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Gergonne Meets Sangaku

Paris Pamfilos

Abstract. In this article we discuss the relation of some hyperbolas, naturally associated with the Gergonne point of a triangle, with the construction of two equal circles known from a Sangaku problem at the temple of Chiba.

1. Hyperbolas related to the Gergonne point

The Gergonne point G of a triangle ABC is the common point of the three cevians $\{AA', BB', CC'\}$ ([8, p. 30]), joining the contact points of the incircle with the opposite vertices (see Figure 1). If we select one of these cevians, AA'

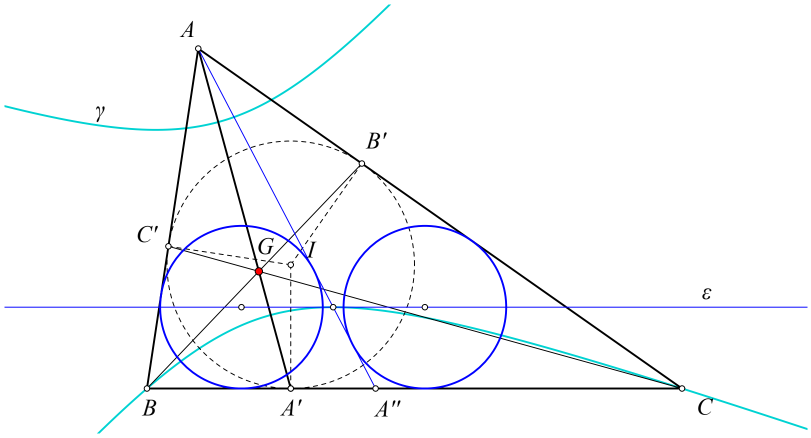

Figure 1. Hyperbola related to the "Gergonne cevian" AA'

say, then there is a unique hyperbola γ with focal points at $\{A, A'\}$, passing through points $\{B, C\}$. It turns out that the two equal "sangaku circles" ([7]), relative to the side BC, have their centers on a tangent ε to this hyperbola, which is parallel to BC.

The present article is devoted to the discussion of this shape, which, among other things, illustrates an easy way to construct the two equal circles. In fact, as it is seen below, the equation of the hyperbola with respect to its axes, of which AA' is per definition its transverse axis, is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{where} \quad 2a = |AB'| = |AC'| \quad \text{and} \quad c^2 = a^2 + b^2 = |AA'|^2/4. \tag{1}$$

Once the hyperbola has been constructed and the parallel ε has been found, the centers of the two circles coincide with the intersections of ε with the bisectors of

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the angles $\{\widehat{B}, \widehat{C}\}$. In dealing with the details, we start with the investigation of general triangles on "focal chords" of hyperbolas. Having enough information and properties of this kind of triangles, we combine them with the properties of the sangaku circles, to obtain the reasoning behind this shape.

Before proceeding further, I must make a note on the naming conventions I am used to. I follow mainly the French naming convention, using the word "principal" or "major", for the circle having diametral points the two vertices of the hyperbola, which in English literature is called "auxiliary". The latter name I reserve instead for the two circles centered at the focal points with radius $2a$, which in French are often called "director circles" ([6, p. 367]) and can be confused with the "director" or "orthoptic circle" ([2, p. 229 II]), which is the locus of points viewing an ellipse or certain hyperbolas under a right angle.

2. Triangles on focal chords

A focal chord of a hyperbola, i.e. a chord PQ, passing through one focal point F, defines, using the other focus, a triangle PQF'. Consider now such a triangle

