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On The Relations Between Truncated Cuboctahedron, Truncated Icosidodecahedron and Metrics

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Abstract. The theory of convex sets is a vibrant and classical field of modern mathematics with rich applications. The more geometric aspects of convex sets are developed introducing some notions, but primarily polyhedra. A polyhedra, when it is convex, is an extremely important special solid in $\mathbb{R}^n$. Some examples of convex subsets of Euclidean 3-dimensional space are Platonic Solids, Archimedean Solids and Archimedean Duals or Catalan Solids. There are some relations between metrics and polyhedra. For example, it has been shown that cube, octahedron, deltoidal icositetrahedron are maximum, taxicab, Chinese Checker's unit sphere, respectively. In this study, we give two new metrics to be their spheres an Archimedean solids truncated cuboctahedron and truncated icosidodecahedron.

1. Introduction

A polyhedron is a three-dimensional figure made up of polygons. When discussing polyhedra one will use the terms faces, edges and vertices. Each polygonal part of the polyhedron is called a face. A line segment along which two faces come together is called an edge. A point where several edges and faces come together is called a vertex. That is, a polyhedron is a solid in three dimensions with flat faces, straight edges and vertices.

A regular polyhedron is a polyhedron with congruent faces and identical vertices. There are only five regular convex polyhedra which are called Platonic solids. A convex polyhedron is said to be semiregular if its faces have a similar configuration of nonintersecting regular plane convex polygons of two or more different types about each vertex. These solids are commonly called the Archimedean solids. Archimedes discovered the semiregular convex solids. However, several centuries passed before their rediscovery by the renaissance mathematicians. Finally, Kepler completed the work in 1620 by introducing prisms and antiprisms as well as four regular nonconvex polyhedra, now known as the Kepler–Poinsot polyhedra. Construction of the dual solids of the Archimedean solids was completed in 1865 by Catalan nearly two centuries after Kepler (See [12]). The duals are known as the Catalan solids. The Catalan solids are all convex. They are face-transitive

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when all its faces are the same but not vertex-transitive. Unlike Platonic solids and Archimedean solids, the face of Catalan solids are not regular polygons.

Minkowski geometry is non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set [13].

Some mathematicians have studied and improved metric geometry in $\mathbb{R}^3$. According to mentioned researches it is found that unit spheres of these metrics are associated with convex solids. For example, unit sphere of maximum metric is a cube which is a Platonic Solid. Taxicab metric's unit sphere is an octahedron, another Platonic Solid. In [1, 2, 4, 5, 7, 8, 9, 10, 11] the authors give some metrics of which the spheres of the 3-dimensional analytical space equipped with these metrics are some of Platonic solids, Archimedian solids and Catalan solids. So there are some metrics which unit spheres are convex polyhedrons. That is, convex polyhedrons are associated with some metrics. When a metric is given, we can find its unit sphere in related space geometry. This enforce us to the question "Are there some metrics whose unit sphere is a convex polyhedron?". For this goal, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solid's surface. In this study, we introduce two new metrics, and show that the spheres of the 3-dimensional analytical space equipped with these metrics are truncated cuboctahedron and truncated icosidodecahedron. Also we give some properties about these metrics.

2. Truncated cuboctahedron and its metric

The story of the rediscovery of the Archimedean polyhedra during the Renaissance is not that of the recovery of a 'lost' classical text. Rather, it concerns the rediscovery of actual mathematics, and there is a large component of human muddle in what with hindsight might have been a purely rational process. The pattern of publication indicates very clearly that we do not have a logical progress in which each subsequent text contains all the Archimedean solids found by its author's predecessors. In fact, as far as we know, there was no classical text recovered by Archimedes. The Archimedean solids have that name because in his Collection, Pappus stated that Archimedes had discovered thirteen solids whose faces were regular polygons of more than one kind. Pappus then listed the numbers and types of faces of each solid. Some of these polyhedra have been discovered many times. According to Heron, the third solid on Pappus' list, the cuboctahedron, was known to Plato. During the Renaissance, and especially after the introduction of perspective into art, painters and craftsmen made pictures of platonic solids. To vary their designs they sliced off the corners and edges of these solids, naturally producing some of the Archimedean solids as a result. For more detailed knowledge, see [3] and [6].

It has been stated in [14], an Archimedean solid is a symmetric, semiregular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. A polyhedron is called semiregular if its faces are all regular polygons and its corners are alike. And, identical vertices are usually means that for two taken vertices there must be an isometry of the entire solid that transforms one vertex to the other.

The Archimedean solids are the only 13 polyhedra that are convex, have identical vertices, and their faces are regular polygons (although not equal as in the Platonic solids).

Five Archimedean solids are derived from the Platonic solids by truncating (cutting off the corners) a percentage less than $1/2$.

Two special Archimedean solids can be obtained by full truncating (percentage $1/2$) either of two dual Platonic solids: the Cuboctahedron, which comes from trucating either a Cube, or its dual an Octahedron. And the Icosidodecahedron, which comes from truncating either an Icosahedron, or its dual a Dodecahedron. Hence their "double name".

The next two solids, the Truncated Cuboctahedron (also called Great Rhombicuboctahedron) and the Truncated Icosidodecahedron (also called Great Rhombicosidodecahedron) apparently seem to be derived from truncating the two preceding ones. However, it is apparent from the above discussion on the percentage of truncation that one cannot truncate a solid with unequally shaped faces and end up with regular polygons as faces. Therefore, these two solids need be constructed with another technique. Actually, they can be built from the original platonic solids by a process called expansion. It consists on separating apart the faces of the original polyhedron with spherical symmetry, up to a point where they can be linked through new faces which are regular polygons. The name of the Truncated Cuboctahedron (also called Great Rhombicuboctahedron) and of the Truncated Icosidodecahedron (also called Great Rhombicosidodecahedron) again seem to indicate that they can be derived from truncating the Cuboctahedron and the Icosidodecahedron. But, as reasoned above, this is not possible.

