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On the Orthogonality of a Median and a Symmedian

Navneel Singhal

Abstract. We give a synthetic proof of F. J. García Capitán's theorem on the lemniscate as the locus of a vertex of a triangle, given the other two vertices, such that the corresponding median and symmedian are orthogonal.

1. Introduction

In his paper [1], F. J. García Capitán proved the following theorem using Cartesian coordinates.

Theorem. *Let B and C be fixed points in the plane. The locus of a point A such that the A-median and the A-symmedian of triangle ABC are orthogonal is the lemniscate of Bernoulli with endpoints at B and C.*

We give a synthetic proof of this theorem, beginning with a series of lemmas.

Lemma 1. *Let $ABCD$ be a cyclic quadrilateral. The points $AB \cap CD$, $BC \cap DA$, $AC \cap BD$ form the vertices of a self polar triangle with respect to the circumcircle of $ABCD$.*

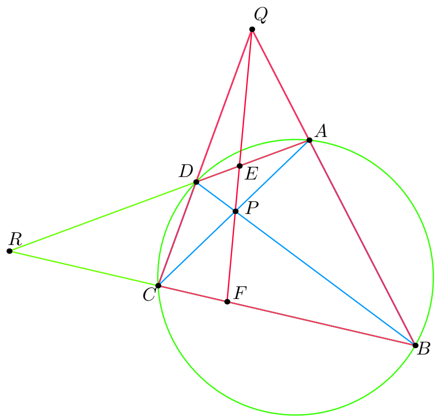

Figure 1

Proof. Let AD meet BC at R, AB meet CD at Q, and AC meet BD at P. Let QP intersect BC, AD at F, E, respectively. We know from triangles DAQ and BCQ that $(R, E; A, D)$ and $(R, F; B, C)$ are harmonic. So it follows that EF is the polar of R. Hence PQ is the polar of R. Similarly PR is the polar of Q and RQ is the polar of P. So PQR is a self polar triangle with respect to the circumcircle of $ABCD$. □

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Lemma 2. *Let ω be the circle with BC as diameter and M as center. Denote by A_1 the inverse of A in ω.*

(a) *The circles ω_1, ω_2 through $\{A, B\}, \{A, C\}$ and tangent to BC pass through A_1.*

(b) *Let H be the orthocenter of ABC. The circle with diameter AH and the circumcircle of BHC meet on A_1.*

(c) *If the A-symmedian cuts Ω, the circumcircle of ABC, at A_2, then A_1, A_2 are the reflections of each other in BC.*

Proof. (a) Since ω_1, ω_2 are tangent to BC, and their radical axis bisects BC, we know that M is on their radical axis. Also the inversion in ω preserves the circles and so A_1 is the other intersection point of ω_1, ω_2.

(b) By (a), $\angle BA_1C = 180^\circ - \angle A_1BC - \angle A_1CB = 180^\circ - \angle BAC$, so A_1 is on the circumcircle of BHC. Let the orthic triangle of ABC be DEF, where D is on BC etc. The inversion at A, and of radius $\sqrt{AH \cdot AD}$ maps BC to the circle with AH as diameter and thus A_1 to the intersection of tangents to that circle at E, F, which is the midpoint of BC. So A_1 is on the circle with AH as diameter as well by inverting back.

(c) Since the circumcircles of BHC and ABC are reflections of each other across BC, so the reflection of A_1 in BC is on Ω. Since we also have $\angle A_2CB = \angle A_2AB = \angle MAC = \angle A_1CB$, so the reflection of A_1 in BC is A_2. □

Remark. A_1 is the vertex of the D-triangle of ABC corresponding to A (see, for example, [2]), and has a lot of interesting properties, which we will not pursue in this paper.

Lemma 3. (a) *All conics through an orthocentric quadruple of points are equilateral* (rectangular) *hyperbolas, and all equilateral hyperbolas through the vertices of a triangle pass through its orthocenter.*

(b) *The locus of the centers of equilateral hyperbolas through the vertices of a triangle is the nine-point circle of the triangle.*

2. Proof of the Main Theorem

2.1. *A on lemniscate $\Rightarrow$ orthogonality of A-median and A-symmedian.*

Let the equilateral hyperbola with BC as transverse axis and passing through B, C be $\mathcal{H}$. It is the inverse image of the lemniscate with B, C as its endpoints in ω. We are going to show that if A_1 is on $\mathcal{H}$, $AA_2 \perp AM$.

Since A_2 is the reflection of A_1 in BC, A_2 is also on $\mathcal{H}$. As the perpendicular from A_1 to BC meets $\mathcal{H}$ at A_2, $\{A_1, A_2, B, C\}$ is an orthocentric quadruple by Lemma 3(a). So $A_1B \cap A_2C, A_1C \cap A_2B$ are on ω.

