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Triangle Constructions Based on Angular Coordinates

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Abstract. Two very different, yet related, triangle constructions are examined, based on a given reference triangle and on a triple of signed angles. These produce triangles that are in perspective with the reference triangle and with each other, using the same center of perspective. The first construction is rather well-known, and produces a Kiepert-Morley-Hofstadter-Kimberling triangle. A new circumconic is associated with this construction. The second construction generalizes work of D. M. Bailey and J. Van Yzeren. A number of known central triangles are obtainable using one or both of these constructions.

1. Introduction

This article is concerned with two very different triangle constructions based on a given reference triangle. Each of these is also based an a triple of signed angles (ψ_1, ψ_2, ψ_3). These two constructions produce triangles that are in perspective with the reference triangle and with each other, using the same point of perspective. If it happens that $\psi_1 + \psi_2 + \psi_3 \equiv 0 \pmod{\pi}$, then the point of perspective will just be the point whose angular coordinates are (ψ_1, ψ_2, ψ_3). The first construction is rather well-known, and produces the Kiepert-Morley-Hofstadter-Kimberling (KMHK) triangle, with ψ_1, ψ_2 and ψ_3 serving as the swing angles. The second construction generalizes work of D. M. Bailey [1] and J. Van Yzeren [7]. It focuses attention on a certain triple of circles, where each circle passes through two of the reference triangle vertices.

Section 2 carefully introduces the notions of "directed angles" and "angular coordinates," in the sense in which we will be using these phrases. Section 3 details the construction of a Kiepert-Morley-Hofstadter-Kimberling triangle. Most of this material is admittedly already presented adequately in Chapter 6 of [4]. However, there is a result at the end of Section 3 here that appears to be new. Section 4 details our extension of [1] and [7], and this results in the construction of another triangle, as mentioned earlier.

In Section 5, straightforward methods are presented for testing the trilinear coordinates of a given triangle to determine whether or not it can be obtained by means of one of the two constructions. In Section 6, the results of thus testing the examples of central triangles in [4] are presented. Many of these central triangles passed

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one or both of these tests. Some of these central triangle were known already to be thus obtainable, but some of the results appear to be new.

2. Directed angles and angular coordinates

We will require the following definition. Let A, B, and P be points in the plane. Define the *directed angle* $\measuredangle APB$ to be the angle through which the line $\overleftrightarrow{AP}$ can be rotated about P to coincide with the line $\overleftrightarrow{BP}$. The angle is signed, with positive values indicating counterclockwise rotation, and is only well-defined modulo π. Any equation involving directed angles should be considered modulo π. We will fix a triangle ΔABC with circumcenter O and circumradius R and with A, B, and C not collinear. The interior angles at A, B, and C will be denoted by θ_1, θ_2, and θ_3, respectively.

Having fixed the triangle ΔABC, define the *angular coordinates* of a point P to be the triple (ϕ_1, ϕ_2, ϕ_3) of directed angles where

$$\phi_1 = \measuredangle BPC, \quad \phi_2 = \measuredangle CPA, \quad \phi_3 = \measuredangle APB. \tag{1}$$

Remark. This agrees with Yzeren's definition in [7]. Some sources (e.g. [2, Chapter II], [6]) define angular coordinates only for points inside ΔABC in terms of absolute angles. Clearly $\phi_1 + \phi_2 + \phi_3 = 0$ (mod π).

Observe that the inscribed angle theorem can be written in terms of directed angles as follows:

Lemma 1. *Let A, B, P, and Q be points in the plane. Then A, B, P, and Q are concyclic if and only if $\measuredangle APB = \measuredangle AQB$ if and only if $\measuredangle PAQ = \measuredangle PBQ$.*

Proof. This follows from the traditional inscribed angle theorem along with the following consideration: If P and Q are on opposite sides of a chord AB of a circle, then $\angle APB = \pi - \angle AQB$. But the directed angles $\measuredangle APB$ and $\measuredangle AQB$ must have opposite orientation in this case, so $\measuredangle APB = \pi + \measuredangle AQB = \measuredangle AQB$. □

The following lemma is a direct consequence of Lemma 1.6 and Corollary 2.8 of [5], so we omit the proof. It also follows from a result in [2, Chapter II], but only for the case that P is inside ΔABC. The condition that P is not on the circumcircle or sidelines is equivalent to the condition that $\phi_i \neq 0, \theta_i$ for each i.

