Forum Geometricorum
Volume 17 (2017) 301–324.

FORUM GEOM
ISSN 1534-1178

# Orthocenters of Simplices on Spheres

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**Abstract**. We consider orthocenters of simplices of the unit sphere of the $n$-dimensional Euclidean space. For $n = 3$, orthocenters always exist for all simplices, but for $n \geqq 4$, they do not necessarily exist. Moreover, unlike the case of the Euclidean space, it is possible that there exist infinite numbers of orthocenters. In this paper, we give characterizations of the existence and the uniqueness of orthocenters.

## 1. Introduction.

For a simplex of the unit sphere $\mathbb{S}^{n-1}$ of the Euclidean space $\mathbb{R}^n$, if great-circles which are passing through vertices and perpendicular to the opposite faces are concurrent, their common point is called an orthocenter. For $n \geqq 4$, orthocenters do not necessarily exist. In particular, the existence of orthocenters is equivalent to

$$(\mathbf{p}_i^* \cdot \mathbf{p}_k^*)(\mathbf{p}_j^* \cdot \mathbf{p}_\ell^*) = (\mathbf{p}_i^* \cdot \mathbf{p}_\ell^*)(\mathbf{p}_j^* \cdot \mathbf{p}_k^*)$$

for arbitrary pairwise distinct vertices $\mathbf{p}_i^*$, $\mathbf{p}_j^*$, $\mathbf{p}_k^*$, $\mathbf{p}_\ell^*$ (Theorem 1). Moreover, if there exist orthocenters, the uniqueness of them is equivalent that there exist at least two pairs of vertices which are not perpendicular, respectively (Theorem 2) (remark that "uniqueness" means the existence of just one pair of orthocenters antipodal each other, because the antipodal of an orthocenter is another orthocenter). Notice that, for a simplex of the Euclidean space, orthocenters exist only at most one point, and the existence of the orthocenter is equivalent to

$$(\mathbf{q}_i - \mathbf{q}_k) \,.\, (\mathbf{q}_j - \mathbf{q}_\ell) = 0$$

for arbitrary pairwise distinct vertices $\mathbf{q}_i$, $\mathbf{q}_j$, $\mathbf{q}_k$, $\mathbf{q}_\ell$ (see, e.g., §1 of [1], (1.1) of [3]).

## 2. Preliminaries.

The following notations are valid throughout this paper. For a spherical simplex $S$ on $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$, i.e., for

$$S = \{\mathbf{x} \in \mathbb{S}^{n-1} : \mathbf{p}_i \,.\, \mathbf{x} \geqq 0,\ {}^\forall i = 0, \ldots, n-1\},$$

where $\mathbf{p}_0, \ldots, \mathbf{p}_{n-1} \in \mathbb{S}^{n-1}$ are linearly independent, let

$$S^* = \{\mathbf{y} \in \mathbb{S}^{n-1} : \mathbf{x} \,.\, \mathbf{y} \geqq 0,\ {}^\forall \mathbf{x} \in S\},$$

Publication Date: June 19, 2017. Communicating Editor: Paul Yiu.

and, for $j = 0, \ldots, n-1$, let $\mathbf{p}_j^* \in \mathbb{S}^{n-1}$ be the unique vector such that

$$\mathbf{p}_i \cdot \mathbf{p}_j^* = 0 \ \text{ for } \ ^\forall i = 0, \ldots, \widehat{j}, \ldots, n-1 \ \text{ and } \ \mathbf{p}_j \cdot \mathbf{p}_j^* > 0,$$

where the circumflex indicates that the term below it has been omitted. Then we have

$$\begin{aligned} S &= \left(\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*\right) \cap \mathbb{S}^{n-1} \\ &= \{\mathbf{x} \in \mathbb{S}^{n-1} : \mathbf{x} \cdot \mathbf{y} \geqq 0, \ ^\forall \mathbf{y} \in S^*\}, \\ S^* &= \left(\mathbb{R}^+ \cdot \mathbf{p}_0 + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}\right) \cap \mathbb{S}^{n-1} \\ &= \{\mathbf{y} \in \mathbb{S}^{n-1} : \mathbf{p}_j^* \cdot \mathbf{y} \geqq 0, \ ^\forall j = 0, \ldots, n-1\}, \end{aligned}$$

