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New Interpolation Inequalities to Euler's $R \geq 2r$

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Abstract. The purpose of this paper is to obtain some interpolation inequalities to the well-known Euler's inequality $R \geq 2r$ in terms of new geometric elements given by the radii R_A, R_B, R_C of the tangent circles at the vertices to the circumcircle of a triangle and to the opposite sides. The main results are given in Theorems 4-8.

1. Introduction

At the first 2015 Romanian IMO Team Selection Test the first author of this paper has proposed the following problem: Let R_A be the radius of the tangent circle at A to the circumcircle of triangle ABC and to the side BC. Similarly, define the radii R_B and R_C. The following inequality holds

$$\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \leq \frac{2}{r},$$

where r is the inradius of triangle ABC.

In this short paper we discuss some proofs to the above inequality and we complete it to the left hand-side in order to get a new interpolation for the well-known Euler's inequality $R \geq 2r$, where R is the circumradius of triangle ABC. Also, we give other interpolation inequalities to the Euler's inequality in terms of the radii R_A, R_B, R_C. For other interpolation and improvements inequalities to the Euler's inequality we refer to the excellent monograph [2].

2. Some auxiliary results

As usual, we denote by a, b, c the lengths of the sides opposite to the vertices A, B, C, respectively, and by $K[ABC]$ the area of triangle ABC. We need the following helpful results.

Lemma 1. *In triangle ABC denote by h_a, h_b, h_c the lengths of the altitudes from the vertices A, B, C, respectively. The relation*

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$$

holds.

Proof. Just use the formula for the area of a triangle. □

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Lemma 2. *If R_A is the radius of the interior (exterior) tangent circle at A to the circum- circle of triangle ABC and to the side BC, then*

$$\frac{r}{R_A} = \frac{a}{s} \cos^2 \frac{B-C}{2},$$

where s denotes the semiperimeter of triangle ABC.

Proof 1. If $B = C$, then clearly we have $R_A = \frac{h}{2}$. Using the relations $K[ABC] = sr = a\frac{h_a}{2}$, the conclusion follows.

When $B \neq C$, let us suppose that $B > C$. Consider T the intersection point of the common tangent line at A to the two circles with the line BC (see Figure 1). In triangle TAB, we have $\hat{T} = B - C$ and from the Law of Sines we obtain

$$\frac{c}{\sin(B-C)} = \frac{TA}{\sin B} \implies TA = \frac{bc}{2Rsin(B-C)}.$$

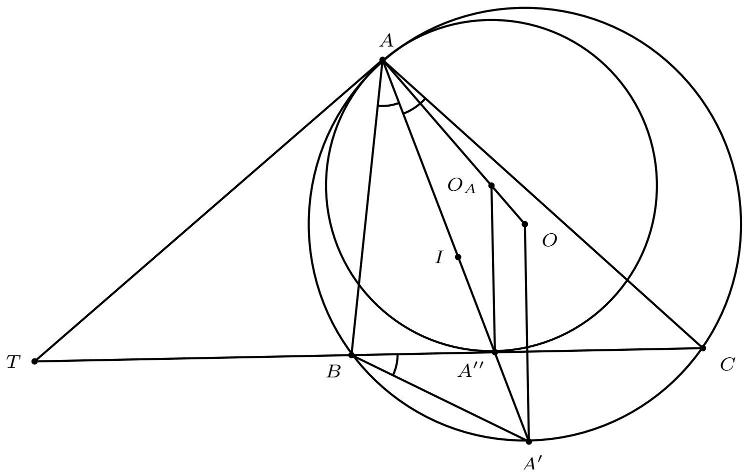

Figure 1

Because $\tan\frac{B-C}{2} = \frac{R_A}{TA}$, it follows that

$$R_A = \frac{bc}{2Rsin(B-C)} \cdot \frac{\sin\frac{B-C}{2}}{\cos\frac{B-C}{2}} = \frac{bc}{2R\cos^2\frac{B-c}{2}}.$$

Therefore,

$$\frac{r}{R_A} = \frac{r}{bc}\cot 4R\cos^2\frac{B-C}{2} = \frac{ar}{4RK}\cos^2\frac{B-c}{2} = \frac{a}{s}\cos^2\frac{B-C}{2},$$

where $K = K[ABC]$, and the proof is complete. □

Proof 2. Let γ_A be the circle tangent at A to the circumcircle of triangle ABC and tangent at T to the line BC. Assume that $R = 1$, and consider the inversion of pole A and unit power. In what follows, X' will denote the image of the point $X \neq A$ by this inversion.