Lemma 2. *Suppose P is not on the circumcircle or sidelines of ΔABC, and that P has angular coordinates (ϕ_1, ϕ_2, ϕ_3). Then P has homogeneous trilinear coordinates*

$$\left[\frac{\sin(\phi_1)}{\sin(\theta_1 - \phi_1)} : \frac{\sin(\phi_2)}{\sin(\theta_2 - \phi_2)} : \frac{\sin(\phi_3)}{\sin(\theta_3 - \phi_3)}\right].$$

3. The first triangle construction

Let us begin by reexamining the construction presented in [4]. This is a generalization of the construction in [3] that is used to define Hofstadter points. Using a reference triangle ΔABC, with directed interior angles $\theta_1, \theta_2, \theta_3$, and given a triple of directed angles (ψ_1, ψ_2, ψ_3), another triangle $\Delta A'B'C'$ is produced that is in perspective to ΔABC. We refer to this resulting triangle as the Kiepert-Morley-Hofstadter-Kimberling triangle. This triangle and the following theorem are illustrated in Figure 1. (The figure also contains some red circles and their intersections that should be ignored for the moment.)

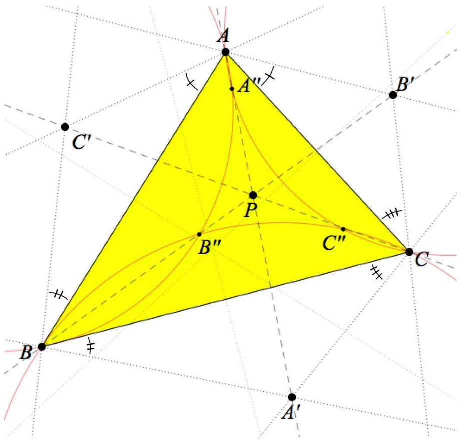

Figure 1. The two constructions

Theorem 3. *Let (ψ_1, ψ_2, ψ_3) be any triple of directed angles such that $\psi_i \neq 0, \theta_i$. Let A', B', and C' be the points satisfying*

$$\begin{aligned} \measuredangle BAC' &= \measuredangle B'AC = \psi_1, \\ \measuredangle CBA' &= \measuredangle C'BA = \psi_2, \\ \measuredangle ACB' &= \measuredangle A'CB = \psi_3. \end{aligned}$$

Then

(i) $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, *and* $\overleftrightarrow{CC'}$ *are concurrent, meeting in a point* P*;*

(ii) *the homogeneous trilinear coordinates of the point* A' *are*

$$\left[\frac{\sin\psi_2 \sin\psi_3}{\sin(\theta_2-\psi_2)\sin(\theta_3-\psi_3)} : \frac{\sin\psi_2}{\sin(\theta_2-\psi_2)} : \frac{\sin\psi_3}{\sin(\theta_3-\psi_3)} \right] \tag{2}$$

and similarly for B' *and* C'*; and*

(iii) P *has homogeneous trilinear coordinates*

$$\left[\frac{\sin\psi_1}{\sin(\theta_1-\psi_1)} : \frac{\sin\psi_2}{\sin(\theta_2-\psi_2)} : \frac{\sin\psi_3}{\sin(\theta_3-\psi_3)} \right]$$