where $\mathbb{R}^+$ is the set of non-negative real numbers, i.e., $\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^*$ and $\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}$ are vertices of $S$ and $S^*$, respectively (see [4] and notice $S^* = -S^\circ$ and $\mathbf{p}_j^* = -\mathbf{p}_j^\circ$). The symbols $\Delta$ and $\Delta^*$ means

$$\Delta\,(k_0 \cdots k_m) = \det \begin{pmatrix} \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_m} \\ \vdots & & \vdots \\ \mathbf{p}_{k_m} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_m} \cdot \mathbf{p}_{k_m} \end{pmatrix},$$

$$\Delta^*\,(k_0 \cdots k_m) = \det \begin{pmatrix} \mathbf{p}_{k_0}^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_{k_0}^* \cdot \mathbf{p}_{k_m}^* \\ \vdots & & \vdots \\ \mathbf{p}_{k_m}^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_{k_m}^* \cdot \mathbf{p}_{k_m}^* \end{pmatrix},$$

for $m = 0, \ldots, n-1$, and pairwise distinct indices $k_0, \ldots, k_m = 0, \ldots, n-1$, and

$$\Delta \begin{pmatrix} k_0 \cdots k_{m-1} & i \\ & j \end{pmatrix} = \det \begin{pmatrix} \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_{m-1}} & \mathbf{p}_{k_0} \cdot \mathbf{p}_j \\ \vdots & & \vdots & \vdots \\ \mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_{k_{m-1}} & \mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_j \\ \mathbf{p}_i \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_i \cdot \mathbf{p}_{k_{m-1}} & \mathbf{p}_i \cdot \mathbf{p}_j \end{pmatrix},$$

$$\Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} & i \\ & j \end{pmatrix} = \det \begin{pmatrix} \mathbf{p}_{k_0}^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_{k_0}^* \cdot \mathbf{p}_{k_{m-1}}^* & \mathbf{p}_{k_0}^* \cdot \mathbf{p}_j^* \\ \vdots & & \vdots & \vdots \\ \mathbf{p}_{k_{m-1}}^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_{k_{m-1}}^* \cdot \mathbf{p}_{k_{m-1}}^* & \mathbf{p}_{k_{m-1}}^* \cdot \mathbf{p}_j^* \\ \mathbf{p}_i^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_i^* \cdot \mathbf{p}_{k_{m-1}}^* & \mathbf{p}_i^* \cdot \mathbf{p}_j^* \end{pmatrix},$$

for $m = 0, \ldots, n-2$, and pairwise distinct indices $k_0, \ldots, k_{m-1}, i, j = 0, \ldots, n-1$. Especially, we set

$$\Delta \begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{p}_i \cdot \mathbf{p}_j, \quad \text{and} \quad \Delta^* \begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{p}_i^* \cdot \mathbf{p}_j^*,$$

for distinct indices $i, j = 0, \ldots, n-1$.

*Remark* 1. From Lemma 5.1 and the second equation of Lemma 3.4 of [4], for an index $k = 0,\dots, n-1$, we have

$$\mathbf{p}_k^* = (-1)^{n-1-k}\varepsilon \frac{\langle\!\langle \mathbf{p}_0, \dots, \widehat{\mathbf{p}_k}, \dots, \mathbf{p}_{n-1}\rangle\!\rangle}{\sqrt{\Delta\left(0\cdots\widehat{k}\cdots n-1\right)}}, \tag{1}$$

