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A Group Theoretic Interpretation of Poncelet's Theorem Part 2: The Complex Case

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Abstract. Poncelet's theorem about polygons that are inscribed in a conic and at the same time circumscribe another one has a greater companion, in which different conics touch the sides of the polygon, while all conics belong to a fixed pencil. In the first part of these investigations on Poncelet's theorem [11] the case of a pencil of circles in the real plane was analyzed. It was shown that the question of whether 'circuminscribed' polygons exist for every starting point can be decided by considering the action of a group. As a continuation of this work we now examine the case of a pencil of conics in the complex plane. It will be shown that in this case the answer to the same question about the existence of a continuous family of 'circuminscribed' polygons depends on the addition of an elliptic curve.

1. Introduction

In [12, p. 285], Jacobi derives a formula relating the 'circuminscribed' polygons in Poncelet's theorem to the repeated addition of a parameter t in the argument of an elliptic function. Further ahead on p. 291 he shows that in the general case of tangents to circles of a pencil one has to add in the argument of the same elliptic function parameters $t, t', t'', \ldots$ depending on the elements of this pencil. In [3], there is given a summary of Jacobi's approach. We present here a proof of Poncelet's theorem in the general case based on the relation of an elliptic curve to Poncelet configurations. These configurations have been studied with other means elsewhere (see e.g. [1], [2], [3], [4], [7], [8], [9], [10], [12], [14], [15], [16], [17] and literature cited there.). As already mentioned in the first part: There is nothing novel about the ingredients but perhaps this particular view could shed a new light on them.

In sections 2 and 3, we collect those results with proofs from [2], [14] and [15], that are important for the understanding of the rest of the article. Lemma 3 takes therein a central place. Its proof shows that the 6 points, where conics of a pencil touch the sides of a triangle inscribed in a conic of this pencil, are the intersecting points of a complete quadrilateral.

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In section 4 and 5 we show how the construction of points where the secants of a conic touch other conics of a pencil can be related to the addition on an elliptic curve. This will be explained in Theorem 8. The main result, however, is Theorem 15. By virtue of the previously defined addition on an elliptic curve this theorem provides a criterion to decide, whether a given set of conics from a pencil forms a Poncelet configuration, which means that starting from each point on a fixed conic we can construct closed 'circuminscribed' polygons whose sides touch the other conics. This result connects the analytic addition of arguments of an elliptic function in Jacobi's article [12] with the geometric addition on an elliptic curve.

2. Preliminaries: Pencils of Quadrics

A *quadric* k in the projective space $\mathbb{P}^n$ (all spaces are over the field of complex numbers) is the zero set of a homogeneous quadratic polynomial $k(X) = \sum_{ij} k_{ij}x_ix_j$, whose coefficients are the entries of a symmetric matrix $(k_{ij}) \in \mathbb{C}^N$, $N = n(n+1)/2$, the zero matrix excluded. A quadric k is called *nondegenerate* if $\det(k_{ij}) \neq 0$. In this case, k is a smooth analytic hypersurface of $\mathbb{P}^n$. Since the multiples of (k_{ij}) are the only symmetric matrices that define the same quadric k, we see that the variety $Q\mathbb{P}^n$ of all quadrics in $\mathbb{P}^n$ is itself a projective space $\mathbb{P}^{N-1}$.

A line $\mathcal{K}$ (one dimensional subspace) in $Q\mathbb{P}^n$ is called a pencil of quadrics. If k_0 and k_1 are two distinct quadrics from $\mathcal{K}$, then

$$\mathcal{K} = \{k_\lambda : \lambda \in \mathbb{P}^1\},\ k_\lambda(X) = \lambda_0 k_0(X) + \lambda_1 k_1(X).$$

A pencil is in general position if it can be defined by two nondegenerate quadrics that intersect transversally. As in this article only the cases $n = 1$ and $n = 2$ are relevant, a discussion of both cases will clarify these definitions.

2.1. *Quadrics in* $\mathbb{P}^1$. There is nothing exciting about zero-dimensional quadrics $k(X) = k_{00}x^2 + 2k_{01}xy + k_{11}y^2 = 0$ in $\mathbb{P}^1$. They correspond to the subsets of $\mathbb{P}^1$, that consist of two points, or one double point in the case of a degenerate quadric, for which $\det(k_{ij}) = k_{00}k_{11} - k_{01}^2 = 0$.

A bit more interesting are pencils of such quadrics that are in general position. Suppose that such a pencil is a line through the nondegenerate quadrics $k_0(X)$ and $k_1(X)$. If we transform the zeros of $k_0(X)$ by a birational map of $\mathbb{P}^1$ into $[0,1]$ and $[1,0]$, the equation of k_0 becomes $k_0(X) = xy = 0$. If the distinct zeros of k_1 are $[u,1]$ and $[v,1]$, its equation is $k_1(X) = (x - uy)(x - vy) = 0$. The fact that k_0 and k_1 intersect transversally means in this case, that $uv \neq 0$. Then we get (see [2, §14.2.8]).

Theorem 1 (Desargues' Involution Theorem in $\mathbb{P}^1$). *For a pencil $\mathcal{K}$ of quadrics in $\mathbb{P}^1$, that is in general position, there exists a birational involution ϕ of $\mathbb{P}^1$, i.e. $\phi^2 = \mathrm{id}$, such that for any $k \in \mathcal{K}$, $k = \{M, N\}$, one has $\phi(M) = N$ (and obviously $\phi(N) = M$).*
Furthermore, in such a pencil there are exactly two degenerate quadrics corresponding to the two fixed points F, F' of the involution ϕ. These fixed points form

harmonic ranges

$$(F, F', M, N) = -1 \tag{1}$$

with each of the nondegenerate quadrics $k = \{M, N\}$.

Proof. From Vieta's formula applied to the zeros $M = [M_0, M_1]$ and $N = [N_0, N_1]$ of $\lambda_0 k_0(X) + \lambda_1 k_1(X) = \lambda_0 xy + \lambda_1(x - uy)(x - vy) = 0$ we get $M_0 N_0 = uvM_1 N_1$. Hence,

$$\phi \begin{pmatrix} M_0 \\ M_1 \end{pmatrix} = \begin{pmatrix} 0 & uv \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \end{pmatrix}$$

is the involution with $\phi(M) = N$ we have been looking for.
The equation (1) follows from $(F, F', N, M) = (\phi(F), \phi(F'), \phi(M), \phi(N)) = (F, F', M, N)$ and the properties of the cross-ratio, to be found in any textbook on projective geometry. $\square$

2.2. *Quadrics in* $\mathbb{P}^2$. A *conic* is the zero set of $k(X) = k_{00}x^2 + k_{11}y^2 + k_{22}z^2 + 2k_{01}xy + 2k_{02}xz + 2k_{12}yz$. This is a one-dimensional quadric in $\mathbb{P}^2$ which degenerates, $\det(k_{ij}) = 0$, iff $k(X)$ factorizes into a product of two lines. This can be seen by diagonalizing the matrix (k_{ij}). The cases of $\operatorname{rank}(k_{ij}) = 1, 2, 3$, that are projectively equivalent to $0 = k(X) = x^2$; $x^2 + y^2$; $x^2 + y^2 + z^2$; respectively, correspond to a double line, two crossing lines and a nondegenerate conic.