Proof. We here follow the same reasoning as in [3]. First, note that a given line through A, B, or C includes all points with some fixed ratio of trilinear coordinates $[\ell_2 : \ell_3]$, $[\ell_1 : \ell_3]$, or $[\ell_1 : \ell_2]$, respectively. Points on $\overleftrightarrow{CA'}$ satisfy

$$[\ell_1 : \ell_2] = [\sin\psi_3 : \sin(\theta_3-\psi_3)]$$

and points on $\overleftrightarrow{BA'}$ satisfy

$$[\ell_1 : \ell_3] = [\,\sin\psi_2 : \sin(\theta_2 - \psi_2)\,].$$

Hence A' has the homogeneous trilinear coordinates claimed in (ii). Moreover, A' satisfies

$$[\ell_2 : \ell_3] = [\,\sin\psi_2 \sin(\theta_3 - \psi_3) : \sin\psi_3 \sin(\theta_2 - \psi_2)\,]. \tag{3}$$

The other points on $\overleftrightarrow{AA'}$ must also have this ratio of trilinear coordinates. Analogous reasoning shows that $\overleftrightarrow{BB'}$ is given by

$$[\ell_1 : \ell_3] = [\,\sin\psi_1 \sin(\theta_3 - \psi_3) : \sin\psi_3 \sin(\theta_1 - \psi_1)\,] \tag{4}$$

and $\overleftrightarrow{CC'}$ is given by

$$[\ell_1 : \ell_2] = [\,\sin\psi_1 \sin(\theta_2 - \psi_2) : \sin\psi_2 \sin(\theta_1 - \psi_1)\,]. \tag{5}$$

The point P with the homogeneous trilinear coordinates given in (iii) satisfies each of (3), (4), and (5), so it must be the common intersection of $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$, thus establishing (i). $\square$

Remark. In the case that $\psi_2 = -\psi_3$, the lines which would intersect to form A' are parallel. In this case the expression (3) gives the line through A parallel to both of these, and the proof continues with this line in place of $\overleftrightarrow{AA'}$. The same principle holds for B' and C'.

In the case that $\psi = r\theta$ and $r \neq 0, 1$, this construction yields the Hofstadter r-point, as defined in [3]. If $\psi_1 = \psi_2 = \psi_3 = -\pi/3$, then P is the first isogonic center. If $\psi_1 = \psi_2 = \psi_3 = \pi/3$, then P is the second isogonic center. By Theorem 3, it follows that the angular coordinates of the first and second isogonic centers are $(-\pi/3, -\pi/3, -\pi/3)$ and $(\pi/3, \pi/3, \pi/3)$, respectively. If $\psi = \theta/2$, then P is the incenter I. It does not follow that the angular coordinates of I are $\psi = \theta/2$, because in this case $\psi_1 + \psi_2 + \psi_3 \neq 0$. Indeed, it is straightforward to deduce that the angular coordinates of I are in fact $\psi = (\theta + \pi)/2$ (and therefore repeating the construction using these angles still produces the incenter I). The following result, illustrated in Figure 2 and Figure 3, appears to be new.

Theorem 4. *Let $\Delta A'B'C'$ be a KMHK triangle (with respect to ΔABC). The orthogonal projections D, E, F of A', B', C' onto the sidelines $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$ are the vertices of a cevian triangle (with respect to ΔABC). Letting Q denote its center of perspective, if P is held fixed, but A', B', C' are allowed to vary, then the point Q traces out a circumconic of ΔABC.*

Proof. Using formulas from [4], if $[\,\ell : m : n\,]$ are the trilinear parameters for the line $\overleftrightarrow{A'D}$, then $\ell = m\cos\theta_3 + n\cos\theta_2$ (perpendicular lines), and $\ell \sin\psi_2 \sin\psi_3 + m \sin\psi_2 \sin(\theta_3 - \psi_3) + n \sin\psi_3 \sin(\theta_2 - \psi_2) = 0$ (line contains A'). So,

$$m \sin\psi_2 \cos\psi_3 \sin\theta_3 + n \sin\psi_3 \cos\psi_2 \sin\theta_2 = 0,$$

and we may thus take $m = \sin\psi_3 \cos\psi_2 \sin\theta_2$ and $n = -\sin\psi_2 \cos\psi_3 \sin\theta_3$.