Under this inversion, the line BC is transformed into a circle $AB'C'$ centered at some point Ω. The circle ABC is transformed into the line $B'C'$, and γ_A is transformed into a line ℓ through T' and parallel to $B'C'$.

Let D be the orthogonal projection of A on the line BC. Then $AD = \frac{1}{AD} = \frac{1}{h_a}$, where h_a is the length of the altitude from the vertex A in the triangle ABC, and $\Omega T' = \Omega A = \frac{1}{2h_a}$.

Next, let A_1 be the antipode of A in circle γ_A, so A_1' is the orthogonal projection of A on line ℓ, and $AA_1' = \frac{1}{AA_1'} = \frac{1}{2R_A}$.

Finally, let O denote the circumcenter of the triangle ABC and notice the angles OAD, $\Omega AA_1'$ are both congruent to the absolute value of the difference of the internal angles of triangle ABC at B and C, to obtain

$$\cos(B - C) = \frac{AA_1' - \Omega T'}{\Omega A} = \frac{\frac{1}{2R_A} - \frac{1}{2h_a}}{\frac{1}{2R_A}} = \frac{h_a}{R_A} - 1 = \frac{2K}{aR_A} - 1,$$

where $K = K[ABC]$ and the desired formula follows after standard transformations. □

Lemma 3. *In every triangle ABC the following inequality holds*

$$\cos^2 \frac{B - C}{2} \geq \frac{2r}{R}.$$

We have equality if and only if $2a = b + c$.

Proof 1. We have

$$\begin{aligned}
\cos \frac{B - C}{2} &= \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \\
&= \cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2} + 2 \sin \frac{B}{2} \sin \frac{C}{2} \\
&= \cos \frac{B + C}{2} + 2 \sin \frac{B}{2} \sin \frac{C}{2} \\
&= \sin \frac{A}{2} + 2 \sin \frac{B}{2} \sin \frac{C}{2}.
\end{aligned}$$

Therefore,

$$\cos \frac{B - C}{2} \geq 2\sqrt{2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = 2\sqrt{2 \cdot \frac{r}{4R}} = \sqrt{\frac{2r}{R}},$$

and the conclusion follows. The equality holds if and only if $\sin \frac{A}{2} = 2 \sin \frac{B}{2} \sin \frac{C}{2}$, that is $a^2 = 4(s - a)^2$, hence $2a = b + c$. □

Proof 2. Let I be the incenter of triangle ABC, and consider A' the intersection point of the ray AI with the circumcircle of triangle ABC. We have

$$A'A^2 = (A'I + AI)^2 \geq 4A'I \cdot AI = 8Rr,$$

where the last equality is obtained from the power of I with respect to the circumcircle of triangle ABC. Clearly, the equality holds if and only if $A'I = AI$. But

$$AA' = 2R\sin\left(B + \frac{A}{2}\right) = 2R\cos\frac{B-C}{2},$$

hence the desired inequality follows. As we already mentioned, the equality holds if and only if $A'I = AI$, that is $AA' = 2IA' = 2BA'$, so

$$\cos\frac{B-C}{2} = 2\sin\frac{A}{2}.$$

We obtain

$$\sin\frac{A}{2} = 2\sin\frac{B}{2}\sin\frac{C}{2},$$

therefore $2a = b + c$. □

3. The main results

The first interpolation result is directly connected to the original problem mentioned in the introduction and it is contained in the following theorem.

Theorem 4. *With the above notations the following inequalities hold*

$$\frac{4}{R} \leq \frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \leq \frac{2}{r}. \tag{1}$$

We have equality if and only if the triangle ABC is equilateral.