$$\mathbf{p}_k = (-1)^{n-1-k}\varepsilon \frac{\langle\!\langle \mathbf{p}_0^*, \dots, \widehat{\mathbf{p}_k^*}, \dots, \mathbf{p}_{n-1}^*\rangle\!\rangle}{\sqrt{\Delta^*\left(0\cdots\widehat{k}\cdots n-1\right)}}, \tag{1$^*$}$$

where $\langle\!\langle \mathbf{q}_0, \dots, \mathbf{q}_{n-2}\rangle\!\rangle$ is the unique vector in $\mathbb{R}^n$ such that

$$\langle\!\langle \mathbf{q}_0, \dots, \mathbf{q}_{n-2}\rangle\!\rangle \cdot \mathbf{q}_{n-1} = \det(\mathbf{q}_0, \dots, \mathbf{q}_{n-2}, \mathbf{q}_{n-1}), \quad \text{for } {}^\forall\mathbf{q}_{n-1} \in \mathbb{R}^n,$$

$(\mathbf{q}_0, \dots, \mathbf{q}_{n-2}, \mathbf{q}_{n-1})$ is the matrix that has $\mathbf{q}_0,\dots, \mathbf{q}_{n-2}, \mathbf{q}_{n-1}$ as column vectors with Cartesian coordinate system, and

$$\varepsilon = \operatorname{sgn}(\det(\mathbf{p}_0, \dots, \mathbf{p}_{n-1})) = \operatorname{sgn}(\det(\mathbf{p}_0^*, \dots, \mathbf{p}_{n-1}^*)) \in \{1, -1\}.$$

Orthocenters are defined as follows:

**Definition 1.** The point $\mathbf{h}$ on $\mathbb{S}^{n-1}$ is called an *orthocenter* if, for each index $i = 0,\dots, n-1$, there exists a great circular arc $C_i$ of $\mathbb{S}^{n-1}$ such that $C_i$ is passing through $\mathbf{p}_i$, $\mathbf{p}_i^*$, and $\mathbf{h}$.

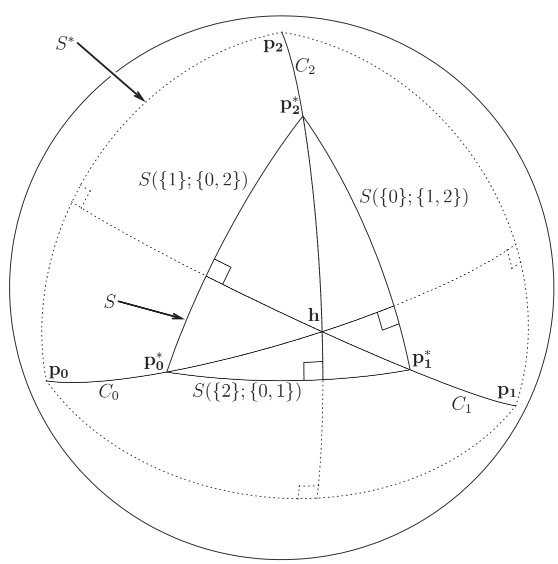


Figure 1.

*Remark* 2. $C_i$ is passing through $\mathbf{p}_i$, so, $C_i$ is perpendicular to the opposite face of the vertex $\mathbf{p}_i^*$:

$$\begin{aligned} & S(\{i\};\{0,\ldots,\widehat{i},\ldots,n-1\}) \\ &= \{\mathbf{x}\in S : \mathbf{p}_i \,.\, \mathbf{x} = 0\} \\ &= \left(\mathbb{R}^+\cdot\mathbf{p}_0^* + \cdots + \widehat{\mathbb{R}^+\cdot\mathbf{p}_i^*} + \cdots + \mathbb{R}^+\cdot\mathbf{p}_{n-1}^*\right)\cap\mathbb{S}^{n-1}. \end{aligned}$$

*Remark* 3. If $\mathbf{h}$ is an orthocenter of $S$, then the antipode $-\mathbf{h}$ is another orthocenter.

*Remark* 4. If $\mathbf{h}$ is an orthocenter of $S$, it is also an orthocenter of $S^*$.