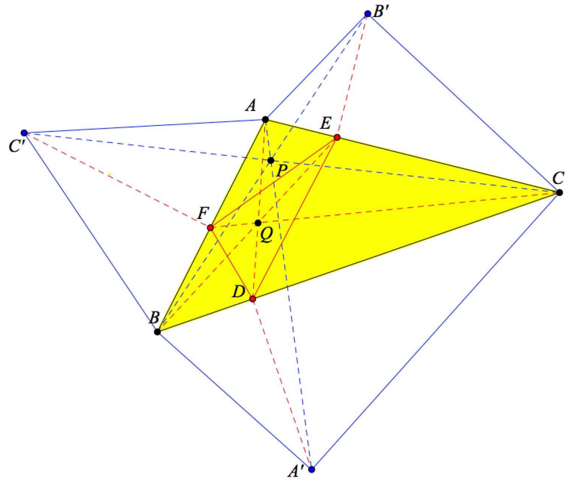

Figure 2. A cevian triangle from a KMHK triangle

Letting $[\,0 : \mu : \nu\,]$ be the trilinear coordinates for D, we may take $\mu = \sin\theta_3 \cot\psi_3$ and $\nu = \sin\theta_2 \cot\psi_2$. The line $\overleftrightarrow{AD}$ has trilinear parameters $[\,0 : \nu : -\mu\,]$. Similarly for the lines $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$. These three lines intersect at a point Q whose trilinear coordinates are $[\,\csc\theta_1 \tan\psi_1 : \csc\theta_2 \tan\psi_2 : \csc\theta_3 \tan\psi_3\,]$.

If we let (ϕ_1, ϕ_2, ϕ_3) be the angular coordinates of P, then its trilinear coordinates are

$$[\,\sin\phi_1/\sin(\theta_1-\phi_1) : \sin\phi_2/\sin(\theta_2-\phi_2) : \sin\phi_2/\sin(\theta_2-\phi_2)\,] =$$

$$[\,\sin\psi_1/\sin(\theta_1-\psi_1) : \sin\psi_2/\sin(\theta_2-\psi_2) : \sin\psi_2/\sin(\theta_2-\psi_2)\,].$$

Therefore, there is a parameter λ such that, for $i = 1, 2, 3$,

$$\frac{\sin\phi_i}{\sin(\theta_i-\phi_i)} = \lambda \cdot \frac{\sin\psi_i}{\sin(\theta_i-\psi_i)}, \text{ and so } \cot\psi_i = \lambda\cot\phi_i + (1-\lambda)\cot\theta_i.$$

The i-th trilinear coordinate of Q thus becomes $1/[\lambda\sin\theta_i\cot\phi_i + (1-\lambda)\cos\theta_i]$, and the isogonal conjugate Q^{-1} of Q has i-th trilinear coordinate $\lambda\sin\theta_i\cot\phi_i + (1-\lambda)\cos\theta_i$. Varying λ, we see that Q^{-1} traces out a line, and therefore Q traces out a circumconic. □

Figure 3 illustrates this circumconic, which is a circumhyperbola here. With P fixed, as A', B', C' are allowed to vary, the point Q moves along the green curve, which is the circumhyperbola.

4. The second triangle construction

Our second triangle construction generalizes a construction studied in [1], [5], [6] and [7]. These studies all essentially concern an arbitrary point P, and the three circles through P that also pass through two of the reference triangle vertices. We

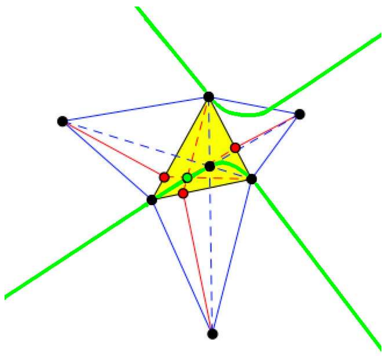

Figure 3. A circumconic associated with the KMHK construction

will instead begin with the reference triangle ΔABC, and with a triple of directed angles (ψ_1, ψ_2, ψ_3), as we did in the first construction.