Let $\mathcal{K}$ be a pencil of conics in general position in the projective plane $\mathbb{P}^2$. This means that all conics of $\mathcal{K}$ have four base points B_1, B_2, B_3, B_4 in common, no three of them collinear. The base points can be moved by a projective transformation of $\mathbb{P}^2$ to $[\pm 1, \pm 1, 1]$, (see Figure 1). The three degenerate conics of the pencil are $k_x : y^2 - z^2 = 0, k_y : z^2 - x^2 = 0, k_z : x^2 - y^2 = 0$. Any conic k_λ of the pencil is the zero set of an equation

$$k_\lambda : \lambda_0(y^2 - z^2) + \lambda_1(z^2 - x^2) = 0,\ \lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1. \tag{2}$$

To a line g, not passing through any base point, Desargues' involution Theorem 1 applies by restriction of the equations of the conics k_λ to g. We get (see [2], [5], [6])

Theorem 2 (Desargues' Involution Theorem in $\mathbb{P}^2$). *Given a pencil $\mathcal{K}$ of conics in $\mathbb{P}^2$ and a line g, not containing any base point of the pencil. Then*

- i *The pairs of intersections of g with the conics of $\mathcal{K}$ generate an involution ϕ on g.*
- ii *The intersections of g with two conics of the pencil determine the involution ϕ uniquely.*
- iii *The fixed points F_1, F_2 of ϕ are the points, where conics from the pencil touch g.*
- iv *The intersections L_1, L_2 of g with any conic k_λ from $\mathcal{K}$ form together with the fixed points F_1, F_2 a harmonic range*

$$(L_1, L_2, F_1, F_2) = -1. \tag{3}$$

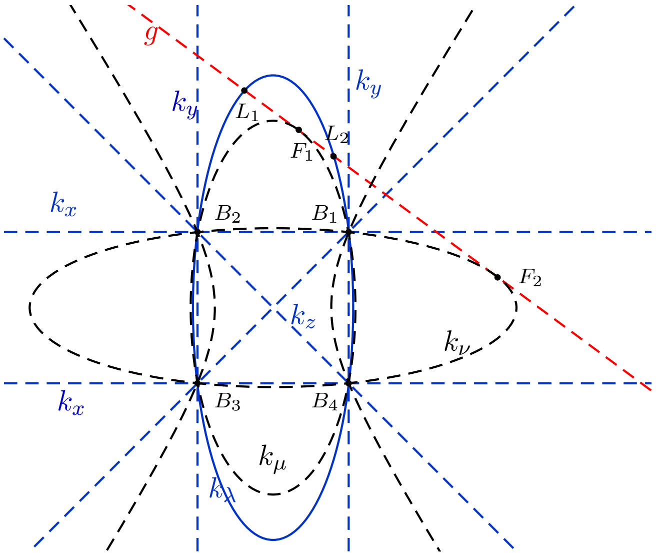

Figure 1

Remark. While in Theorem 1 it is important that the pencil is in general position - if not, all quadrics would have a common point - the essential condition of Theorem 2 is that the line g avoids the base points. The theorem remains true if some base points coincide.

In Figure 2 two generators of pencils of types I-V are represented from left to right. These different types reflect coincidences between some of the base points (cf. [2, §16.5]). Sure, these are just one-dimensional sketches of surfaces of real dimension two in a space of real dimension four.

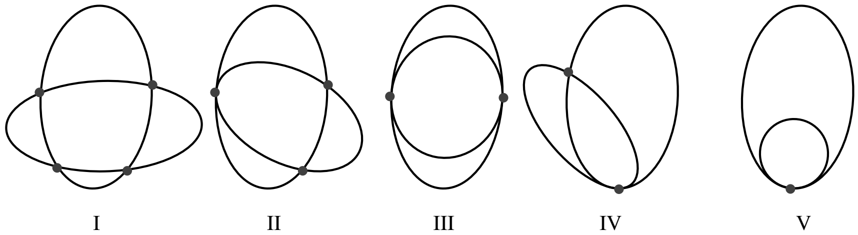

Figure 2

3. The Interplay of Involutions

According to Desargues' Theorem 2, a pencil $\mathcal{K}$ of conics in $\mathbb{P}^2$ induces a biratonal involution on every line not meeting any base point of the pencil. How these involutions on different lines fit together will be explained in the following lemmas[1]. For the first two see Figure 3.

Lemma 3 (**Collinearity of Fixed Points**). *On a conic k_0 from a pencil $\mathcal{K}$ in general position[2], take three points P_0, Q_0, P, distinct from the base points B_i. The intersections with $\mathcal{K}$ induce involutions on the sides of the triangle P_0Q_0P. Let $T_0 \in P_0Q_0$ and $R \in P_0P$ be fixed points of these involutions. Then $S = T_0R \cap PQ_0$ is a fixed point on PQ_0.*

Proof. The conic k_1 touches P_0Q_0 at T_0 and the conic k_λ touches P_0P at R. The line $g = T_0R$ intersects k_1 again at T. Let $\mathcal{L}$ be the pencil containing k_1 and the degenerate conic $P_0Q_0 \cup PT$. Since P_0Q_0 touches k_1, there are two possibilities: Either PT intersects k_1 at two points T, T' and $\mathcal{L}$ is a pencil of type II, or PT is tangent to k_1 at T and $\mathcal{L}$ is a pencil of type III, see Figure 2. We show that the latter occurs and that the double line g is in $\mathcal{L}$.

Step 1: The restrictions of $\mathcal{K}$ and $\mathcal{L}$ induce the same involution on P_0P.

Indeed, one common zero-dimensional quadric is $\{P_0, P\}$ and the other is the intersection of P_0P and the conic k_1 from $\mathcal{K} \cap \mathcal{L}$. Hence, besides the conic $k_\lambda \in \mathcal{K}$, tangent to P_0P at R by assumption, there must be a conic $l \in \mathcal{L}$, tangent to P_0P at this point R.

Step 2: The pencil $\mathcal{L}$ contains the double line g.

The line g intersects the conic l at three points T_0, T and R. Obviously, T is distinct from T_0. Indeed, otherwise g would have a double intersection with k_1 at T_0 and would hence coincide with P_0Q_0, which is absurd. If $T = R$, since there is only one conic from the pencil $\mathcal{K}$ through this point, then $k_1 = k_\lambda$ and $\mathcal{L}$ is of type III containing the double line g.