Finally, there are two special solids which have two chiral (specular symmetric) variations: the Snub Cube and the Snub Dodecahedron. These solids can be constructed as an alternation of another Archimedean solid. This process consists on deleting alternated vertices and creating new triangles at the deleted vertices.

One of the Archimedean solids is the truncated cuboctahedron. It has 12 square faces, 8 regular hexagonal faces, 6 regular octagonal faces, 48 vertices and 72 edges. Since each of its faces has point symmetry (equivalently, 180 rotational symmetry), the truncated cuboctahedron is a zonohedron ([15]).

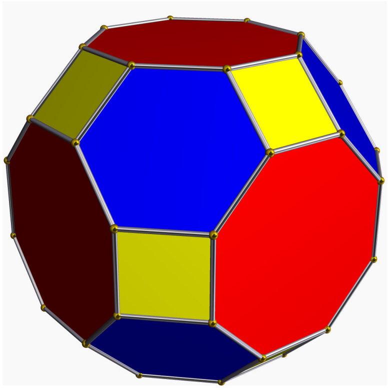

Figure 1(a) truncated cuboctahedron

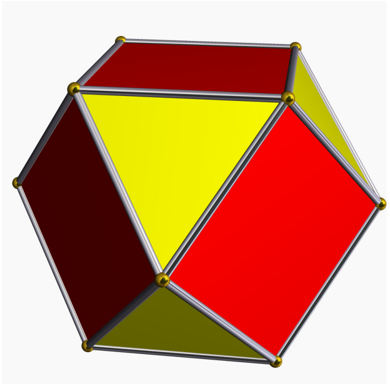

Figure 1(b) Cuboctahedron

We describe the metric that unit sphere is truncated cuboctahedron as following:

Definition 1. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.The distance function $d_{TC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated cuboctahedron distance between P_1 and P_2 is defined by

$$d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l} X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}X_{12},\frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}, \\ Y_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}Y_{12},\frac{3\sqrt{2}+2}{14}(X_{12}+Z_{12}),\\ Y_{12}+\frac{3\sqrt{2}+3}{2}Z_{12},Y_{12}+\frac{3\sqrt{2}+3}{2}X_{12}\end{array}\right\}, \\ Z_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}Z_{12},\frac{3\sqrt{2}+2}{14}(X_{12}+Y_{12}),\\ Z_{12}+\frac{3\sqrt{2}+3}{2}X_{12},Z_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}\end{array}\right\}\end{array}\right\}$$

where $X_{12} = |x_1 - x_2|$, $Y_{12} = |y_1 - y_2|$, $Z_{12} = |z_1 - z_2|$.

According to truncated cuboctahedron distance, there are three different paths from P_1 to P_2. These paths are

(i) a line segment which is parallel to a coordinate axis.

(ii) union of three line segments which one is parallel to a coordinate axis and other line segments makes $\arctan(\frac{195}{28})$ angle with another coordinate axes.

(iii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{3}{4})$ angle with another coordinate axis.

Thus truncated cuboctahedron distance between P_1 and P_2 is for (i) Euclidean lengths of line segment, for (ii) $\frac{3-\sqrt{2}}{3}$ times the sum of Euclidean lengths of mentioned three line segments, and for (iii) $\frac{10-\sqrt{2}}{14}$ times the sum of Euclidean lengths of mentioned two line segments.

Figure 2 illustrates truncated cuboctahedron way from P_1 to P_2 if maximum value is $|y_1 - y_2|$, $\frac{3-\sqrt{2}}{3}(|y_1 - y_2| + \frac{1}{14}(|x_1 - x_2| + |z_1 - z_2|))$, $\frac{10-\sqrt{2}}{14}(|y_1 - y_2| + \frac{1}{2}|z_1 - z_2|)$, or $\frac{10-\sqrt{2}}{14}\left(|y_1 - y_2| + \frac{1}{2}|x_1 - x_2|\right)$.

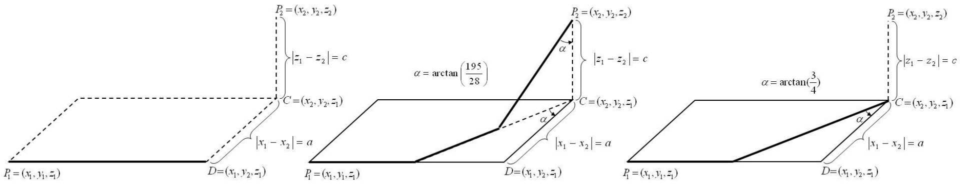

Figure 2: TC way from P_1 to P_2

Lemma 1. *Let $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ be distinct two points in $\mathbb{R}^3$. X_{12}, Y_{12}, Z_{12} denote $|x_1-x_2|$, $|y_1-y_2|$, $|z_1-z_2|$, respectively. Then*

$$d_{TC}(P_1,P_2) \geq \frac{3-\sqrt{2}}{3}\left(X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12},\frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}\right),$$

$$d_{TC}(P_1,P_2) \geq \frac{3-\sqrt{2}}{3}\left(Y_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Y_{12},\frac{3\sqrt{2}+2}{14}(X_{12}+Z_{12}),\\ Y_{12}+\frac{3\sqrt{2}+3}{2}Z_{12},Y_{12}+\frac{3\sqrt{2}+3}{2}X_{12}\end{array}\right\}\right),$$

$$d_{TC}(P_1,P_2) \geq \frac{3-\sqrt{2}}{3}\left(Z_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Z_{12},\frac{3\sqrt{2}+2}{14}(X_{12}+Y_{12}),\\ Z_{12}+\frac{3\sqrt{2}+3}{2}X_{12},Z_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}\end{array}\right\}\right).$$