Starting with the triple of directed angles, construct three circles as follows: Let P be any point for which $\measuredangle BPC = \psi_1$ and let $\mathcal{C}_X$ denote the circle BPC. By Lemma 1, this construction is well-defined. Using C and A (*resp.* A and B) in place of B and C, we obtain a circle $\mathcal{C}_Y$ (*resp.* $\mathcal{C}_Z$). Finally, let X, Y, and Z denote the centers of $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$, respectively.

If the three circles $\mathcal{C}_X$, $\mathcal{C}_Y$, and $\mathcal{C}_Z$ have a common point of intersection, then by definition that point has angular coordinates (ψ_1, ψ_2, ψ_3) and so $\psi_1 + \psi_2 + \psi_3 \equiv 0 \pmod{\pi}$. We do not assume, however, that our original triple of directed angles satisfies this equation, and so the three circles do not generally have a common point of intersection.

Let A'' (*resp.* B'', *resp.* C'') be the point of intersection of $\mathcal{C}_Y$ and $\mathcal{C}_Z$ (*resp.* $\mathcal{C}_Z$ and $\mathcal{C}_X$, *resp.* $\mathcal{C}_X$ and $\mathcal{C}_Y$), other than A (*resp.* B, *resp.* C). Figure 1 shows these circles and their intersections, and also illustrates the theorem to be presented concerning these.

Theorem 5. *For a triangle ΔABC, and for a triple of directed angles (ψ_1, ψ_2, ψ_3) such that $\psi_i \neq 0, \theta_i$, let A'', B'', C'' be the circle intersection points considered above. Let A', B', C' be the points in Theorem 3. Then*

(i) *A, A' and A'' are collinear, as are B, B' and B'', as are C, C' and C'';*
(ii) *the homogeneous trilinear coordinates of the point A'' are*

$$\left[\frac{\sin(\psi_2 + \psi_3)}{\sin(\psi_2 + \psi_3 - \theta_2 - \theta_3)} : \frac{\sin(\psi_2)}{\sin(\theta_2 - \psi_2)} : \frac{\sin(\psi_3)}{\sin(\theta_3 - \psi_3)} \right] \tag{6}$$

and similarly for B'' and C''; and

(iii) *if $\psi_1 + \psi_2 + \psi_3 \equiv 0$ (mod π), then $A'' = B'' = C'' = P$ (with P as in Theorem 3), and (ψ_1, ψ_2, ψ_3) are the angular coordinates of P.*

Proof. Observe that the point A'' has angular coordinates $(\psi_1'', \psi_2, \psi_3)$ for some value ψ_1'': the last two angular coordinates are known by Lemma 1 and the construction of A''. It follows that $\psi_1'' = -\psi_2 - \psi_3$. Part (ii) is then established by converting angular to trilinear coordinates (Lemma 2) and replacing θ_1 with $\pi - \theta_2 - \theta_3$.

Part (iii) follows immediately from (ii).

Since the ratio $[\ell_2 : \ell_3]$ is shared by points A' and A'' (as in Figure 2 and Figure 6), it must be the case that A' and A'' are on the same line through A. This is part (i). □

Following a simple lemma, a characterization is now presented of triangles that can be obtained via the second construction, using the same center of perspective.

Lemma 6. *Let E, G, H and F be concyclic points, occurring in this cyclic order. Let X and Y be points such that E, G and X are collinear, F, H and Y are collinear, and the lines $\overleftrightarrow{GH}$ and $\overleftrightarrow{XY}$ are parallel. Then, E, X, Y and F are concyclic points. Conversely, if U and V are points such that E, U, V and F are concyclic, E, G and U are collinear, and F, H and V are collinear, then $\overleftrightarrow{GH}$ and $\overleftrightarrow{UV}$ are parallel.*