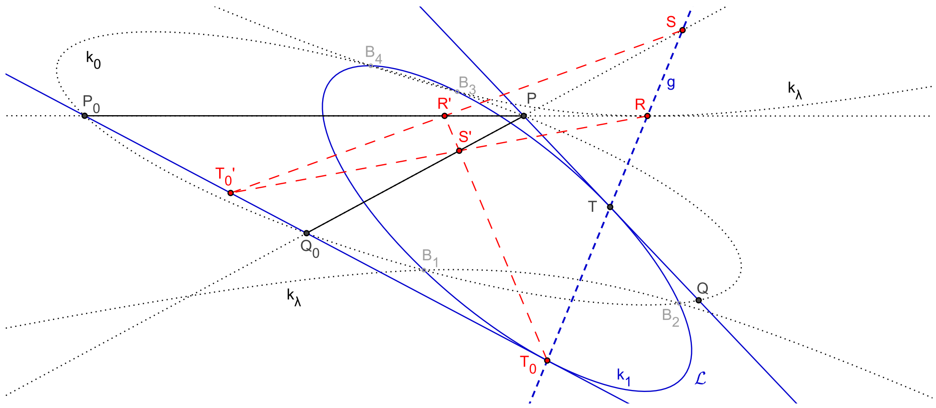

Figure 3

If $T \neq R$, then the line g is a part of l, both having three intersections, so l is degenerate. Like all conics of $\mathcal{L}$, l has a double intersection with P_0Q_0 at T_0. By

[1]The origin of the ideas behind the statements of this section is [15]. A more recent presentation is in [2].

[2]The letter $\mathcal{K}$ from now on will always denote a pencil in general position.

definition, l is tangent to P_0P at R, hence intersects it twice at R. Therefore, l is the double line g. In either way, $\mathcal{L}$ is of type III. The double line g connects the tangency points of P_0Q_0 and PT with k_1.

Step 3: The intersection S of g and PQ_0 is a fixed point on PQ_0.

Applying the argument from step 1 again, we see that the involutions induced on the line PQ_0 by the intersections with $\mathcal{L}$ and $\mathcal{K}$ are the same. Hence, the intersection S of the double line $g = T_0R$ from $\mathcal{L}$ with PQ_0 must be a fixed point of both pencils. So we conclude that the fixed points T_0, R and S are collinear. □

Corollary 4. *For fixed points T_0, R, S of the involutions on the sides of the triangle P_0Q_0P we have the following dichotomy: Either these points are collinear* (*Menelaus configuration*) *or their respective connexions with the vertices P, Q_0, P_0 are concurrent* (*Ceva configuration*).

Proof. From $(Q_0, P, S, S') = -1$, cf. Figure 3 and (3), it follows that

$$\frac{P_0T_0}{T_0Q_0}\frac{Q_0S}{SP}\frac{PR}{RP_0} = -\frac{P_0T_0}{T_0Q_0}\frac{Q_0S'}{S'P}\frac{PR}{RP_0}.$$

One triplet T_0, R, S or T_0, R, S' is collinear (Menelaus configuration) according to Lemma 3. The other triplet forms a Ceva configuration by the theorems of Menelaus and Ceva. □

The second intersection Q of PT with k_0 produces a quadrangle P_0Q_0PQ inscribed in the conic k_0. Both P_0Q_0 and PQ touch k_1. We call $\mathcal{K}-$*parallelograms* quadrangles inscribed in k_0 with a pair of opposite sides touching the same conic of the pencil $\mathcal{K}$. This naming will become transparent after Definition 1 below. In $\mathcal{K}-$parallelograms not only one but each pair of opposite sides is tangent to one conic of the pencil. This is the content of Lemma 5, see Figure 4.

Lemma 5. *The $\mathcal{K}-$parallelogram P_0Q_0PQ is inscribed in the conic k_0 with P_0Q_0 and PQ tangent to k_1 at T_0 and T, respectively. Then PQ_0 and QP_0 touch a conic $k_\mu \in \mathcal{K}$ and the tangency points S and U are on the line $g = T_0T$. The same is true for the other pair of opposite sides PP_0 and QQ_0.*

Proof. The same reasoning as in Lemma 3 shows that the involutions induced by the pencil $\mathcal{L}$, containing $P_0Q_0 \cup PQ, k_1$ and the double line g, and the pencil $\mathcal{K}$ on each of the lines PQ_0 and QP_0 must coincide. Hence, the intersections S and U with the double line $g \in \mathcal{L}$ are tangency points of the lines PQ_0 and QP_0 with conics $k_\mu, k_\nu \in \mathcal{K}$, possibly different. Let us show that $k_\mu = k_\nu$ by introducing the pencil $\mathcal{M}$ of all conics through P_0, Q_0, P, Q. The same involutions are induced by $\mathcal{M}$ and $\mathcal{K}$ on g. In fact, $P_0Q_0 \cup PQ \in \mathcal{M}$ as well as $k_1 \in \mathcal{K}$, intersect g at $\{T_0, T\}$ and k_0 is in both pencils. So there must be one conic in $\mathcal{K}$ going through the points $\{S, U\}$, where g intersects $PQ_0 \cup QP_0 \in \mathcal{M}$. This conic is $k_\mu = k_\nu$. □

Lemma 6 (**Completing to a $\mathcal{K}-$parallelogram**). *If T_0 and R are fixed points of the involutions induced on P_0Q_0 and P_0P by the pencil $\mathcal{K}$, then there exists one and only one $\mathcal{K}-$parallelogram P_0Q_0PQ for which T_0 and R are amongst the points where pairs of opposite sides touch a conic of the pencil.*

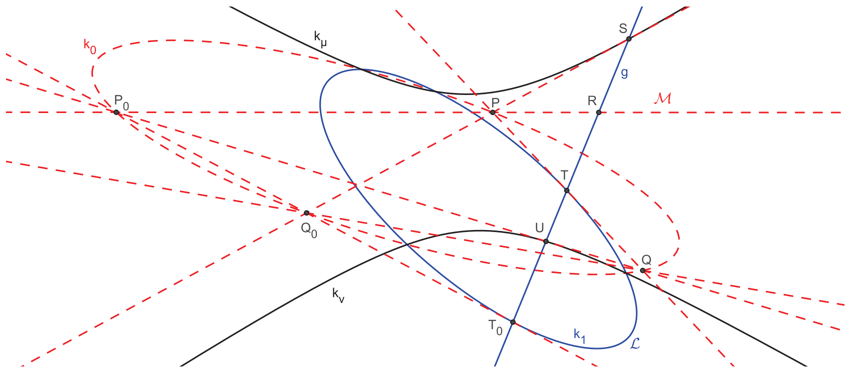

Figure 4

Proof. Since the line $g = T_0R$ contains all tangency points where pairs of opposite sides of $\mathcal{K}-$parallelograms completing P_0Q_0P touch the same conic of the pencil according to Lemma 5, the construction of T and Q carried out in the proof to Lemma 3 yields a necessary condition to construct the $\mathcal{K}-$parallelogram. Lemma 5 shows that the completion of P_0Q_0P by Q does indeed give the desired result. $\square$

In the next section, when we define a parallelism of Poncelet structures we need this parallelism to be transitive. This transitivity is related to the content of the next lemma.