Proof. Proof is trivial by the definition of maximum function. □

Theorem 2. *The distance function d_{TC} is a metric. Also according to d_{TC}, the unit sphere is a truncated cuboctahedron in $\mathbb{R}^3$.*

Proof. Let $d_{TC}:\mathbb{R}^3\times\mathbb{R}^3\to[0,\infty)$ be the truncated cuboctahedron distance function and P_1=(x_1,y_1,z_1), P_2=(x_2,y_2,z_2) and P_3=(x_3,y_3,z_3) are distinct three points in $\mathbb{R}^3$. X_{12}, Y_{12}, Z_{12} denote $|x_1-x_2|$, $|y_1-y_2|$, $|z_1-z_2|$, respectively. To show that d_{TC} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3\in\mathbb{R}^3$.

(M1) $d_{TC}(P_1,P_2)\geq 0$ and $d_{TC}(P_1,P_2)=0$ iff $P_1=P_2$
(M2) $d_{TC}(P_1,P_2)=d_{TC}(P_2,P_1)$
(M3) $d_{TC}(P_1,P_3)\leq d_{TC}(P_1,P_2)+d_{TC}(P_2,P_3)$.

Since absolute values is always nonnegative value $d_{TC}(P_1,P_2)\geq 0$. If $d_{TC}(P_1,P_2)=0$ then there are possible three cases. These cases are

(1) $d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\left(X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12},\frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}\right)$

(2) $d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\left(Y_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Y_{12},\frac{3\sqrt{2}+2}{14}(X_{12}+Z_{12}),\\ Y_{12}+\frac{3\sqrt{2}+3}{2}Z_{12},Y_{12}+\frac{3\sqrt{2}+3}{2}X_{12}\end{array}\right\}\right)$

(3) $d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\left(Z_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Z_{12},\frac{3\sqrt{2}+2}{14}(X_{12}+Y_{12}),\\ Z_{12}+\frac{3\sqrt{2}+3}{2}X_{12},Z_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}\end{array}\right\}\right)$.

Case I: If

$$d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\left(X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12},\frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}\right),$$

then

$$\frac{3-\sqrt{2}}{3}\left(X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12},\frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}\right)=0$$

$$\Leftrightarrow X_{12}=0 \text{ and } \frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12},\frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}=0$$

$$\Leftrightarrow x_1=x_2, y_1=y_2, z_1=z_2$$

$$\Leftrightarrow (x_1,y_1,z_1)=(x_2,y_2,z_2)$$

$$\Leftrightarrow P_1=P_2$$

The other cases can be shown by similar way in Case I. Thus we get $d_{TC}(P_1,P_2)=0$ iff $P_1=P_2$.

Since $|x_1-x_2|=|x_2-x_1|$, $|y_1-y_2|$=$|y_2-y_1|$ and $|z_1-z_2|=|z_2-z_1|$, obviously $d_{TC}(P_1,P_2)=d_{TC}(P_2,P_1)$. That is, d_{TC} is symmetric.

$X_{13},Y_{13},Z_{13},X_{23},Y_{23},Z_{23}$ denote $|x_1-x_3|, |y_1-y_3|, |z_1-z_3|, |x_2-x_3|, |y_2-y_3|, |z_2-z_3|$, respectively.

$$d_{TC}(P_1,P_3)$$

$$=\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l}X_{13}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{13},\frac{3\sqrt{2}+2}{14}(Y_{13}+Z_{13}),\\ X_{13}+\frac{3\sqrt{2}+3}{2}Y_{13},X_{13}+\frac{3\sqrt{2}+3}{2}Z_{13}\end{array}\right\},\\ Y_{13}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Y_{13},\frac{3\sqrt{2}+2}{14}(X_{13}+Z_{13}),\\ Y_{13}+\frac{3\sqrt{2}+3}{2}Z_{13},Y_{13}+\frac{3\sqrt{2}+3}{2}X_{13}\end{array}\right\},\\ Z_{13}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Z_{13},\frac{3\sqrt{2}+2}{14}(X_{13}+Y_{13}),\\ Z_{13}+\frac{3\sqrt{2}+3}{2}X_{13},Z_{13}+\frac{3\sqrt{2}+3}{2}Y_{13}\end{array}\right\}\end{array}\right\}$$

$$\leq\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l}X_{12}+X_{23}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}(X_{12}+X_{23}),\\ \frac{3\sqrt{2}+2}{14}(Y_{12}+Y_{23}+Z_{12}+Z_{23}),\\ X_{12}+X_{23}+\frac{3\sqrt{2}+3}{2}(Y_{12}+Y_{23}),\\ X_{12}+X_{23}+\frac{3\sqrt{2}+3}{2}(Z_{12}+Z_{23})\end{array}\right\},\\ Y_{12}+Y_{23}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}(Y_{12}+Y_{23}),\\ \frac{3\sqrt{2}+2}{14}(X_{12}+X_{23}+Z_{12}+Z_{23}),\\ Y_{12}+Y_{23}+\frac{3\sqrt{2}+3}{2}(Z_{12}+Z_{23}),\\ Y_{12}+Y_{23}+\frac{3\sqrt{2}+3}{2}(X_{12}+X_{23})\end{array}\right\},\\ Z_{12}+Z_{23}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}(Z_{12}+Z_{23}),\\ \frac{3\sqrt{2}+2}{14}(X_{12}+X_{23}+Y_{12}+Y_{23}),\\ Z_{12}+Z_{23}+\frac{3\sqrt{2}+3}{2}(X_{12}+X_{23}),\\ Z_{12}+Z_{23}+\frac{3\sqrt{2}+3}{2}(Y_{12}+Y_{23})\end{array}\right\}\end{array}\right\}$$