The following two theorems are main purposes of this paper.

**Theorem 1.** *The followings are equivalent:*

(a) *there exists an orthocenter* $\mathbf{h}$*;*

(b) *the equation* $\Delta\begin{pmatrix} i \\ k\end{pmatrix}\Delta\begin{pmatrix} j \\ \ell\end{pmatrix} = \Delta\begin{pmatrix} i \\ \ell\end{pmatrix}\Delta\begin{pmatrix} j \\ k\end{pmatrix}$ *holds for arbitrary pairwise distinct indices* $i, j, k, \ell = 0,\ldots, n-1$*;*

(b)$^*$ *the equation* $\Delta^*\begin{pmatrix} i \\ k\end{pmatrix}\Delta^*\begin{pmatrix} j \\ \ell\end{pmatrix} = \Delta^*\begin{pmatrix} i \\ \ell\end{pmatrix}\Delta^*\begin{pmatrix} j \\ k\end{pmatrix}$ *holds for arbitrary pairwise distinct indices* $i, j, k, \ell = 0,\ldots, n-1$.

**Theorem 2.** *The followings are equivalent:*

(c) *there exist just two orthocenters and they are antipodal each other,* $\pm\mathbf{h}$*;*

(d) (b) *holds and there exist at least two pairs of distinct indices* $i$, $k$ *and* $j$, $\ell$ *such that neither* $\Delta\begin{pmatrix} i \\ k\end{pmatrix}$ *nor* $\Delta\begin{pmatrix} j \\ \ell\end{pmatrix}$ *is equal to* $0$*;*

(d)$^*$ (b)$^*$ *holds and there exist at least two pairs of distinct indices* $i$, $k$ *and* $j$, $\ell$ *such that neither* $\Delta^*\begin{pmatrix} i \\ k\end{pmatrix}$ *nor* $\Delta^*\begin{pmatrix} j \\ \ell\end{pmatrix}$ *is equal to* $0$.

*Remark* 5. Assume (d). Then, we have that either there exist pairwise distinct indices $i$, $j$, $k$, $\ell$ such that

$$\Delta\begin{pmatrix} i \\ k\end{pmatrix} \neq 0 \neq \Delta\begin{pmatrix} j \\ \ell\end{pmatrix}, \tag{2}$$

or there exist pairwise distinct indices $i$, $k$, $\ell$ such that

$$\Delta\begin{pmatrix} i \\ k\end{pmatrix} \neq 0 \neq \Delta\begin{pmatrix} i \\ \ell\end{pmatrix}. \tag{3}$$

If (2) holds, we have

$$0 \neq \Delta\begin{pmatrix} i \\ k\end{pmatrix}\Delta\begin{pmatrix} j \\ \ell\end{pmatrix} = \Delta\begin{pmatrix} i \\ \ell\end{pmatrix}\Delta\begin{pmatrix} j \\ k\end{pmatrix},$$

so (3) holds, i.e., (d) implies (3) for some pairwise distinct indices $i$, $k$, $\ell$. Similarly, (d)$^*$ implies

$$\Delta^*\begin{pmatrix} i \\ k\end{pmatrix} \neq 0 \neq \Delta^*\begin{pmatrix} i \\ \ell\end{pmatrix}, \tag{3$^*$}$$

for some pairwise distinct indices $i$, $k$, $\ell$.

*Remark* 6. Assume (b) and the negation of (d). Then, if there exists no pair of distinct indices $i$, $j$ with $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$, the set of orthocenters is the whole unit sphere $\mathbb{S}^{n-1}$. Otherwise, i.e., if there exists the unique pair of distinct indices $i$, $j$ with $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$, the set of orthocenters is the unit circle passing through $\mathbf{p}_i$ and $\mathbf{p}_j$ (see the proof of Lemma 5).

*Remark* 7. From Theorem 1, orthocenters always exist for $n = 3$.