Lemma 7 (Transitivity). *Let P_0, P_1, P_2 be distinct points on a conic k_0 so that P_iT_i are tangent at T_i to the same conic k_1. Then the intersections $R_{i+2} = T_iT_{i+1} \cap P_iP_{i+1}$ (mod 3), being fixed points of the involutions induced by the pencil $\mathcal{K}$ through k_0 and k_1 on P_iP_{i+1} by Lemma 5, form a Ceva configuration on the sides of the triangle $P_0P_1P_2$,* (*cf.* *Corollary 4*).

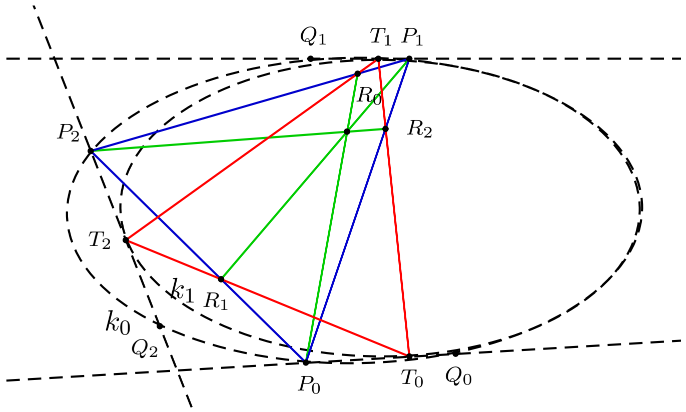

Figure 5

Proof. If we set $Q_i = P_iT_i \cap k_0$, then $P_iQ_iP_{i+1}Q_{i+1}$ are $\mathcal{K}-$parallelograms to which Lemma 5 applies and shows that R_{i+2} are fixed points of the involutions on the respective sides of $P_0P_1P_2$, see Figure 5. Apart from cases of coinciding tangency points if $P_2 \to Q_0$ and $Q_2 \to P_0$, see Figure 6, that we kindly ask the reader to clarify by himself, the points T_i are distinct. Tangents at three distinct points to a conic are never concurrent, hence the triangles $P_0P_1P_2$ and $T_0T_1T_2$ lack a center of perspectivity. According to Desargues' two triangle theorem ([5, §2.3]) they lack an axis of perspectivity too. This excludes the possibility for the intersections R_i of corresponding sides of these triangles to be collinear, so they must form a Ceva configuration according to Corollary 4. □

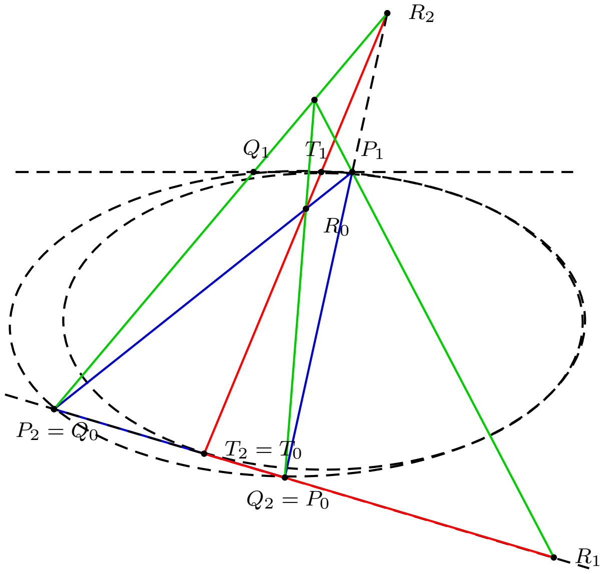

Figure 6

4. The Cubic

From now on let $\mathcal{K}$ be the pencil of conics through the base points $B_i = [\pm 1, \pm 1, 1]$, and let $k_\lambda : \lambda_0(y^2 - z^2) + \lambda_1(z^2 - x^2) = 0$ be one of the nondegenerate conics from $\mathcal{K}$ and $O = [u, v, w]$ a point of k_λ, which is not a base point.

If tangents are drawn from O to all conics of $\mathcal{K}$, their contact points $T = [x, y, z]$ are on a cubic E_O. To obtain its equation

$$E_O(x, y, z) = ux(y^2 - z^2) + vy(z^2 - x^2) + wz(x^2 - y^2) = 0 \qquad (4)$$

we have to eliminate the parameters μ_i from the conic $\mu_0(y^2 - z^2) + \mu_1(z^2 - x^2) = 0$ and the equation of the polar line $\mu_0(vy - wz) + \mu_1(wz - ux) = 0$ of the pole O (for polars cf. [2, §14.5]). This cubic already appeared in Lebesgue's article, section II, although without coordinates, see [15, p.121]. It is curious that

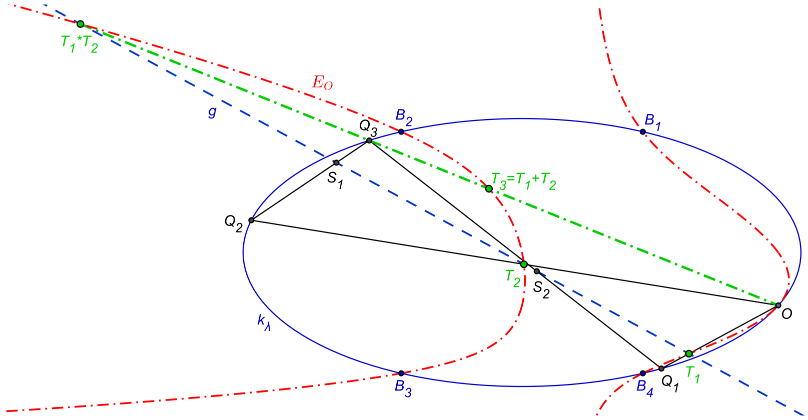

Figure 7

its properties as an elliptic curve with an addition closely related to the properties of Poncelet configurations were not exploited there.

The Weierstrass form (see [13, p.56]) of E_O is, thanks to Maple,

$$\begin{aligned} y^2 &= x^3 + Ax + B, \\ A &= -\frac{1}{6}(\Delta_{uv}^2 + \Delta_{vw}^2 + \Delta_{wu}^2), \\ B &= \frac{1}{27}(\Delta_{uv} - \Delta_{vw})(\Delta_{vw} - \Delta_{wu})(\Delta_{wu} - \Delta_{uv}), \end{aligned} \tag{5}$$

with the shortcuts $\Delta_{uv} = u^2 - v^2$, $\Delta_{vw} = v^2 - w^2$, $\Delta_{wu} = w^2 - u^2$. From (5) we obtain, using the relation $\Delta_{uv} + \Delta_{vw} + \Delta_{wu} = 0$, the discriminant Δ of the cubic E_O as

$$\Delta = -16(4A^3 + 27B^2) = 16\Delta_{uv}^2\Delta_{vw}^2\Delta_{wu}^2.$$

Consequently, the discriminant Δ vanishes on k_λ only at the base points. Since O is not a base point, the curve E_O is nonsingular, see [13, p.58], hence an elliptic curve[3].