$$=I.$$

Therefore one can easily find that $I\leq d_{TC}(P_1,P_2)+d_{TC}(P_2,P_3)$ from Lemma 1. So $d_{TC}(P_1,P_3)\leq d_{TC}(P_1,P_2)+d_{TC}(P_2,P_3)$. Consequently, truncated cuboctahedron distance is a metric in 3-dimensional analytical space.

Finally, the set of all points $X=(x,y,z)\in\mathbb{R}^3$ that truncated cuboctahedron distance is 1 from $O=(0,0,0)$ is $S_{TC}=$

$$\left\{(x,y,z): \frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l} |x|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|x|,\frac{3\sqrt{2}+2}{14}(|y|+|z|),\\ |x|+\frac{3\sqrt{2}+3}{2}|y|,|x|+\frac{3\sqrt{2}+3}{2}|z|\end{array}\right\},\\ |y|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|y|,\frac{3\sqrt{2}+2}{14}(|x|+|z|),\\ |y|+\frac{3\sqrt{2}+3}{2}|z|,|y|+\frac{3\sqrt{2}+3}{2}|x|\end{array}\right\},\\ |z|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|z|,\frac{3\sqrt{2}+2}{14}(|x|+|y|),\\ |z|+\frac{3\sqrt{2}+3}{2}|x|,|z|+\frac{3\sqrt{2}+3}{2}|y|\end{array}\right\}\end{array}\right\}=1\right\}.$$

Thus the graph of S_{TC} is as in Figure 3:

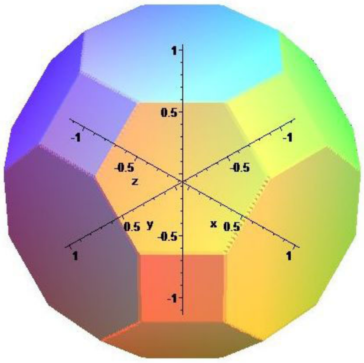

Figure 3 The unit sphere in terms of d_{TC}: Truncated cuboctahedron

□

Corollary 3. *The equation of the truncated cuboctahedron with center* (x_0,y_0,z_0) *and radius* r *is*

$$\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l} |x-x_0|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|x-x_0|,\frac{3\sqrt{2}+2}{14}(|y-y_0|+|z-z_0|),\\ |x-x_0|+\frac{3\sqrt{2}+3}{2}|y-y_0|,|x-x_0|+\frac{3\sqrt{2}+3}{2}|z-z_0|\end{array}\right\},\\ |y-y_0|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|y-y_0|,\frac{3\sqrt{2}+2}{14}(|x-x_0|+|z-z_0|),\\ |y-y_0|+\frac{3\sqrt{2}+3}{2}|z-z_0|,|y-y_0|+\frac{3\sqrt{2}+3}{2}|x-x_0|\end{array}\right\},\\ |z-z_0|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|z-z_0|,\frac{3\sqrt{2}+2}{14}(|x-x_0|+|y-y_0|),\\ |z-z_0|+\frac{3\sqrt{2}+3}{2}|x-x_0|,|z-z_0|+\frac{3\sqrt{2}+3}{2}|y-y_0|\end{array}\right\}\end{array}\right\}=r$$

which is a polyhedron which has 26 *faces and* 48 *vertices. Coordinates of the vertices are translation to* (x_0,y_0,z_0) *all permutations of the three axis components and all possible* $+/-$ *sign components of the points* $(\frac{\sqrt{2}+3}{7}r,\frac{2\sqrt{2}-1}{7}r,r)$.

Lemma 4. *Let* l *be the line through the points* $P_1=(x_1,y_1,z_1)$ *and* $P_2=(x_2,y_2,z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector* (p,q,r), *then*

$$d_{TC}(P_1,P_2)=\mu(P_1P_2)d_E(P_1,P_2)$$

where

$$\mu(P_1P_2)=\frac{\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{c}|p|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|p|,\frac{3\sqrt{2}+2}{14}(|q|+|r|),\\|p|+\frac{3\sqrt{2}+3}{2}|q|,|p|+\frac{3\sqrt{2}+3}{2}|r|\end{array}\right\},\\|q|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|q|,\frac{3\sqrt{2}+2}{14}(|p|+|r|),\\|q|+\frac{3\sqrt{2}+3}{2}|r|,|q|+\frac{3\sqrt{2}+3}{2}|p|\end{array}\right\},\\|r|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|r|,\frac{3\sqrt{2}+2}{14}(|p|+|q|),\\|r|+\frac{3\sqrt{2}+3}{2}|p|,|r|+\frac{3\sqrt{2}+3}{2}|q|\end{array}\right\}\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1-x_2=\lambda p$, $y_1-y_2=\lambda q$, $z_1-z_2=\lambda r$, $r\in\mathbb{R}$. Thus,$d_{TC}(P_1,P_2)$ is equal to