Later, theorems above are proved completely and the pair of orthocenters are represented explicitly if three conditions of Theorem 2 hold. Now we can prove (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (b)$^*$ immediately by the following two lemmas:

**Lemma 3.** *The condition* (a) *implies*

$$(\mathbf{h}\cdot\mathbf{p}_i)(\mathbf{p}_j\cdot\mathbf{p}_k) = (\mathbf{h}\cdot\mathbf{p}_j)(\mathbf{p}_i\cdot\mathbf{p}_k), \quad (\textit{resp.}\ (\mathbf{h}\cdot\mathbf{p}_i^*)(\mathbf{p}_j^*\cdot\mathbf{p}_k^*) = (\mathbf{h}\cdot\mathbf{p}_j^*)(\mathbf{p}_i^*\cdot\mathbf{p}_k^*))$$

*for arbitrary pairwise distinct indices* $i, j, k = 0, \ldots, n-1$.

*Proof.* If $\mathbf{p}_k = \mathbf{p}_k^*$, we have $\mathbf{p}_i\cdot\mathbf{p}_k = \mathbf{p}_j\cdot\mathbf{p}_k = 0$. So the both sides of the conclusion are equal to $0$. Otherwise, i.e., if $\mathbf{p}_k \neq \mathbf{p}_k^*$, there exists a great circular arc $C_k$ passing through $\mathbf{p}_k$, $\mathbf{p}_k^*$, and $\mathbf{h}$, so, we can represent $\mathbf{h} = A\mathbf{p}_k + A^*\mathbf{p}_k^*$. Hence, we have

$$(\mathbf{h}\cdot\mathbf{p}_i)(\mathbf{p}_j\cdot\mathbf{p}_k) = A(\mathbf{p}_k\cdot\mathbf{p}_i)(\mathbf{p}_j\cdot\mathbf{p}_k) = (\mathbf{p}_k\cdot\mathbf{p}_i)(\mathbf{p}_j\cdot\mathbf{h}).$$

□

**Lemma 4.** *If there exists a point* $\mathbf{n} \in \mathbb{S}^{n-1}$ *which satisfies*

$$(\mathbf{n}\cdot\mathbf{p}_i)(\mathbf{p}_j\cdot\mathbf{p}_k) = (\mathbf{n}\cdot\mathbf{p}_j)(\mathbf{p}_i\cdot\mathbf{p}_k) \quad (\textit{resp.}\ (\mathbf{n}\cdot\mathbf{p}_i^*)(\mathbf{p}_j^*\cdot\mathbf{p}_k^*) = (\mathbf{n}\cdot\mathbf{p}_j^*)(\mathbf{p}_i^*\cdot\mathbf{p}_k^*))$$

*for arbitrary pairwise distinct indices* $i$, $j$, $k$, *then the condition* (b) (*resp.* (b)$^*$) *holds.*

*Proof.* Let $i$, $j$, $k$, and $\ell$ be pairwise distinct indices. Then, fix an index $h$ with $\mathbf{n}\cdot\mathbf{p}_h \neq 0$. If $h = i$, we have

$$(\mathbf{p}_i\cdot\mathbf{p}_k)(\mathbf{p}_j\cdot\mathbf{p}_\ell) = \frac{\mathbf{n}\cdot\mathbf{p}_j}{\mathbf{n}\cdot\mathbf{p}_i}(\mathbf{p}_i\cdot\mathbf{p}_k)(\mathbf{p}_i\cdot\mathbf{p}_\ell) = (\mathbf{p}_j\cdot\mathbf{p}_k)(\mathbf{p}_i\cdot\mathbf{p}_\ell).$$