The next theorem is fundamental. It establishes the link between Poncelet configurations and the addition on the elliptic curve E_O with origin O.

Theorem 8. *Let $T_1, T_2 \in E_O$ be tangency points on the secants OQ_1 and OQ_2, respectively. Completing OQ_1Q_2 to a $\mathcal{K}-$parallelogram yields $OQ_1Q_2Q_3$ as in Lemma 6. The line $g = T_1T_2$ intersects Q_2Q_3 at S_1 where this secant touches the same conic k_μ as OQ_1 at T_1. The same is true for the intersection $S_2 = g \cap Q_1Q_3$ and a conic k_ν touching OQ_2 and Q_1Q_3 at T_2 and S_2. The addition on E_O produces $T_3 = T_1 + T_2$ on OQ_3. This point T_3 forms Ceva configurations together with T_1 and S_2 on the sides of OQ_1Q_3, and together with T_2 and S_1 on the sides of OQ_2Q_3, see Figure 7.*

[3]Who neither trusts the program Maple and is not willing either to check the Weierstrass form by hand, may apply Proposition 11 with the not yet defined addition on E_O substituted by Lemma 6 to convince himself that E_O is an elliptic curve.

Proof. The line $g = T_1T_2$ intersects all sides of the quadrangle $OQ_1Q_2Q_3$ at tangency points with conics from $\mathcal{K}$. Its intersection $T_1 * T_2$ with the line OQ_3 is the third intersection of T_1T_2 with E_O. The sum $T_3 = T_1 + T_2$ is the third intersection of the line through O and $T_1 * T_2$ with E_O. From Corollary 4 the remaining assertions of the theorem follow easily. □

The addition on the elliptic curve E_O can be made more visual if we identify tangency points on elliptic curves with different origins, see Figure 8.

Definition 1. Let $T_1, T_2 \in E_O$ be tangency points on the secants OQ_1 and OQ_2. We say that T_1 and a point $S_1 \in E_{O_2}$ are equivalent $(S_1 \sim T_1)$ or that the lines OT_1 and Q_2S_1 are $\mathcal{K} - parallel$ along OT_2Q_2 if T_1, T_2, S_1 are collinear and OT_1 and Q_2S_1 touch the same conic.

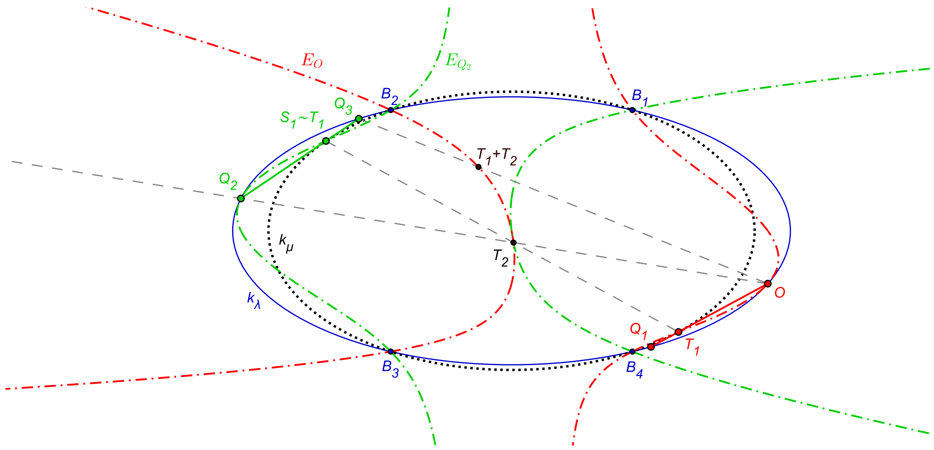

Figure 8

With this identification, writing the tangency points that form a Ceva configuration between the start points and end points, we have

$$OT_2Q_2 + Q_2T_1Q_3 = O(T_1 + T_2)Q_3 \tag{6}$$

as if this were an addition of vectors. From Theorem 8 it follows with a look at Figure 7 that T_2 and S_2 are equivalent too. More precisely, they are equivalent along OT_1Q_1, and

$$OT_1Q_1 + Q_1T_2Q_3 = O(T_1 + T_2)Q_3. \tag{7}$$

Corollary 9. *Let $T_1, T_2 \in E_O$ be tangency points on the secants OQ_1, OQ_2 and its sum $T_3 = T_1 + T_2$ the corresponding tangency point on the secant OQ_3. The tangency point S_1 on Q_2Q_3 is equivalent to T_1 if and only if S_1, T_2 and T_3 form a Ceva configuration on the sides of OQ_2Q_3.*

Proof. Look at $T_1 + T_2$ and $T_1 * T_2$ in the proof of Theorem 8 and apply Corollary 4. □

This definition of $\mathcal{K}-$parallelism explains also the name of $\mathcal{K}-$parallelograms: They are quadrangles with $\mathcal{K}-$parallel opposite sides. The next proposition says that for secants these two concepts are equivalent: touching the same conic and being $\mathcal{K}-$parallel.

Proposition 10. *If secants OQ_1 and Q_2Q_3 of k_λ touch the same conic of the pencil $\mathcal{K}$ at T_1 and S_1 then there exists a unique point $T_2 \in E_O$ so that T_1 and S_1 are equivalent and the secants are $\mathcal{K}-$parallel along OT_2Q_2.*

Proof. From Lemma 5 it follows that T_1S_1 intersects OQ_2 at a point T_2, that is one of the fixed points of the involution induced by $\mathcal{K}$ on OQ_2 and hence belongs to E_O. Then T_1 and S_1 are equivalent along OT_2Q_2 by definition. The uniqueness of T_2 is obvious. □

Griffiths and Harris showed ([9, p.38]) by abstract considerations that the incidence variety

$$E = \{(X, \xi) \in k_\lambda \times k_\mu^* : X \in \xi\} \tag{8}$$

of all pairs of secants ξ with starting point $X \in k_\lambda$, which are tangent to a fixed conic k_μ (hence elements of k_μ^*) is an elliptic curve. Here we have a concrete bijection from E to E_O:

Proposition 11. *Let $T_1 \in E_O$ be a tangency point so that OT_1 is tangent to the conic k_μ. Further let $p : k_\mu^* \to k_\mu$ be the pole map restricted to k_μ^*. Then a bijection from E onto E_O is*

$$(X, \xi) \mapsto T = T_1 p(\xi) \cap OX. \tag{9}$$

Proof. Let $q : E_O \to k_\lambda$ associate to each tangency point $T \in E_O$ the second intersection of OT with k_λ. Then $T \mapsto (q(T), q(T)q(T + T_1))$ is the inverse of (9), see Figure 9.