$$|\lambda|\left(\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{c}|p|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|p|,\frac{3\sqrt{2}+2}{14}(|q|+|r|),\\|p|+\frac{3\sqrt{2}+3}{2}|q|,|p|+\frac{3\sqrt{2}+3}{2}|r|\end{array}\right\},\\|q|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|q|,\frac{3\sqrt{2}+2}{14}(|p|+|r|),\\|q|+\frac{3\sqrt{2}+3}{2}|r|,|q|+\frac{3\sqrt{2}+3}{2}|p|\end{array}\right\},\\|r|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}|r|,\frac{3\sqrt{2}+2}{14}(|p|+|q|),\\|r|+\frac{3\sqrt{2}+3}{2}|p|,|r|+\frac{3\sqrt{2}+3}{2}|q|\end{array}\right\}\end{array}\right\}\right)$$

and $d_E(A,B)=|\lambda|\sqrt{p^2+q^2+r^2}$ which implies the required result. □

The above lemma says that d_{TC}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 5. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1,X)=d_E(P_2,X)$ if and only if $d_{TC}(P_1,X)=d_{TC}(P_2,X)$.*

Corollary 6. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{TC}(X,P_1)\;/\;d_{TC}(X,P_2)=d_E(X,P_1)\;/\;d_E(X,P_2)\;.$$

That is, the ratios of the Euclidean and $d_{TC}-$distances along a line are the same.

3. Truncated icosadodecahedron and its metric

The truncated icosidodecahedron is an Archimedean solid, one of thirteen convex isogonal nonprismatic solids constructed by two or more types of regular polygon faces. It has 30 square faces, 20 regular hexagonal faces, 12 regular decagonal faces, 120 vertices and 180 edges more than any other convex nonprismatic uniform polyhedron. Since each of its faces has point symmetry (equivalently, 180 rotational symmetry), the truncated icosidodecahedron is a zonohedron ([16]; see Figure 4(a)).

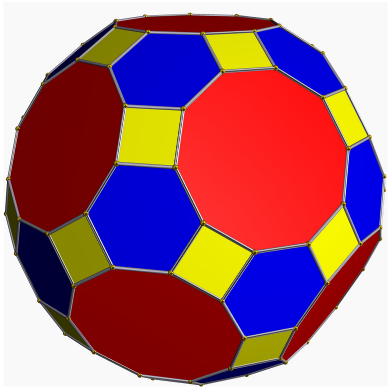

Figure 4(a)Truncated icosidodecahedron

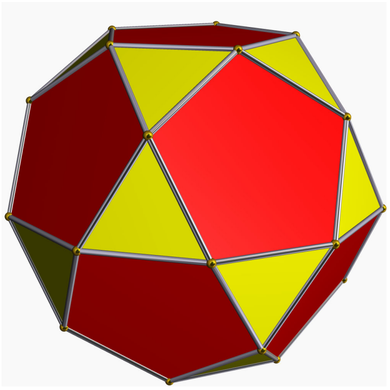

Figure 4(b) Icosidodecahedron

We describe the metric that unit sphere is truncated icosidodecahedron as following:

Definition 2. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.The distance function $d_{TI} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated icosidodecahedron distance between P_1 and P_2 is defined by
$d_{TI}(P_1, P_2) =$

$$\max\left\{\begin{array}{l} \frac{4-\sqrt{5}}{3}X_{12}+\frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}X_{12},\ \frac{15\sqrt{5}+17}{19}\left(Y_{12}+Z_{12}\right),\ \frac{\sqrt{5}+29}{19}\left(X_{12}+Z_{12}\right), \\ X_{12}+\frac{21\sqrt{5}+39}{38}Z_{12}+\frac{9\sqrt{5}+33}{38}Y_{12},\ \frac{18\sqrt{5}+104}{95}X_{12}+\frac{48\sqrt{5}+138}{95}Y_{12} \end{array}\right\}, \\ \frac{4-\sqrt{5}}{3}Y_{12}+\frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}Y_{12},\ \frac{15\sqrt{5}+17}{19}\left(X_{12}+Z_{12}\right),\ \frac{\sqrt{5}+29}{19}\left(X_{12}+Y_{12}\right), \\ Y_{12}+\frac{21\sqrt{5}+39}{38}X_{12}+\frac{9\sqrt{5}+33}{38}Z_{12},\ \frac{18\sqrt{5}+104}{95}Y_{12}+\frac{48\sqrt{5}+138}{95}Z_{12} \end{array}\right\}, \\ \frac{4-\sqrt{5}}{3}Z_{12}+\frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}Z_{12},\ \frac{15\sqrt{5}+17}{19}\left(X_{12}+Y_{12}\right),\ \frac{\sqrt{5}+29}{19}\left(Y_{12}+Z_{12}\right), \\ Z_{12}+\frac{21\sqrt{5}+39}{38}Y_{12}+\frac{9\sqrt{5}+33}{38}X_{12},\ \frac{18\sqrt{5}+104}{95}Z_{12}+\frac{48\sqrt{5}+138}{95}X_{12} \end{array}\right\} \end{array}\right\}$$

where $X_{12} = |x_1 - x_2|$, $Y_{12} = |y_1 - y_2|$, $Z_{12} = |z_1 - z_2|$.

According to truncated icosidodecahedron distance, there are five different paths from P_1 to P_2. These paths are

(i) a line segment which is parallel to a coordinate axis,

(ii) union of three line segments each of which is parallel to a coordinate axis,

(iii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{\sqrt{5}}{2})$ angle with another coordinate axis,

(iv) union of three line segments which two of them are parallel to a coordinate axis and other line segment makes $\arctan(\frac{937-215\sqrt{5}}{1824})$ angle with other coordinate axis,

(v) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{1}{2})$ angle with another coordinate axis.