Proof. $\measuredangle YFE = \measuredangle HFE = -\measuredangle EGH = \measuredangle HGX = -\measuredangle GXY = -\measuredangle EXY$. Therefore, E, X, Y and F are concyclic. (Euclid's theorem on cyclic quadrilaterals is used in both directions.) Also, $\measuredangle GUV = \measuredangle EUV = -\measuredangle VFE = -\measuredangle HFE = \measuredangle EGH = -\measuredangle HGU$. So $\overleftrightarrow{GH}$ and $\overleftrightarrow{UV}$ are parallel. □

Theorem 7. *Suppose that $\Delta A''B''C''$ can be obtained from the reference triangle ΔABC, using the second construction. Let P denote the center of perspective. Suppose that ΔXYZ is another triangle, homothetic to $\Delta A''B''C''$, with P as the homothetic center. Then ΔXYZ can also be obtained from ΔABC by means of the second construction. Conversely, all triangles obtainable via the second construction, and having P as the center of perspective, are related to $\Delta A''B''C''$ in this manner.*

Proof. A, B, A'', B'' are concyclic. A, A'', X are collinear, as are B, B'', Y, with the two lines intersecting at P. The lines $\overleftrightarrow{A''B''}$ and $\overleftrightarrow{XY}$ are parallel. So by the lemma, A, B, X, Y are concyclic. Similarly, B, C, Y, Z are concyclic, and C, A, Z, X are concyclic. This reasoning can be reversed to establish that all triangles obtainable via the second construction, and having P as the center of perspective, are related to $\Delta A''B''C''$ in this manner. □

Remark. Let O denote the circumcenter of the reference triangle ΔABC. In [5], it is demonstrated that the triangle whose vertices are the centers of $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$, is in an orthological relation with the reference triangle, with P as the orthology center of the latter with respect to the former, and with O as the orthology center of the former with respect to the latter. Conversely, given a triangle T with this orthological relation to the reference triangle (still using P and O as orthology centers), the vertices of the reference triangle can be reflected about the corresponding sides of T to obtain the vertices of a triangle that is also obtainable via the construction discussed in this section.

5. Triangles obtainable via the two constructions

The question of whether a given triangle can or cannot be obtained from the reference triangle by means of one of the two constructions discussed above shall now be considered. Here we will suppose that we are presented with the homogeneous trilinear coordinates of the vertices of some triangle. As is customary, we will assume this is presented in the form of a 3×3 matrix with each row providing the trilinear coordinates of a vertex:

$$L = \begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}.$$

We wish to know if this is the trilinear coordinates matrix of a KMHK triangle. We know that any KMHK triangle has the following as its trilinear coordinates matrix:

$$M = \begin{pmatrix} \rho_2\rho_3 & \rho_2 & \rho_3 \\ \rho_1 & \rho_3\rho_1 & \rho_3 \\ \rho_1 & \rho_2 & \rho_1\rho_2 \end{pmatrix},$$

where $\rho_i = \sin\psi_i / \sin(\theta_i - \psi_i)$, with θ_i and ψ_i as before. It is required therefore that $\ell_{12}\ell_{23}\ell_{31} = \ell_{21}\ell_{32}\ell_{31}$. If this is so then the rows of L can easily be rescaled (each row being multiplied by a scalar) to cause $\ell_{21} = \ell_{31}$, $\ell_{12} = \ell_{32}$ and $\ell_{13} = \ell_{23}$. Assume that this has been done already. If L is indeed the trilinear coordinate matrix of a KMHK triangle, then it must equal λM for some scalar λ. But this means that $\ell_{12}\ell_{13}/\ell_{11} = \ell_{21}\ell_{23}/\ell_{22} = \ell_{31}\ell_{32}/\ell_{33}$. Conversely, if this condition concerning the entries of L is satisfied, then it is straightforward to see that L is indeed the trilinear coordinates matrix of a KMHK triangle.