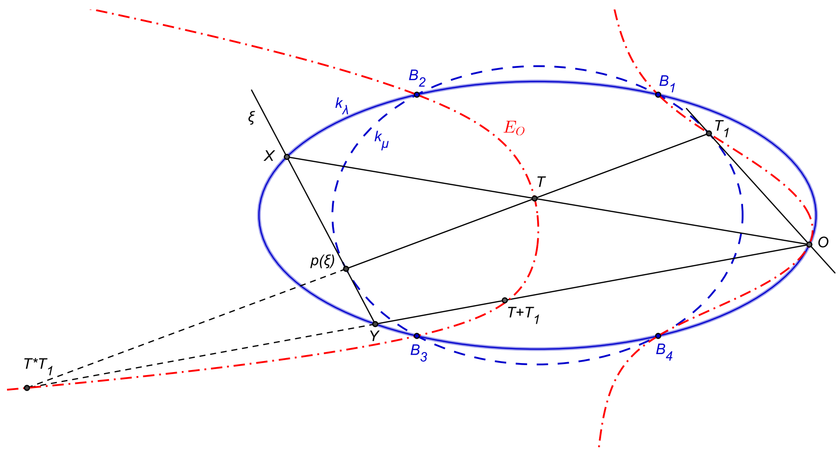

Figure 9

□

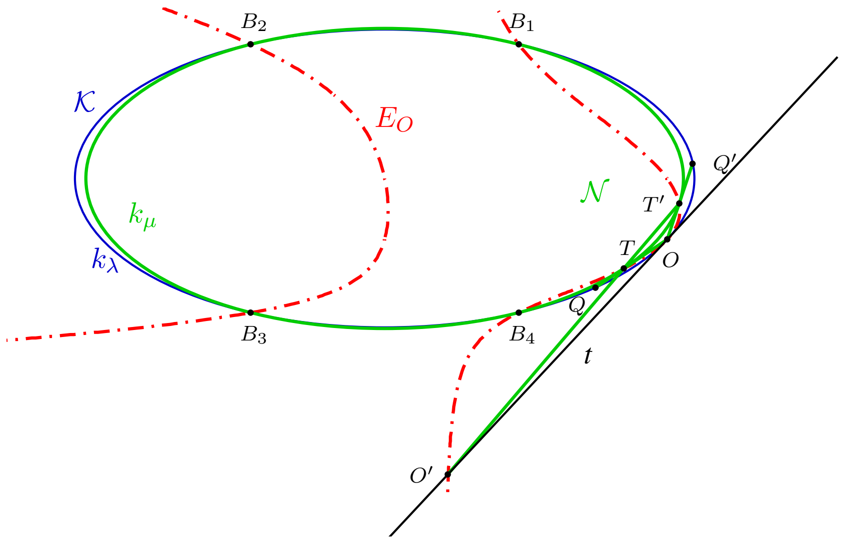

Figure 10

Just two remarks: First, although equivalence mappings between the elliptic curve E_O and the cubics E_{B_i} cannot exist since the equations $\pm x(y^2 - z^2) \pm y(z^2 - x^2) + z(x^2 - y^2) = 0$ of the latter factorize into the product of three lines through B_i, one has the mappings

$$T \mapsto q(B_i)q(B_i + T) = B_i q(B_i + T) \tag{10}$$

from E_O onto the line bundle through B_i which extend continuously the $\mathcal{K}-$ parallelism to the secants through the base points. Below it will be shown that OT and $O(-T)$ touch the same conic, hence $B_i q(B_i + T)$ and $B_i q(B_i - T)$ too. Then $B_i q(B_i + T) = B_i q(B_i - T)$, since B_i belongs to all conics and through a point on a conic only one tangent can be drawn. Therefore, the mapping (10) is a double cover of the $\mathbb{P}^1$-isomorphic line bundle through B_i, branched over the four lines B_iB_j.

Second, in Proposition 11 it was shown that all elliptic curves E_{O_i} are isomorphic to the elliptic curve E from (8). Before proving that the equivalences of tangency points along secants via a tangency point are isomorphisms of the elliptic curves E_{O_i} we keep the promise and confirm that OT and $O(-T)$ touch the same conic.

Proposition 12. *If OT is tangent to k_μ at $T \in E_O$ then $(-T)$ is the tangency point of the other tangent from O to k_μ.*

Proof. Let OQ and OQ' with tangency points T and T' be the tangents from O to a conic $k_\mu \in \mathcal{K}$, see Figure 10. The main observation is that k_λ and E_O share the tangent line t at O. To see this, we can consider a secant OP with a tangency point S close to O and let P and S approach O or calculate the tangent to E_O in O from the equation (4) and compare it with the tangent at this point to the conic k_λ.

Let $\mathcal{N}$ be the pencil of type III containing k_μ, the degenerate conic $OT \cup OT'$ and therefore also the double line TT'. Both pencils $\mathcal{K}$ and $\mathcal{N}$ induce by intersections the same one-dimensional pencil on t because both produce the double point O and the intersections with k_μ. Consequently, the intersection O' of the double line TT' with t must also be a tangency point with a conic from $\mathcal{K}$, hence an element of E_O. Since t already has a double intersection with E_O in O, the point $O' = O * O$ is the third intersection of t with the cubic and $T + T' = O$. $\square$

Corollary 13. *For $T \in E_O$ let $q(T) = Q$ be the second intersection of OT with k_λ. The other preimage of Q under q is the third intersection $O * T$ of the line OT with E_O, given by*

$$O * T = O * O - T. \tag{11}$$

Proposition 14. *The equivalence $\epsilon_{OT_2Q_2}$ of tangency points from E_O via T_2 with those from E_{Q_2} according to Definition 1. is an isomorphism of these elliptic curves*[4]. *The composition of equivalences is transitive* (*cf.* *also (6) and (7) and contemplate Figure 8*).

$$\epsilon_{Q_2T_1Q_3} \circ \epsilon_{OT_2Q_2} = \epsilon_{O(T_1+T_2)Q_3}, \tag{12}$$

Proof. All elliptic curves E_{O_i} go through seven points: four base points $B_i = [\pm 1, \pm 1, 1]$ and the respective singular points $X = [1,0,0], Y = [0,1,0], Z = [0,0,1]$ of the singular conics $k_x : y^2 - z^2 = 0, k_y : z^2 - x^2 = 0, k_z : x^2 - y^2 = 0$ of $\mathcal{K}$. The remaining two from the nine intersections of E_O and E_{Q_2} are obviously the tangency points T_2 and T_2' on the line OQ_2. If we take any point $T \in E_O$, the line TT_2 will intersect E_{Q_2} in three points: T_2, S and S'. To choose the correct image of T under $\epsilon_{OT_2Q_2}$ among S and S', i.e. the $\mathcal{K}-$equivalent one, we take the second intersection of TT_2 with the conic through T. This intersection is uniquely defined and can be found by eliminating the parameters μ_i from $\mu_0 k_x(T) + \mu_1 k_y(T) = 0$ and intersecting $k_\mu(X) = k_y(T)k_x(X) - k_x(T)k_y(X) = 0$ with the line TT_2, see Lemma 6 and Figure 8. Another way to see that $S = \epsilon_{OT_2Q_2}(T)$ is obtained from T by a regular mapping is to write it as

$$\epsilon_{OT_2Q_2}(T) = q(T_2)q(T + T_2) \cap TT_2,$$

with q defined in the proof to Proposition 11.