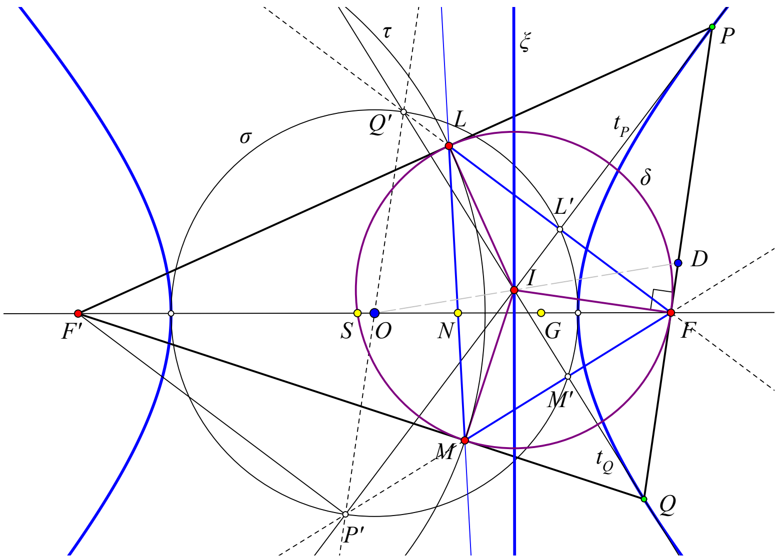

Figure 2. Triangle PQF' on the focal chord PFQ

with respect to the hyperbola with center O, given by equation (1). Let $\{t_P, t_Q\}$ denote the tangents at points $\{P, Q\}$ and $\{\sigma(O, a), \tau(F', 2a)\}$ be respectively the major and the auxiliary circle relative to F' (see Figure 2). Let further ξ be the directrix relative to the focus F. The following lemma formulates the main properties of these triangles. Its proof can be found in almost any book on conics or analytic geometry ([4, p. 76], [2, p. 226, II], [1, p. 8]).

Lemma 1. *Under the previous conventions, the following are valid properties.*

(1) *The tangents $\{t_P, t_Q\}$ are respectively bisectors of the angles $\{\widehat{P}, \widehat{Q}\}$.*

(2) *The projections* $\{L', M'\}$ *of* F *on these tangents are points of the major circle* $\sigma(O, a)$.
(3) *The reflected points* $\{L, M\}$ *of* F *on these tangents are points of the auxiliary circle* $\tau(F', 2a)$.
(4) *The orthogonal to* PQ *at* F *intersects the directrix* ξ *at a point* I*, which is the incenter of triangle* PQF'.
(5) *The radii* $\{IL, IM\}$ *of the incircle* δ *of* PQF' *are tangent to the auxiliary circle* τ.

3. The Gergonne point

The following theorem supplies some less known properties of these triangles PQF' (see Figure 2), that seem to me interesting for their own sake, the last one being of use also in our particular problem. In this $E = (AB, CD)$ denotes the intersection of lines $\{AB, CD\}$, $E(ABCD)$ denotes the pencil of four lines or "rays" $\{EA, EB, EC, ED\}$ through E and $D = C(A, B)$ denotes the harmonic conjugate of C relative to $\{A, B\}$.

Theorem 2. *Continuing with the previous conventions, the following are valid properties.*

(1) *Lines* $\{LM, \xi, PQ\}$ *are concurrent at a point* N' *(not shown in Figure 2).*
(2) *The pole* N *of the directrix* ξ*, relative to the auxiliary circle* τ*, coincides with the intersection* $N = (LM, FF')$*. Hence* LM *passes through the fixed* N*, depending only on the hyperbola and not on the particular direction of the chord* PQ *through* F.
(3) *The incircle* δ *intersects the transverse axis a second time at a point* S*, which is fixed and independent of the direction of* PQ.
(4) *Similarly, the Gergonne point* G *of the triangle* PQF' *is fixed on the transverse axis* FF' *and independent of the direction of* PQ *through* F.
(5) *The center* O *of the hyperbola, the incenter* I *of triangle* PQF' *and the middle* D *of side* PQ *are collinear.*

Proof. Nr-1. can be proved by defining initially N' to be the intersection $N' = (\xi, PQ)$ and considering the two pencils $\{F'(N'FQP), F(N'F'ML)\}$, which have the ray FF' in common. Using the characteristic property of the polar ξ, it is easy to see, that both pencils are harmonic. Hence they define the same cross ratio on every line intersecting them and not passing through their centers $\{F, F'\}$. Applying then the well known property of such pencils with a common ray ([5, p. 90]), we see that the three other pairs of corresponding rays intersect at collinear points $N' = (F'N', FN'), M = (F'Q, FM')$ and $L = (F'P, FL')$.

Nr-2. can be proved by first observing that the incenter I is the pole of LM relative to the circle τ and lies also on line ξ. Then, by the reciprocity of pole-polar ([5, p. 166]), the pole of ξ relative to τ must also be contained in LM. Since this pole is also on the orthogonal FF' to line ξ, it must coincide with $N = (LM, FF')$.