We turn now to the question of whether or not L is the trilinear coordinates matrix of some triangle that can be obtained using the second construction, the one based on intersecting circles. The approach taken to answering this question differs substantially from the approach used for the first construction. However, it is again clear that $\ell_{12}\ell_{23}\ell_{31} = \ell_{21}\ell_{32}\ell_{13}$ is still a necessary condition, so we will assume that this is the case. Let O be the circumcenter of the reference triangle. Let A'', B'' and C'' be the points having the first, second and third rows of L as their trilinear coordinates. Essentially following the notation used in [5], let

$$c_{ij} = R \cdot \frac{\ell_{i2}\ell_{i3}\sin\theta_1 + \ell_{i3}\ell_{i1}\sin\theta_2 + \ell_{i2}\ell_{i3}\sin\theta_3}{\ell_{i1}\sin\theta_1 + \ell_{i2}\sin\theta_2 + \ell_{i3}\sin\theta_3} \cdot \frac{1}{\ell_{ij}},$$

where R is the circumradius of the reference triangle. In [5] it is demonstrated that $|c_{11}|$ is the distance between O and the center of the circle containing A'', B, C. Similarly, $|c_{12}|$ is the distance between O and the center of the circle containing A'', C, A, and $|c_{13}|$ is the distance between O and the center of the circle containing

A'', A, B. Likewise for $|c_{21}|$, $|c_{22}|$ and $|c_{23}|$ ($|c_{31}|$, $|c_{32}|$ and $|c_{33}|$) with B'' (C'') taking the place of A''.

Now, the triangle $A''B''C''$ is obtained from the triangle ABC using the second construction if and only if B, C, B'' and C'' are concyclic, and C, A, C'' and A'' are concyclic, and A, B, A'' and B'' are concyclic. This is so if and only if $|c_{21}| = |c_{31}|$, $|c_{32}| = |c_{12}|$ and $|c_{13}| = |c_{23}|$. If we divide both sides of these three equations by R, we obtain three equations that can easily be used to test whether or not L is the trilinear coordinates matrix of a triangle that can be obtained using the second construction.

6. Relationship with central triangles

The center of perspective P used in the two constructions will henceforth be assumed to be a triangle center for the reference triangle. The trilinear coordinates of the vertices produced by the two constructions, as presented in Sections 3 and 4, make it clear that the constructed triangle is a central triangle of type 1, as defined in Chapter 2 of [4]. Recall that this means that the matrix has the form

$$\begin{pmatrix} f(a,b,c) & g(b,c,a) & g(c,a,b) \\ g(a,b,c) & f(b,c,a) & g(c,a,b) \\ g(a,b,c) & g(b,c,a) & f(c,a,b) \end{pmatrix}.$$

for triangle center functions f and g, where a, b, c are the triangle side lengths. More explicitly, f and g must be homogeneous and must be invariant under a swapping of their second and third arguments.

We now ask, which of the central triangles presented in Chapter 6 of [4] can be obtained using one of the two constructions? Many of the central triangles there are presented using a trilinear coordinates matrix that manifests the triangle to be of type 1. These triangles can be tested immediately using the tests given in the previous section. Most of the other triangles are presented using a trilinear coordinate matrix whose rows can be rescaled so as to produce a matrix of the above form. This then shows that the triangle is actually of type 1, and provides a matrix that can be used in the tests in the previous section. The matrices in Chapter 6 that can be adjusted in this way all have the form

$$\begin{pmatrix} * & \gamma\alpha'\beta'' & \beta\alpha'\gamma'' \\ \gamma\beta'\alpha'' & * & \alpha\beta'\gamma'' \\ \beta\gamma'\alpha'' & \alpha\gamma'\beta'' & * \end{pmatrix}.$$

where $\alpha = \alpha(a,b,c)$, $\alpha' = \alpha'(a,b,c)$ and $\alpha'' = \alpha''(a,b,c)$ are triangle center functions, and $\beta = \alpha(b,c,a), \beta' = \alpha'(b,c,a), \beta'' = \alpha''(b,c,a), \gamma = \alpha(c,a,b), \gamma' = \alpha'(c,a,b), \gamma'' = \alpha''(c,a,b)$. To bring this matrix into the desired form, just divide the first row by $\alpha'\beta\gamma$, divide the second row by $\alpha\beta'\gamma$, and divide the third row by $\alpha\beta\gamma'$. This yields the following matrix:

$$\begin{pmatrix} * & \beta''/\beta & \gamma''/\gamma \\ \alpha''/\alpha & * & \gamma''/\gamma \\ \alpha''/\alpha & \beta''/\beta & * \end{pmatrix}.$$