In view of Lemma 1, A_1, A_2 are conjugate points with respect to ω. Now as A is the inverse of A_1 in ω, the line through A and perpendicular to AM is the polar of A_1 with respect to ω, which passes through A_2. Therefore, the A-median is perpendicular to the A-symmedian.

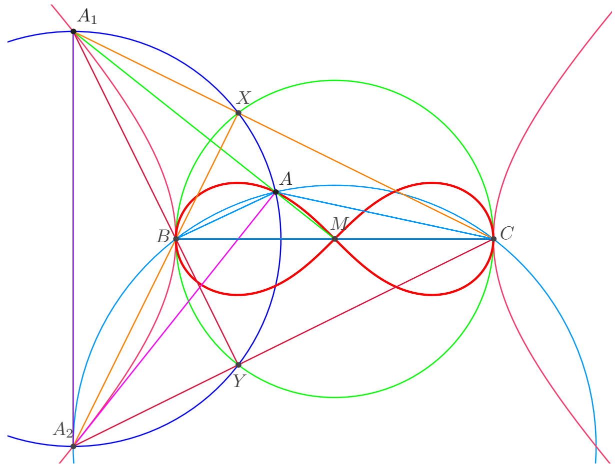

Figure 2

2.2. *Orthogonality of A-median and A-symmedian $\Rightarrow$ A on lemniscate.*

Now we suppose that $AM \perp AA_2$. Since A is the inverse of A_1 in ω, AA_2 is the polar of A_1 with respect to ω. Let $A_2B \cap A_1C = X$. Then

$$\begin{aligned} \angle BXC &= 180^\circ - \angle XBC - \angle XCB \\ &= \angle A_2BC - \angle A_1CB \\ &= \angle A_1BC - \angle A_1CB \\ &= \angle BAM - \angle CAM \\ &= 90^\circ + \angle BAA_2 - \angle MAC \\ &= 90^\circ \end{aligned}$$

wherein we have used Lemma 2 repeatedly.

So $X \in \omega$. Similarly, $A_1B \cap A_2C \in \omega$.

Thus $\{A_1, A_2, B, C\}$ is an orthocentric quadruple. Invoking Lemma 3(b), as M is on the nine point circle of A_1BC, let $\mathcal{H}$ be the hyperbola through A_1, A_2, B, C and with center M.

Now that $MA_1 = MA_2$, A_1, A_2 are on a circle with M as center, so the perpendicular bisector of A_1A_2 is an axis of $\mathcal{H}$. Thus BC is the transverse axis of $\mathcal{H}$ (as B, C are on $\mathcal{H}$), and so A_1 is on the fixed hyperbola $\mathcal{H}$, thereby establishing the fact that A is on the lemniscate with B, C as endpoints by inversion in ω.

This completes the proof of the Main Theorem.

3. Some interesting properties

Property 1. *$A_1A_2 \cap BC$ is the inverse of the trace of the symmedian point on BC in ω.*

Proof. The pole of AA_2 with respect to Ω is A_1 and that of BC is the point at ∞ in the direction perpendicular to BC, so the claim follows. □

Property 2. *The line A_1A_2 is tangent to Ω.*

Proof. Follows from the previous property and the fact that the pole of AA_2 with respect to Ω is the intersection of the tangents to Ω at A, A_2 and BC. □

Property 3. *Irrespective of the condition of orthogonality of the median and the symmedian, the points A, X, Y, A_1, A_2 are on the A-Apollonius circle.*

Proof. Since the quadrilateral $BCAA_2$ is harmonic, we know that A_2 is on the A-Apollonius circle. Now the A-Apollonius circle is symmetric with respect to BC, so A_1 is on it as well. $\angle YA_2A = \angle ACB = \angle AA_1C$, so Y and analogously X are on the A-Apollonius circle. □

Property 4. *Irrespective of the condition of orthogonality of the median and the symmedian, if AC and AB meet the A-Apollonius circle at A_b, A_c respectively, then the arcs A_1A_c and A_2A_b are congruent.*

Proof. Simple angle chasing. □

Property 5. *Irrespective of the condition of orthogonality of the median and the symmedian, the tangent to Ω at A_2, the A-Apollonius circle and the line through A and parallel to BC are concurrent.*

Proof. Using cross ratios,

$$\begin{aligned} -1 &= (B, C; A, A_2) \\ &\stackrel{A_2}{=} (X, Y; A, A_2A_2 \cap \odot(AXY)) \end{aligned}$$

By projecting this through A_1, we have our conclusion. □

References

[1] F. J. García Capitán, Lemniscates and a locus related to a pair of median and symmedian, *Forum Geom.*, 15 (2015) 123–125.

[2] E. W. Weisstein, D-Triangle, From *Math World–A Wolfram Web Resource*, `http://mathworld.wolfram.com/D-Triangle.html`.

Navneel Singhal: 493, Mantola, Paharganj, New Delhi - 110055, India
E-mail address: `navneel.singhal@ymail.com`