Nr-3. follows from *nr-2.* by comparing powers relative to the circles $\{\sigma, \delta\}$. In fact, the power of N relative to δ : $|NS||NF| = |NL||NM| = p$ is also the power of N relative to τ, shown above to be fixed. Hence p is constant, which implies the claim.

Nr-4. follows from the coincidence of the Gergonne point of PQF' with the symmedian point of the triangle FLM, which has PQF' as its tangential triangle ([8, p. 56]). But it is well known that the symmedian point G of FLM is the harmonic conjugate $G = F'(N, F)$ of F' relative to $\{N, F\}$ ([3, p. 160]).

Nr-5. follows by inspecting the quadrilateral $PQP'Q'$. Here $\{P', Q'\}$ denote the second intersections of the principal circle σ respectively with lines $\{FL', FM'\}$. By Lemma 1, follows that the tangents $\{t_P, t_Q\}$ pass respectively through $\{P', Q'\}$ and that $P'Q'$ is a diameter of the circle σ, seen from $\{L', M'\}$ under a right angle. This means that $\{Q'M', P'L'\}$ are two altitudes of the triangle $FP'Q'$ meeting at I. Hence FI extended is also an altitude of this triangle, and, consequently, orthogonal to $P'Q'$. Since FI is also orthogonal to PQ the two lines $\{PQ, P'Q'\}$ are parallel. Hence, $PQP'Q'$ is a trapezium, its diagonals meeting at I, which is collinear with the middles $\{O, D\}$ of its parallel sides $\{P'Q', PQ\}$. □

The next lemma simply stresses the fact that every triangle can be considered to be based on a focal chord of an appropriate hyperbola.

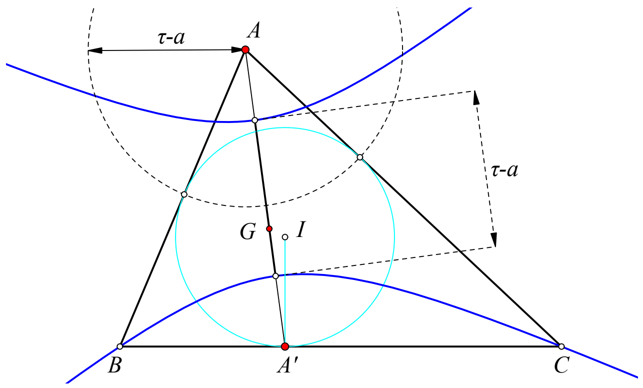

Figure 3. The hyperbola associated to the "Gergonne cevian" AA'

Lemma 3. *For every triangle ABC and each one of the three "Gergonne cevians", i.e. cevians like AA', passing through the Gergonne point G, there is precisely one hyperbola with focal points at $\{A, A'\}$ and passing through the other vertices $\{B, C\}$.*

Proof. By Figure (see Figure 3). Since the hyperbola is uniquely defined by its focal points $\{A, A'\}$ and the property

$$||BA| - |BA'|| = ||CA| - |CA'|| = \tau - a, \tag{2}$$

where τ denotes now the semi-perimeter of the triangle and $a = |BC|$. □

4. The critical tangent

Continuing with the notation established in the preceding sections, we examine now the tangent ε to the hyperbola, which is parallel to the side BC of the triangle.

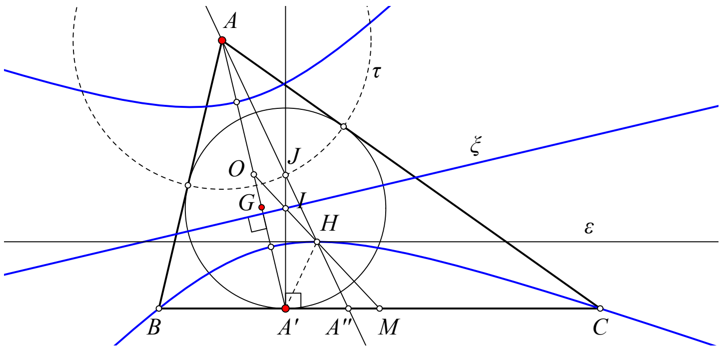

Figure 4. The tangent ε parallel to BC

Lemma 4. *Let M be the middle of the side BC and H be the intersection of the hyperbola with line OM. Then the tangent ε of the hyperbola at H is parallel to BC, points $\{O, I, H, M\}$ are collinear and points $\{A, J, H\}$ are also collinear, J being the symmetric of A' relative to ε.*