These mappings $\epsilon_{OT_2Q_2}$ are non-constant isogenies (biholomorphic mappings, respecting the origins) of elliptic curves, hence isomorphisms of the underlying group structure, see [13, chap VI.4], and [18, chap III.4]. This can be seen by lifting the mapping to the universal cover $\mathbb{C}$ of the elliptic curves, where its derivative is an everywhere bounded analytic function, hence constant. So, the covering mapping, being linear and respecting the origins, is a multiplication with a complex number and as such an isomorphism of the group structure. This remains true if the covering mapping descends from the cover to the elliptic curves.

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[4]The mere fact that all $E_O, O \in k_\lambda \setminus \{B_i\}$, are isomorphic, without indicating explicit isomorphisms, can also be deduced from their $j-$invariants, expressing them with the Weierstrass form (5) only by the parameter $\lambda = [\lambda_0, \lambda_1]$ of k_λ. In order to do this, we only have to eliminate the coordinates $[u, v, w]$ using equation (2). For $j-$invariant see [18, Proposition 1.4]

The equation (12) is a consequence of the transitivity explained in Lemma 7 and of Corollary 9. □

Note that there is another isomorphism from E_O to E_{Q_2}, i.e. $\epsilon_{OT_2'Q_2}$ which goes via the second intersection T_2' of E_O and E_{Q_2}. By Proposition 12, each of them is the negative of the other

$$\epsilon_{OT_2'Q_2} = -\epsilon_{OT_2Q_2}. \tag{13}$$

5. Poncelet Configurations

As before, on a nondegenerate conic k_λ from the pencil $\mathcal{K}$ a point O, distinct from the base points of $\mathcal{K}$, is fixed. Let E_O be the elliptic curve defined in (4).

Definition 2. We call the conics $k_0, \ldots, k_{n-1} \in \mathcal{K}$ a *Poncelet configuration* if for every $Q_0 \in k_\lambda$ there is a closed polygonal chain $(Q_0, \ldots, Q_{n-1})$ inscribed in k_λ so that its secants $Q_iQ_{i+1}, i = 0, 1, \ldots (\operatorname{mod} n)$ touch all conics $k_0, \ldots, k_{n-1}$ in some order.

Theorem 15. *In the same settings as above, let $k_0, \ldots, k_{n-1} \in \mathcal{K}$ be given together with points $\Delta T_i \in E_O \cap k_i$, where tangents from O touch k_i. These conics form a Poncelet configuration if and only if some of the sums $\Sigma = \pm\Delta T_0 \pm \cdots \pm \Delta T_{n-1}$ vanish.*

Proof. Suppose that any of the above sums vanishes. Since ΔT_i and $-\Delta T_i$ are the points where tangents from O touch k_i (Proposition 12), let us assume, by switching signs if necessary, that $\Sigma = \Delta T_0 + \cdots + \Delta T_{n-1} = 0$. For $Q_0 \in k_\lambda$ take a $T_0 \in E_O$ with $q(T_0) = Q_0$, for the mapping q see Corollary 13, and define inductively $T_{i+1} = T_i + \Delta T_i$ and $Q_{i+1} = q(T_{i+1})$. By Proposition 14, both secants $q(T_i)q(T_{i+1})$ and $O\Delta T_i$ are $\mathcal{K}-$parallel and therefore touch the same conic k_i. This argument is valid if among the end points of the secants there is no base point. By the continuous extension (10), the tangency to the same conic persists if some of the end points of the secants are base points. The polygonal chain $(Q_0, \ldots, Q_n)$ is closed since $T_n = T_1 + \Sigma = T_1$.

Now suppose that none of the sums is zero. We will show that $\Sigma = \Delta T_0 + \cdots + \Delta T_{n-1} \neq 0$ implies that there are only four $T_0 \in E_O$ for which a polygonal chain, starting in $Q_0 = q(T_0)$ and going through consecutive points Q_i via tangency points S_i that are equivalent to ΔT_i, will close after n steps at Q_0. The choice of the other tangent from Q_i to k_i results in a sign change of ΔT_i (Proposition 12). Since none of these sign changes yields the sum zero of the $\pm\Delta T_i$, due to the assumption made above, it follows that there are at most $4 \cdot 2^n$ starting points, all possible sign changes taken into account and all permutations of the k_i neglected for having no effect on the end point, for which the polygonal chain is closed. But for a Poncelet configuration there should be a closed chain for every starting point.

Drawing from $Q_0 = q(T_0)$ the tangent to k_0 through the point $\epsilon_{OT_0Q_0}(\Delta T_0) \in E_{Q_0}$ of tangency we arrive by (6) at the point $Q_1 = q(T_1), T_1 = T_0 + \Delta T_0$. These formulas for Q_1 and T_1 remain true even if T_0 is one of the base points and there is no isomorphism from E_O onto E_{Q_0}, see (10). Eventually, we arrive at $Q_{n-1} = q(T_{n-1}), T_{n-1} = T_0 + \Delta T_0 + \cdots + \Delta T_{n-2}$. For the closing of the polygon by a tangent from Q_{n-1} to k_{n-1} through the tangency point $S_{n-1} = \epsilon_{OT_{n-1}Q_{n-1}}(\Delta T_{n-1}) \in E_{Q_{n-1}}$ it is necessary that

$$q(T_{n-1} + \Delta T_{n-1}) = q(T_0 + \Sigma) = q(T_0).$$

Since $T_0 + \Sigma \neq T_0$, the line $T_0(T_0 + \Sigma)$ contains O, see Figure 11. Hence, $O = T_0 * (T_0 + \Sigma)$ and