Proof. From Lemma 2 we have $\frac{r}{R_A} \leq \frac{a}{s}$, with equality if and only if $B = C$. Similarly, $\frac{r}{R_B} \leq \frac{b}{s}$ with equality when $C = A$, and $\frac{r}{R_C} \leq \frac{c}{s}$ with equality when $A = B$. Summing up these inequalities it follows the right hand-side inequality, with equality if and only if $A = B = C$, that is the triangle is equilateral.

From Lemma 3 and Lemma 2 we have $\frac{r}{R_A} \geq \frac{a}{s} \cdot \frac{2r}{R}$, with equality if and only if $2a = b + c$, and two analogous inequalities for the radii R_B and R_C. Summing up these inequalities we obtain

$$\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \geq \frac{2}{R} \cdot \frac{a+b+c}{s} = \frac{4}{R},$$

and we are done. □

Remark. (1) It is possible to give a direct geometric argument for the right hand-side inequality in (1). Consider OA to be the center of the tangent circle at A to the circumcircle of triangle ABC and to the side BC, and A'' the tangency point of this circle with the line BC (see Figure 1). Using the triangle inequality in triangle AO_AA'' we have $h_a \leq AA''leAO_A + O_AA'' = 2R_A$, hence we obtain $\frac{1}{2R_A} \leq \frac{1}{h_a}$, and other two similar inequalities for R_B and R_C. Summing up these inequalities the conclusion follows from Lemma 1.

Theorem 5. *With the above notations the following inequalities hold*

$$\frac{2r}{R} \leq \frac{K[ABC]}{\sqrt[3]{abcR_AR_BR_C}} \leq 1. \tag{2}$$

We have equality if and only if the triangle ABC is equilateral.

Proof. Multiplying the inequalities obtained from Lemma 2, we obtain

$$\frac{r^3}{R_AR_BR_C} \leq \frac{abc}{s^3},$$

hence

$$\frac{K[ABC]}{\sqrt[3]{abcR_AR_BR_C}} \leq 1.$$

On the other hand, multiplying the inequalities obtained from Lemma 2 and using Lemma 3, it follows that

$$\frac{abc}{s^3} \cdot \frac{8r^3}{R^3} \leq \frac{r^3}{R_AR_BR_C}.$$

That is

$$\frac{2r}{R} \leq \frac{K[ABC]}{\sqrt[3]{abcR_AR_BR_C}},$$

and we complete the left hand-side of (2). Clearly, the equality holds if and only if the triangle ABC is equilateral. □

From the relation $\frac{r}{R_A} = \frac{a}{s} \cdot \cos^2 \frac{B-C}{2}$ proved in Lemma 2, we obtain

$$R_A = \frac{K}{a \cos^2 \frac{B-C}{2}}.$$

In the second proof of Lemma 3 we have shown that $AA' = 2R \sin\left(B + \frac{A}{2}\right) = 2R \cos \frac{B-C}{2}$, hence $\cos \frac{B-C}{2} = \frac{AA'}{2R}$. It is clear that the point A'' is the feet of the bisector of the angle A of triangle ABC. Denote by ℓ_a the length of bisector of angle A of triangle ABC, i.e. the length of the segment $[AA'']$. Triangles $AA''O_A$ and $AA'O$ are similar, therefore we obtain

$$\frac{R_A}{R} = \frac{\ell_a}{AA'} = \frac{\ell_a^2}{\ell_a \cdot AA'}.$$

From the Law of Sines in triangle ACA'', it follows that

$$\frac{\ell_a}{\sin C} = \frac{b}{\sin\left(C + \frac{A}{2}\right)}.$$

But, clearly we have

$$\sin\left(C + \frac{A}{2}\right) = \sin\left(B + \frac{A}{2}\right) = \cos \frac{B-C}{2},$$

hence $\ell_a \cdot AA' = 2Rb \sin C$. We obtain

$$R_A = \frac{\ell_a^2}{2b \sin C} = \frac{\ell_a^2}{2h_a} = \frac{a \cdot \ell_a^2}{4K}, \tag{3}$$

where $K = K[ABC]$ is the area of triangle ABC.