Proof. The parallelism of ε to BC is obvious, since line OM, joining the center O of the hyperbola with the middle of the chord BC defines the conjugate direction to that of BC. By Theorem 2, I is on OM, hence the first claimed collinearity. By lemma-1, the symmetric J relative to ε is on τ and points $\{A, J, H\}$ are collinear. □

The connection with the sangaku circles results now from the fact, that the intersection point $A'' = (AJ, BC)$ defines the two "subtriangles" $\{ABA'', AA''C\}$, the incircles of which are the two equal sangaku circles of the triangle ABC relative to the side BC ([7]). Thus, it remains to show that the centers of these circles are on the line ε. But this is a general property of the hyperbola, as seen by the following lemma.

Lemma 5. *With the notation adopted so far, the incenter of the triangle ABA'' is on the line ε.*

Proof. Consider the intersection R of the bisector t_B of the angle $\widehat{B}$ with line ε. It suffices to show that the other bisector, of angle $\widehat{BA''A}$, passes through R. To see this, examine the triangle $A'HR$. Because R is the intersection point of two tangents $\{t_B, \varepsilon\}$ of the hyperbola at points $\{B, H\}$, the angle $\widehat{BA'H}$ from the focus A' to the points of tangency is bisected by $A'R$ ([1, p. 12]). Since $\{BC, \varepsilon\}$ are parallel, we have also $\widehat{RA'B} = \widehat{A'RH}$. Thus, $A'HR$ is isosceles and $|HR| = |HA'| = |HA''|$. Last equality being valid because $JA'A''$ is a right

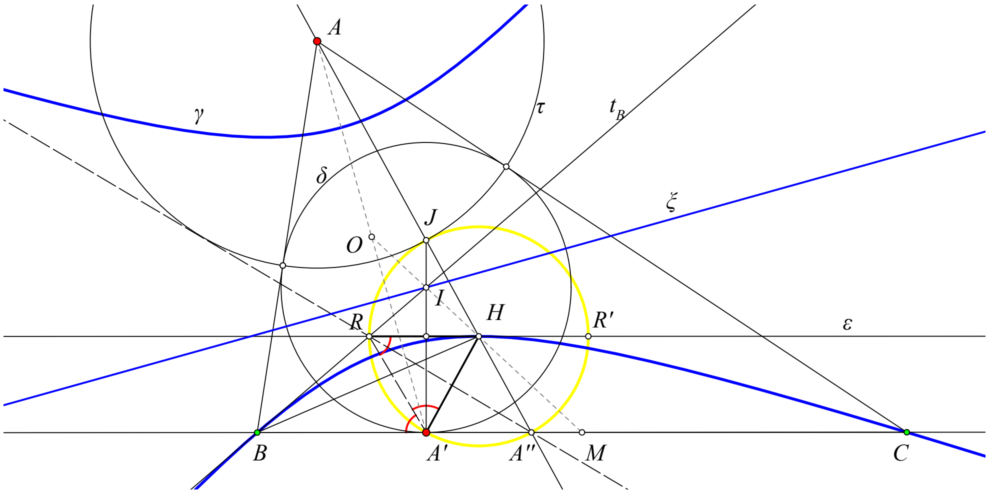

Figure 5. Incenter of triangle ABA''

triangle and H is the middle of its hypotenuse. This shows that triangle RHA'' is also isosceles and $\widehat{HA''R} = \widehat{HRA''} = \widehat{RA''A'}$, thereby proving that $A''R$ is a bisector of the angle $\widehat{BA''A}$. $\square$

From the arguments, used in the proof given above, results also an alternative way to find the centers of the two sangaku circles, expressed by the following corollary.

Corollary 6. *The centers of the two equal sangaku circles relative to the side BC of the triangle are the diametral points $\{R, R'\}$ of a diameter parallel to BC of the circle with diameter JA''.*

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Paris Pamfilos: University of Crete, Greece
E-mail address: `pamfilos@math.uoc.gr`