Otherwise, i.e., if $h \neq i$, we have

$$\begin{aligned}(\mathbf{p}_i\cdot\mathbf{p}_k)(\mathbf{p}_j\cdot\mathbf{p}_\ell) &= \frac{\mathbf{n}\cdot\mathbf{p}_k}{\mathbf{n}\cdot\mathbf{p}_h}(\mathbf{p}_i\cdot\mathbf{p}_h)(\mathbf{p}_j\cdot\mathbf{p}_\ell) \\ &= \frac{\mathbf{n}\cdot\mathbf{p}_\ell}{\mathbf{n}\cdot\mathbf{p}_h}(\mathbf{p}_i\cdot\mathbf{p}_h)(\mathbf{p}_j\cdot\mathbf{p}_k) \\ &= (\mathbf{p}_i\cdot\mathbf{p}_\ell)(\mathbf{p}_j\cdot\mathbf{p}_k).\end{aligned}$$

□

The following lemma is useful to prove the equivalence (d) $\Leftrightarrow$ (d)$^*$.

**Lemma 5.** *We have the following equivalences:*

($\alpha$) *there exists no pair of distinct indices* $i$, $j$ *with* $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$ *if and only if there exists no pair of distinct indices* $i'$, $j'$ *with* $\Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix} \neq 0$,

($\beta$) *there exists the unique pair of distinct indices* $i$, $j$ *with* $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$ *if and only if there exists the unique pair of distinct indices* $i'$, $j'$ *with* $\Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix} \neq 0$,

($\gamma$) *there exist at least two pairs of distinct indices* $i$, $k$ *and* $j$, $\ell$ *with* $\Delta\begin{pmatrix} i \\ k \end{pmatrix} \neq 0 \neq \Delta\begin{pmatrix} j \\ \ell \end{pmatrix}$ *if and only if there exist at least two pairs of distinct indices* $i'$, $k'$ *and* $j'$, $\ell'$ *with* $\Delta^*\begin{pmatrix} i' \\ k' \end{pmatrix} \neq 0 \neq \Delta^*\begin{pmatrix} j' \\ \ell' \end{pmatrix}$.

*Notice that, if the conditions of* ($\beta$) *hold, the pair* $i$, $j$ *is equal to the pair* $i'$, $j'$.

*Proof.* ($\alpha$): If there exists no pair $i$, $j$ with $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$, then $\mathbf{p}_k = \mathbf{p}_k^*$ holds for an arbitrary index $k = 0,\ldots, n-1$ (so, for each $\mathbf{x} \in \mathbb{S}^{n-1}$, there exists a great circular arc $C_k$ passing through $\mathbf{p}_k$, $\mathbf{p}_k^*$, and $\mathbf{x}$. See Remark 6). Hence there exists no pair $i'$, $j'$ with $\Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix} \neq 0$. The proof of the converse is similar. ($\beta$): Assume that there exists the unique pair $i$, $j$ with $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$. Then, $\mathbf{p}_k = \mathbf{p}_k^*$ holds for each index $k \neq i, j$ (see Remark 6 again). Hence we have $\Delta^*\begin{pmatrix} i'' \\ j'' \end{pmatrix} = 0$ for each pair of distinct indices $i''$, $j''$ except the pair $i$, $j$ (we have $\Delta^*\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$ because, if not, we have $\Delta\begin{pmatrix} i \\ j \end{pmatrix} = 0$ from the equivalence ($\alpha$)). The proof of the converse is similar. ($\gamma$): It is from ($\alpha$) and ($\beta$). □

## 3. Equivalence of (b) and (b)$^*$, and Equivalence of (d) and (d)$^*$

In this section, we prove the equivalences (b) ⇔ (b)$^*$ and (d) ⇔ (d)$^*$. To prove them, we need the following two lemmas. Notice that equations except (8) do not require the assumption (b).