Theorem 6. *With the above notations the following inequalities hold*

$$\frac{9}{2}r \le R_A + R_B + R_C \le \frac{9}{4}R. \tag{4}$$

We have equality if and only if the triangle ABC is equilateral.

Proof. The left hand-side inequality can be proved using the inequality

$$R_A = \frac{K}{a\cos^2\frac{B-C}{2}} = \frac{sr}{a\cos^2\frac{B-C}{2}} \ge \frac{sr}{a},$$

where the equality holds if and only if $B = C$, and other two similar inequalities for R_B and R_C. We obtain

$$R_A + R_B + R_C \ge rs\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \frac{r}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge \frac{9}{2}r,$$

with equality if and only if $a = b = c$.

From (3) and from the well-known formula $\ell_a^2 = \frac{4bc}{(b+c)^2}s(s-a)$, the right hand-side inequality is equivalent to

$$\sum_{\text{cyclic}} \frac{4abc}{4K}\cdot\frac{s(s-a)}{(b+c)^2} \le \frac{9}{4}R,$$

hence X

$$\sum_{\text{cyclic}} \frac{4RK}{K}\cdot\frac{s(s-a)}{(b+c)^2} \le \frac{9}{4}R,$$

that is

$$\sum_{\text{cyclic}} \frac{s(s-a)}{(b+c)^2} \le \frac{9}{16}. \tag{5}$$

The inequality (5) is equivalent to

$$\sum_{\text{cyclic}} \frac{1-\frac{a}{s}}{\left(\frac{b}{s}+\frac{c}{s}\right)^2} \le \frac{9}{16}.$$

Let $\frac{a}{s} = 2x$, $\frac{b}{s} = 2y$, $\frac{c}{s} = 2z$, where x, y, $z > 0$ and $x+y+z = 1$. The inequality (5) os equivalent to

$$\sum_{\text{cyclic}} \frac{1-2x}{(y+z)^2} \le \frac{9}{4},$$

for every x, y, $z > 0$ with $x+y+z = 1$. Hence, it is reduced to

$$\sum_{\text{cyclic}} \frac{1-2x}{(1-x)^2} \le \frac{9}{4},$$

for every x, y, $z > 0$ with $x+y+z = 1$.

The function $f:\ (0,1) \to \mathbb{R}$ defined by $f(t) = \frac{1-2t}{(1-t)^2}$ has second derivative

$$f''(t) = \frac{-4t-2}{(1-t)^4} < 0.$$

That is, it is concave on the interval $(0,1)$. From Jensen's inequality it follows that

$$f(x)+f(y)+f(z) \leq 3f\left(\frac{x+y+z}{3}\right) = 3f\left(\frac{1}{3}\right) = \frac{9}{4}, \tag{6}$$

and the result is completely proved. □

The function f in the proof of Theorem 6 satisfies $f'' < -2$ on the interval $(0,1)$. Therefore, using the result of [1], the function $g: (0,1) \to \mathbb{R}$ defined by $g(t) = f(t) + t^2$ is concave on $(0,1)$. Applying the Jensen's inequality for g, we get the following inequality for f:

$$f(x)+f(y)+f(z) \leq 3f\left(\frac{x+y+z}{3}\right) - \frac{1}{3}\left((x-y)^2+(y-z)^2+(z-x)^2\right). \tag{7}$$

Considering $x = \frac{a}{2s}$, $y = \frac{b}{2s}$, $z = \frac{c}{2s}$, the inequality (7) is equivalent to

$$\sum_{\text{cyclic}} \frac{4s(s-a)}{(b+c)^2} \leq \frac{9}{4} - \frac{1}{12s^2}\left((a-b)^2+(b-c)^2+(c-a)^2\right),$$

that is

$$\sum_{\text{cyclic}} \frac{4RK}{K} \cdot \frac{4s(s-a)}{(b+c)^2} \leq \frac{9}{4}R - \frac{R}{12s^2}\left((a-b)^2+(b-c)^2+(c-a)^2\right),$$

and we obtain the following refinement of right-hand side inequality in Theorem 6:

Theorem 7. *With the above notations the following inequality holds*

$$R_A + R_B + R_C \leq \frac{9}{4}R - \frac{R}{12s^2}\left((a-b)^2+(b-c)^2+(c-a)^2\right), \tag{8}$$

with equality if and only if the triangle ABC is equilateral.