Thus truncated cuboctahedron distance between P_1 and P_2 is for (i) Euclidean length of line segment, for (ii) $\frac{4-\sqrt{5}}{3}$ times the sum of Euclidean lengths of mentioned three line segments, for (iii) $\frac{3\sqrt{5}-1}{6}$ times the sum of Euclidean lengths of two line segments, for (iv) $\frac{\sqrt{5}+1}{4}$ times the sum of Euclidean lengths of three line

segments, and for (v) $\frac{\sqrt{5}+7}{10}$ times the sum of Euclidean lengths of two line segments. Figure 5 shows that the path between P_1 and P_2 in case of the maximum is
$|y_1 - y_2|$,
$\frac{4-\sqrt{5}}{3}\left(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right)$,
$\frac{3\sqrt{5}-1}{6}\left(|y_1 - y_2| + \frac{3-\sqrt{5}}{2}|x_1 - x_2|\right)$,
$\frac{\sqrt{5}+1}{4}\left(|y_1 - y_2| + |z_1 - z_2| + \frac{12\left(\sqrt{5}-1\right)}{19}|x_1 - x_2|\right)$, or
$\frac{\sqrt{5}+7}{10}\left(|y_1 - y_2| + \frac{\sqrt{5}-1}{2}|z_1 - z_2|\right)$.

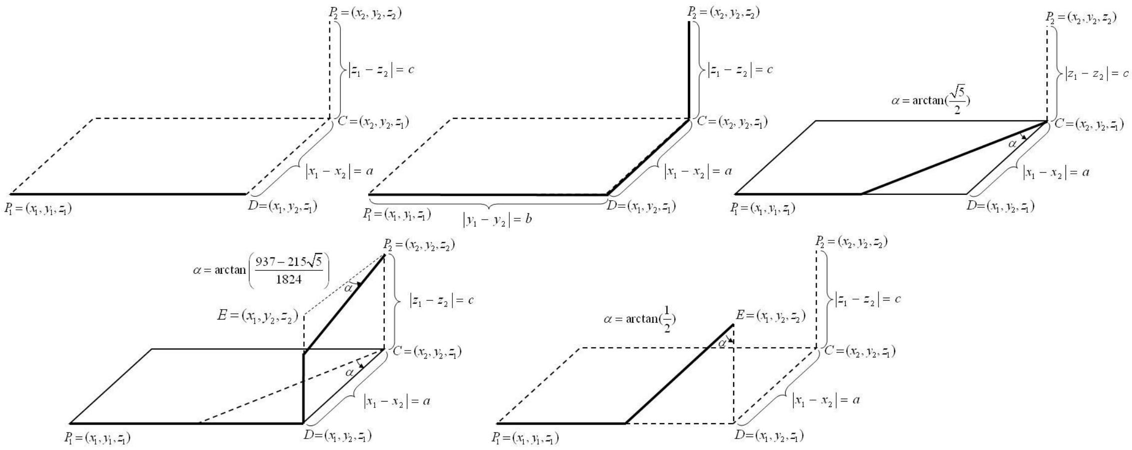

Figure 5: TI way from P_1 to P_2

Lemma 7. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. Then*

$$d_{TI}(P_1,P_2) \geq \frac{4-\sqrt{5}}{3}X_{12} + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}X_{12},\ \frac{15\sqrt{5}+17}{19}\left(Y_{12}+Z_{12}\right),\ \frac{\sqrt{5}+29}{19}\left(X_{12}+Z_{12}\right), \\ X_{12} + \frac{21\sqrt{5}+39}{38}Z_{12} + \frac{9\sqrt{5}+33}{38}Y_{12},\ \frac{18\sqrt{5}+104}{95}X_{12} + \frac{48\sqrt{5}+138}{95}Y_{12} \end{array}\right\}$$

$$d_{TI}(P_1,P_2) \geq \frac{4-\sqrt{5}}{3}Y_{12} + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}Y_{12},\ \frac{15\sqrt{5}+17}{19}\left(X_{12}+Z_{12}\right),\ \frac{\sqrt{5}+29}{19}\left(X_{12}+Y_{12}\right), \\ Y_{12} + \frac{21\sqrt{5}+39}{38}X_{12} + \frac{9\sqrt{5}+33}{38}Z_{12},\ \frac{18\sqrt{5}+104}{95}Y_{12} + \frac{48\sqrt{5}+138}{95}Z_{12} \end{array}\right\}$$

$$d_{TI}(P_1,P_2) \geq \frac{4-\sqrt{5}}{3}Z_{12} + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}Z_{12},\ \frac{15\sqrt{5}+17}{19}\left(X_{12}+Y_{12}\right),\ \frac{\sqrt{5}+29}{19}\left(Y_{12}+Z_{12}\right), \\ Z_{12} + \frac{21\sqrt{5}+39}{38}Y_{12} + \frac{9\sqrt{5}+33}{38}X_{12},\ \frac{18\sqrt{5}+104}{95}Z_{12} + \frac{48\sqrt{5}+138}{95}X_{12} \end{array}\right\}.$$

where X_{12}=$|x_1 - x_2|$, Y_{12}=$|y_1 - y_2|$, Z_{12}=$|z_1 - z_2|$.

Proof. Proof is trivial by the definition of maximum function. □

Theorem 8. *The distance function d_{TI} is a metric. Also according to d_{TI}, unit sphere is a truncated icosidodecahedron in $\mathbb{R}^3$.*

Proof. One can easily show that the truncated icosidodecahedron distance function satisfies the metric axioms by similar way in Theorem 2.

