc. America, 1995.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, avaiable at
`http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[5] C. Pohoata, A short proof of Lemoine's theorem, *Forum Geom.*, 8 (2008) 97–98.

[6] Q. H. Tran, A simple synthetic proof of Lemoine's theorem, *Forum Geom.*, 17 (2017) 93–95.

[7] B. Wolk, Hyacinthos message 19, December 27, 1999.

[8] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, 2013, available at
`http://math.fau.edu/Yiu/Geometry.html`.

Nguyen Tien Dung: No 24, 3 Quarter, Phuc Loi Ward, Long Bien District, Hanoi, Vietnam
E-mail address: `tiendung12121993@gmail.com`