Remark. (2) The radius R_A can be expressed in terms of the exradius r_a of the triangle ABC as follows:

$$R_A = \frac{\ell_a^2}{2h_a} = \frac{4bc}{(b+c)^2} \cdot \frac{s(s-a)}{4K/a} = \frac{abcs}{r_a(b+c)^2},$$

and similar formulas for R_B and R_C. We obtain the following formula connecting all the radii R_A, R_B, R_C, R, r:

$$\frac{1}{\sqrt{R_A r_a}} + \frac{1}{\sqrt{R_B r_b}} + \frac{1}{\sqrt{R_C r_c}} = \frac{2}{\sqrt{Rr}}. \tag{9}$$

Using the Cauchy-Schwarz inequality and formula (9) we can write

$$\begin{aligned}\frac{4}{Rr} &= \left(\frac{2}{\sqrt{Rr}}\right)^2 = \left(\frac{1}{\sqrt{R_A r_a}} + \frac{1}{\sqrt{R_B r_b}} + \frac{1}{\sqrt{R_C r_c}}\right)^2 \\ &\leq \left(\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C}\right)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \\ &= \left(\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C}\right)\frac{1}{r},\end{aligned}$$

where we have used the well-known formula $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$. Therefore, we have obtained a new proof to the left-hand side inequality in Theorem 4.

The last result contains two weighted interpolation results.

Theorem 8. *With the above notations the following inequalities hold:*

$$6r \le \frac{a}{a+b+c}R_A + \frac{b}{a+b+c}R_B + \frac{c}{a+b+c}R_C \le 3R; \tag{10}$$

$$\frac{s}{2R} \le \frac{R_A}{a} + \frac{R_B}{b} + \frac{R_C}{c} \le \frac{s}{4r}. \tag{11}$$

We have equality if and only if the triangle ABC is equilateral.

Proof. From Lemma 2 we have a $aR_A = \frac{rs}{a\cos^2\frac{B-C}{2}}$, and using the inequality in Lemma 3, it follows that $rs \le aR_A \le \frac{sR}{2}$, and two similar inequalities for R_B and R_C. Summing up these inequalities we get (10).

For the right-hand side inequality in (11), from $\frac{R_A}{a} = \frac{\ell_a}{4s}$, using the inequality $\ell_a^2 \le s(s-a)$, we obtain

$$\frac{R_A}{a} + \frac{R_B}{b} + \frac{R_C}{c} \le \sum_{\text{cyclic}} \frac{s(s-a)}{4s} = \frac{s}{4r}.$$

For the left-hand side inequality in (11), we use $R_A = \frac{rs}{a\cos^2\frac{B-C}{2}} \ge \frac{rs}{a}$, and we obtain $\frac{R_A}{a} \ge \frac{rs}{a^2}$, and two similar inequalities for R_B and R_C. Then

$$\begin{aligned}\frac{R_A}{a} + \frac{R_B}{b} + \frac{R_C}{c} &\ge rs\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\\ &\ge rs\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)\\ &= rs \cdot \frac{2s}{abc} = rs \cdot \frac{2s}{4Rrs} = \frac{s}{2R},\end{aligned}$$

and we are done. □

References

[1] D. Andrica and I. Raşa, The Jensen inequality: refinements and applications, *Mathematica, L'Analyse Numerique et la Theorie de l'Approximation*, 14 (1985) 105–108.

[2] D. S. Mitrinovic, J. Pecaric, and V. Volonec, *Recent Advances in Geometric Inequalities*, (Mathematics and its Applications), Springer, Softcover 2011 reprint of the original 1st ed. 1989 edition.

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