Consequently, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that truncated icosidodecahedron distance is 1 from $O = (0, 0, 0)$ is $S_{TI} =$

$$\left\{ (x,y,z) \colon \max \left\{ \begin{array}{l} \frac{4-\sqrt{5}}{3}|x| + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{15\sqrt{5}+17}{19}(|y|+|z|), \frac{\sqrt{5}+29}{19}(|x|+|z|), \\ \frac{6\sqrt{5}+22}{19}|x|, |x| + \frac{21\sqrt{5}+39}{38}|z| + \frac{9\sqrt{5}+33}{38}|y|, \\ \frac{18\sqrt{5}+104}{95}|x| + \frac{48\sqrt{5}+138}{95}|y| \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3}|y| + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{15\sqrt{5}+17}{19}(|x|+|z|), \frac{\sqrt{5}+29}{19}(|x|+|y|), \\ \frac{6\sqrt{5}+22}{19}|y|, |y| + \frac{21\sqrt{5}+39}{38}|x| + \frac{9\sqrt{5}+33}{38}|z|, \\ \frac{18\sqrt{5}+104}{95}|y| + \frac{48\sqrt{5}+138}{95}|z| \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3}|z| + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{15\sqrt{5}+17}{19}(|x|+|y|), \frac{\sqrt{5}+29}{19}(|y|+|z|), \\ \frac{6\sqrt{5}+22}{19}|z|, |z| + \frac{21\sqrt{5}+39}{38}|y| + \frac{9\sqrt{5}+33}{38}|x|, \\ \frac{18\sqrt{5}+104}{95}|z| + \frac{48\sqrt{5}+138}{95}|x| \end{array} \right\} \end{array} \right\} = 1 \right\}.$$

Thus the graph of S_{TI} is as in Figure 6:

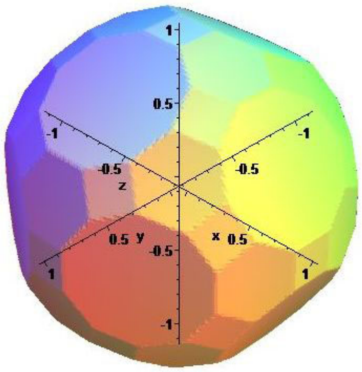

Figure 6 The unit sphere in terms of d_{TI}: Truncated icosidodecahedron

□

Corollary 9. *The equation of the truncated icosidodecahedron with center* (x_0, y_0, z_0) *and radius* r *is*

$$\max \left\{ \begin{array}{l} \frac{4-\sqrt{5}}{3}|x-x_0| + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19}|x-x_0|, \frac{15\sqrt{5}+17}{19}(|y-y_0|+|z-z_0|), \\ \frac{\sqrt{5}+29}{19}(|x-x_0|+|z-z_0|), \\ |x-x_0| + \frac{21\sqrt{5}+39}{38}|z-z_0| + \frac{9\sqrt{5}+33}{38}|y-y_0|, \\ \frac{18\sqrt{5}+104}{95}|x-x_0| + \frac{48\sqrt{5}+138}{95}|y-y_0| \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3}|y-y_0| + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19}|y-y_0|, \frac{15\sqrt{5}+17}{19}(|x-x_0|+|z-z_0|), \\ \frac{\sqrt{5}+29}{19}(|x-x_0|+|y-y_0|), \\ |y-y_0| + \frac{21\sqrt{5}+39}{38}|x-x_0| + \frac{9\sqrt{5}+33}{38}|z-z_0|, \\ \frac{18\sqrt{5}+104}{95}|y-y_0| + \frac{48\sqrt{5}+138}{95}|z-z_0| \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3}|z-z_0| + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19}|z-z_0|, \frac{15\sqrt{5}+17}{19}(|x-x_0|+|y-y_0|), \\ \frac{\sqrt{5}+29}{19}(|y-y_0|+|z-z_0|), \\ |z-z_0| + \frac{21\sqrt{5}+39}{38}|y-y_0| + \frac{9\sqrt{5}+33}{38}|x-x_0|, \\ \frac{18\sqrt{5}+104}{95}|z-z_0| + \frac{48\sqrt{5}+138}{95}|x-x_0| \end{array} \right\} \end{array} \right\} = r$$

which is a polyhedron which has 62 *faces and* 120 *vertices. Coordinates of the vertices are translation to* (x_0, y_0, z_0) *all possible* $+/-$ *sign components of the points*