$$T_0 + (T_0 + \Sigma) = O * O. \tag{14}$$

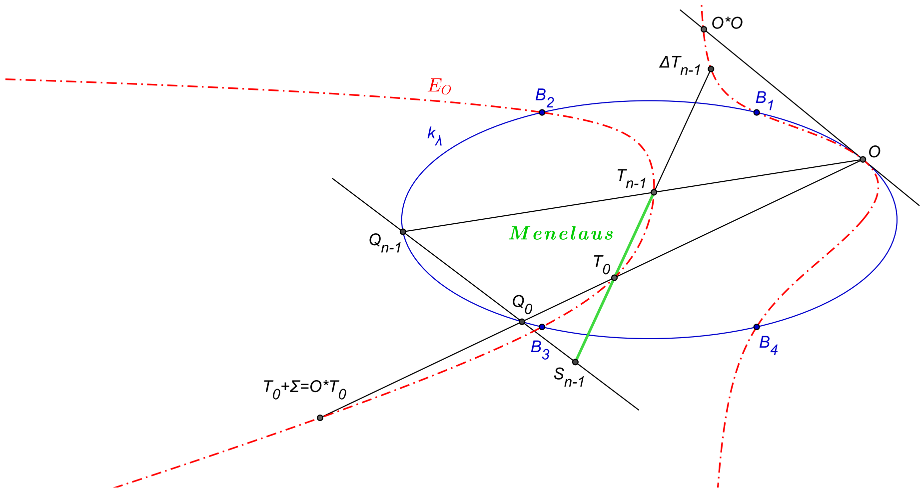

Figure 11

But (14) has just four solutions: $T_0 = \frac{O*O-\Sigma}{2} + U$, where $U \in E_O$ is a point of order two. It follows from $U = -U$ and Proposition 12. that there exists a conic k_μ from $\mathcal{K}$ such that OU is the unique tangent from O to k_μ. If k_μ is nondegenerate, then $k_\mu = k_\lambda$ and $U = O$. If k_μ is one of the degenerate conics k_x, k_y, k_z of $\mathcal{K}$ formed by two lines, then U is their respective intersection $X[1,0,0], Y[0,1,0]$ or $Z[0,0,1]$.

The equation $2U = O$ for the points U of order two can be solved as well by the following observation. Its four solutions are those points U for which the secant line $O'U$ through U and the point $O' = O * O = [1/u, 1/v, 1/w]$ is tangent to E_O at U. It can easily be verified that this occurs at the previously determined points $U = O, X[1,0,0], Y[0,1,0], Z[0,0,1]$.

This proves the claim about the existence of at most four starting points $Q_0 = q(T_0)$ for which we get a closed polygonal chain along tangents to k_i, that are parallel to ΔT_i. $\square$

The next proposition states a numerical criterion to check out if a polygon $Q_0, \ldots, Q_{n-1}$ produces via given tangency points points $S_i \in Q_iQ_{i+1}$ the conics of a Poncelet configuration.

Proposition 16. *In the same settings as above, let $Q_0, \ldots, Q_{n-1} \in k_\lambda \setminus \{B_i\}$ be given (the base points are excluded since (15) makes no sense for these points) and on each secant Q_iQ_{i+1} a fixed point S_i of the involution induced by $\mathcal{K}$ on it. Then*

$$\frac{Q_0S_0}{S_0Q_1}\frac{Q_1S_1}{S_1Q_2}\cdot \ldots \cdot \frac{Q_{n-1}S_{n-1}}{S_{n-1}Q_0} = \pm 1. \tag{15}$$

If the result is 1, the conics k_i tangent to Q_iQ_{i+1} at S_i form a Poncelet configuration.

If the result is -1, take an odd number of points S_i on Q_iQ_{i+1} and substitute them by the other tangent point S_i' to change the sign in (15). With an appropriate change of the correponding conics we obtain again a Poncelet configuration.

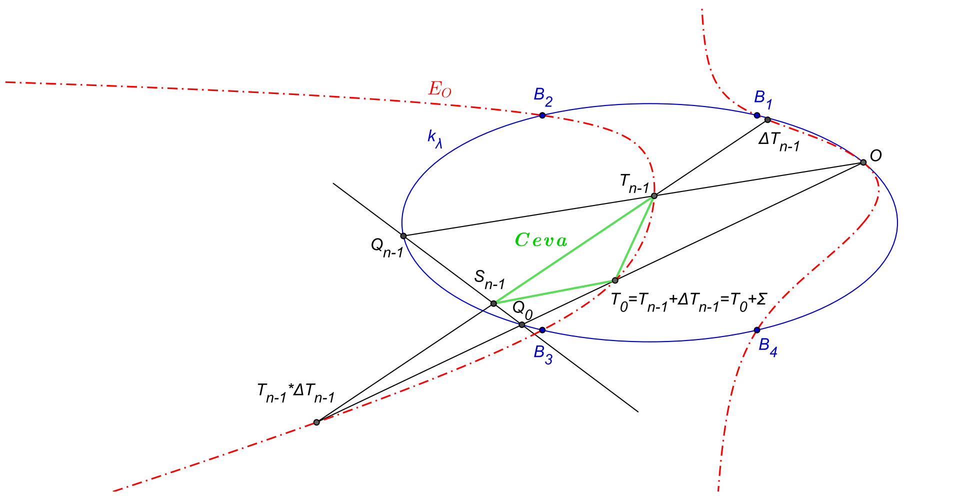

Figure 12

Proof. Go through the construction of T_i with $q(T_i) = Q_i$ as before and take always the next T_{i+1} that forms a Ceva configuration together with T_i and S_i on the sides of OQ_iQ_{i+1}. By Corollary 9. we have $T_{i+1} = T_i + \Delta T_i$ and $q(T_{i+1}) = Q_{i+1}$ for $\Delta T_i = \epsilon_{Q_iT_iO}(S_i)$. You can close up the polygon and get $\sum \Delta T_i = 0$ if and only if the last triangle OQ_0Q_{n-1} with T_0, T_{n-1} and S_{n-1} on its sides forms a Ceva configuration, see Figure 12. In this case (15) is obtained by multiplying the Ceva relations for all triangles OQ_iQ_{i+1}. If the last triangle is a Menelaus configuration, (15) gives -1 from multiplying this Menelaus relation with the other Ceva relations. The substitution of a tangency point on Q_iQ_{i+1} by the other one changes the sign in (15) because of (3). □

In order to conclude this article, let us ask if there could be defined an action of the group E_O on polygons inscribed in k_λ, because something similar was done in the real case, see [11]. It can be seen very quickly that this is impossible in the complex case. There is no reasonable group action of E_O on k_λ. The above mentioned points $O, X[1,0,0]$, $Y[0,1,0]$, $Z[0,0,1]$ of order two are the only elements S of E_O for which the translation $\tau_S : T \mapsto S+T$ of E_O descends via $q : E_O \to k_\lambda$ to a well defined mapping τ_S on the conic. From Corollary 13 we have that $q(O*O-T) = q(T)$. If we want τ_S to descend from E_O to a mapping on k_λ, then we must have $q(S+O*O-T) = q(S+T)$ for all T. This is possible only if $S+O*O-T = O*O-(S+T)$, i.e. if $S = -S$ is an element of order two. Only the four translations $\tau_O, \tau_X, \tau_Y, \tau_Z$ can be transferred to the conic k_λ. They give the Frégier involutions through the Frégier points X, Y, Z (see [2, §16.3]), or in the concrete example of $\mathcal{K}$ the reflections across the $x-$, $y-$ and $z-$ axis that form together with the identity a very small group, the Klein four-group.

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