MATHEMATICA was used to conduct the tests on the triangles in Chapter 6 of [4]. Cevian triangles are trivially KHMK triangles, or at least limiting cases of such as the swing angles go to zero in some fixed proportion. Similarly, circumcervian triangles are trivially examples of the second construction since their vertices and those of the reference triangle are concyclic. The triangles in the table on page 198 of [4] are, as stated there, KHMK triangles. Apart from these, our testing also determined that the excentral triangle (6.7 of [4]), the hexyl triangle (6.36 of [4]), the half-altitude triangle (6.38 of [4]), and the BCI triangle (6.39 of [4]) are KHMK triangles. For the excentral triangle, the claim is known and it is straightforward to check that $\psi_i = \frac{1}{2}(\pi - \theta_i)$ $(i = 1, 2, 3)$. For the half-altitude triangle, the claim is also known and it is straightforward to check that $\tan \psi_i = \frac{1}{2} \tan \theta_i$ $(i = 1, 2, 3)$. The facts concerning the other two triangles are less immediate.

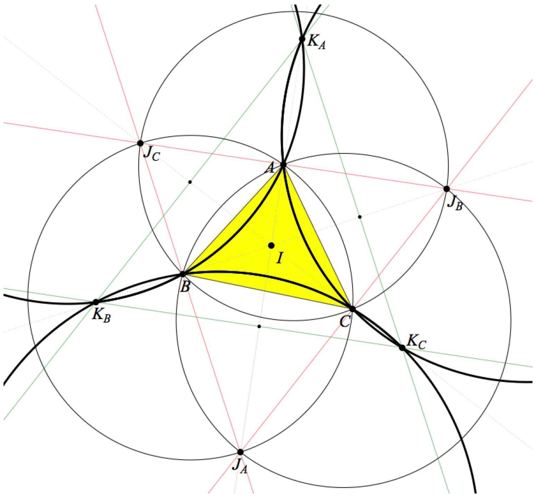

Figure 4. Circles for the excenters and their reflections about the incenter

Skipping the reference triangle itself, and the circumcevian examples, the only other triangles listed in Chapter 6 of [4] that can be obtained using our second construction are as follows: the orthic triangle (6.4 of [4]), the excentral triangle (6.7 of [4]), the reflections of the circumcenter about the reference triangle vertices (6.13 of [4]), and the reflections of the excenters about the incenter (6.42 of [4]). This fact concerning the orthic triangle and the excentral triangle are well-known, and indeed they are related since the excentral (orthic) triangle of the orthic (excentral) triangle is the reference triangle. Figure 4 exhibits the situation for the excentral triangle and for the reflections of the excenters about the incenter. Here I is the incenter, J_A, J_B, J_C are the excenters, and K_A, K_B, K_C are their reflections about I.

We may deduce that the triangle obtained by reflecting the feet of the altitudes about the vertices of the reference triangle can also be obtained via the second construction. This is so since this triangle has the same inverse relationship to the triangle obtained by reflecting the excenters about the incenter that the orthic triangle has to the excentral triangle. The triangle whose vertices are the reflected excenters of the reference triangle, has the reference triangle's incenter I as its orthocenter. The feet of its altitudes can be seen in Figure 4 as small dots. The reflection of these about I are just the vertices of the reference triangle.

Similarly, but rather trivially, the triangle obtained by reflecting the circumcenter about the reference triangle vertices has the same inverse relation to the triangle obtained from the reference triangle by taking as vertices the midpoints on the segments connecting the reference triangle's circumcenter to its vertices. Therefore, the latter is also an example of a central triangle obtainable by means of the second construction. Alternatively, Theorem 7 can be used to establish this and similar claims.

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