$$\begin{array}{lll}
\left(\frac{2\sqrt{5}-3}{11}r, \frac{2\sqrt{5}-3}{11}r, r\right), & \left(r, \frac{2\sqrt{5}-3}{11}r, \frac{2\sqrt{5}-3}{11}r\right), & \left(\frac{2\sqrt{5}-3}{11}r, r, \frac{2\sqrt{5}-3}{11}r\right), \\
\left(\frac{4\sqrt{5}-6}{11}r, \frac{3\sqrt{5}+1}{22}r, \frac{5\sqrt{5}+9}{22}r\right), & \left(\frac{5\sqrt{5}+9}{22}r, \frac{4\sqrt{5}-6}{11}r, \frac{3\sqrt{5}+1}{22}r\right), & \left(\frac{3\sqrt{5}+1}{22}r, \frac{5\sqrt{5}+9}{22}r, \frac{4\sqrt{5}-6}{11}r\right), \\
\left(\frac{2\sqrt{5}-3}{11}r, \frac{\sqrt{5}+4}{11}r, \frac{5\sqrt{5}-2}{11}r\right), & \left(\frac{5\sqrt{5}-2}{11}r, \frac{2\sqrt{5}-3}{11}r, \frac{\sqrt{5}+4}{11}r\right), & \left(\frac{\sqrt{5}+4}{11}r, \frac{5\sqrt{5}-2}{11}r, \frac{2\sqrt{5}-3}{11}r\right), \\
\left(\frac{7\sqrt{5}-5}{22}r, \frac{7-\sqrt{5}}{11}r, \frac{\sqrt{5}+15}{22}r\right), & \left(\frac{\sqrt{5}+15}{22}r, \frac{7\sqrt{5}-5}{22}r, \frac{7-\sqrt{5}}{11}r\right), & \left(\frac{7-\sqrt{5}}{11}r, \frac{\sqrt{5}+15}{22}r, \frac{7\sqrt{5}-5}{22}r\right), \\
\left(\frac{3\sqrt{5}+1}{22}r, \frac{21-3\sqrt{5}}{22}r, \frac{3\sqrt{5}+1}{11}r\right), & \left(\frac{3\sqrt{5}+1}{11}r, \frac{3\sqrt{5}+1}{22}r, \frac{21-3\sqrt{5}}{22}r\right), & \left(\frac{21-3\sqrt{5}}{22}r, \frac{3\sqrt{5}+1}{11}r, \frac{3\sqrt{5}+1}{22}r\right).
\end{array}$$

Lemma 10. *Let* l *be the line through the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector* (p, q, r)*, then*

$$d_{TI}(P_1, P_2) = \mu(P_1P_2)d_E(P_1, P_2)$$

where

$$\mu(P_1P_2) = \frac{\max\left\{\begin{array}{l}
\frac{4-\sqrt{5}}{3}|p| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l}\frac{15\sqrt{5}+17}{19}(|q|+|r|), \frac{\sqrt{5}+29}{19}(|p|+|r|), \\ \frac{6\sqrt{5}+22}{19}|p|, |p| + \frac{21\sqrt{5}+39}{38}|r| + \frac{9\sqrt{5}+33}{38}|q|, \\ \frac{18\sqrt{5}+104}{95}|p| + \frac{48\sqrt{5}+138}{95}|q|\end{array}\right\}, \\
\frac{4-\sqrt{5}}{3}|q| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l}\frac{15\sqrt{5}+17}{19}(|p|+|r|), \frac{\sqrt{5}+29}{19}(|p|+|q|), \\ \frac{6\sqrt{5}+22}{19}|q|, |q| + \frac{21\sqrt{5}+39}{38}|p| + \frac{9\sqrt{5}+33}{38}|r|, \\ \frac{18\sqrt{5}+104}{95}|q| + \frac{48\sqrt{5}+138}{95}|r|\end{array}\right\}, \\
\frac{4-\sqrt{5}}{3}|r| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l}\frac{15\sqrt{5}+17}{19}(|p|+|q|), \frac{\sqrt{5}+29}{19}(|q|+|r|), \\ \frac{6\sqrt{5}+22}{19}|r|, |r| + \frac{21\sqrt{5}+39}{38}|q| + \frac{9\sqrt{5}+33}{38}|p|, \\ \frac{18\sqrt{5}+104}{95}|r| + \frac{48\sqrt{5}+138}{95}|p|\end{array}\right\}
\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $r \in \mathbb{R}$. Thus,

$$d_{TI}(P_1, P_2) = |\lambda| \max\left\{\begin{array}{l}
\frac{4-\sqrt{5}}{3}|p| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l}\frac{15\sqrt{5}+17}{19}(|q|+|r|), \frac{\sqrt{5}+29}{19}(|p|+|r|), \\ \frac{6\sqrt{5}+22}{19}|p|, |p| + \frac{21\sqrt{5}+39}{38}|r| + \frac{9\sqrt{5}+33}{38}|q|, \\ \frac{18\sqrt{5}+104}{95}|p| + \frac{48\sqrt{5}+138}{95}|q|\end{array}\right\}, \\
\frac{4-\sqrt{5}}{3}|q| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l}\frac{15\sqrt{5}+17}{19}(|p|+|r|), \frac{\sqrt{5}+29}{19}(|p|+|q|), \\ \frac{6\sqrt{5}+22}{19}|q|, |q| + \frac{21\sqrt{5}+39}{38}|p| + \frac{9\sqrt{5}+33}{38}|r|, \\ \frac{18\sqrt{5}+104}{95}|q| + \frac{48\sqrt{5}+138}{95}|r|\end{array}\right\}, \\
\frac{4-\sqrt{5}}{3}|r| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l}\frac{15\sqrt{5}+17}{19}(|p|+|q|), \frac{\sqrt{5}+29}{19}(|q|+|r|), \\ \frac{6\sqrt{5}+22}{19}|r|, |r| + \frac{21\sqrt{5}+39}{38}|q| + \frac{9\sqrt{5}+33}{38}|p|, \\ \frac{18\sqrt{5}+104}{95}|r| + \frac{48\sqrt{5}+138}{95}|p|\end{array}\right\}
\end{array}\right\}$$

and $d_E(A, B) = |\lambda| \sqrt{p^2 + q^2 + r^2}$ which implies the required result. □

The above lemma says that d_{TI}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 11. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{TI}(P_1, X) = d_{TI}(P_2, X)$.*

Corollary 12. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{TI}(X, P_1) \ / \ d_{TI}(X, P_2) = d_E(X, P_1) \ / \ d_E(X, P_2) \ .$$

That is, the ratios of the Euclidean and $d_{TI}-$distances along a line are the same.

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