**Lemma 6.** *Let* $\{k_0, \ldots, k_{n-1}\}$ *be a permutation of the set of all indices* $\{0, \ldots, n-1\}$. *Then we have*

$$\Delta^*(k_0 \cdots k_m) = \frac{\Delta(k_{m+1} \cdots k_{n-1})}{\Delta(0 \cdots n-1)} \prod_{r=0}^{m} (\mathbf{p}_{k_r} \cdot \mathbf{p}_{k_r}^*)^2, \tag{4}$$

*for* $m = 0,\ldots, n-1$, *and*

$$\Delta^*\left(k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix}\right) = \frac{-\Delta\left(k_m \cdots \widehat{i} \cdots \widehat{j} \cdots k_{n-1} \begin{matrix} i \\ j \end{matrix}\right)}{\Delta(0 \cdots n-1)} (\mathbf{p}_i \cdot \mathbf{p}_i^*)(\mathbf{p}_j \cdot \mathbf{p}_j^*) \prod_{r=0}^{m-1} (\mathbf{p}_{k_r} \cdot \mathbf{p}_{k_r}^*)^2, \tag{5}$$

*for* $m = 0,\ldots, n-2$ *and distinct indices* $i$, $j = k_m,\ldots, k_{n-1}$ *(We can use the notation* $k_m \cdots \widehat{k_s} \cdots \widehat{k_t} \cdots k_{n-1}$ *whether* $s < t$ *or not. If* $s > t$*, it means* $k_m \cdots \widehat{k_t} \cdots \widehat{k_s} \cdots k_{n-1}$*), and*

$$\mathbf{p}_k \cdot \mathbf{p}_k^* = \frac{\sqrt{\Delta(0 \cdots n-1)}}{\sqrt{\Delta\left(0 \cdots \widehat{k} \cdots n-1\right)}}, \tag{6}$$

*for an index* $k = 0,\ldots, n-1$.

*Proof.* We have (6) by

$$\begin{aligned}
\mathbf{p}_k^* \cdot \mathbf{p}_k &= \frac{(-1)^{n-1-k} \varepsilon \det(\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1}, \mathbf{p}_k)}{\sqrt{\Delta\left(0 \cdots \widehat{k} \cdots n-1\right)}} \\
&= \frac{\varepsilon \det(\mathbf{p}_0, \ldots, \mathbf{p}_{n-1})}{\sqrt{\Delta\left(0 \cdots \widehat{k} \cdots n-1\right)}} \\
&= \frac{|\det(\mathbf{p}_0, \ldots, \mathbf{p}_{n-1})|}{\sqrt{\Delta\left(0 \cdots \widehat{k} \cdots n-1\right)}} \\
&= \frac{\sqrt{\Delta(0 \cdots n-1)}}{\sqrt{\Delta\left(0 \cdots \widehat{k} \cdots n-1\right)}},
\end{aligned}$$

where the first equality is from (1). The equations (4) and (5) are from (6) and Lemma 5.3 of [4]:

$$\Delta^*(k_0 \cdots k_m) = \frac{(\Delta(0 \cdots n-1))^m}{\prod_{r=0}^{m-1} \Delta\left(0 \cdots \widehat{k_r} \cdots n-1\right)} \cdot \frac{\Delta(k_{m+1} \cdots k_{n-1})}{\Delta\left(0 \cdots \widehat{k_m} \cdots n-1\right)},$$

$$\Delta^*\left(k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix}\right) = \frac{(\Delta(0 \cdots n-1))^m}{\prod_{r=0}^{m-1} \Delta\left(0 \cdots \widehat{k_r} \cdots n-1\right)} \cdot \frac{-\Delta\left(k_m \cdots \widehat{i} \cdots \widehat{j} \cdots k_{n-1} \begin{matrix} i \\ j \end{matrix}\right)}{\sqrt{\Delta\left(0 \cdots \widehat{i} \cdots n-1\right)}\sqrt{\Delta\left(0 \cdots \widehat{j} \cdots n-1\right)}}. \qquad \square$$

*Remark* 8. We also have similar equations:

$$\Delta\left(k_0\cdots k_m\right)=\frac{\Delta^*\left(k_{m+1}\cdots k_{n-1}\right)}{\Delta^*\left(0\cdots n-1\right)}\prod_{r=0}^{m}(\mathbf{p}_{k_r}\cdot\mathbf{p}^*_{k_r})^2, \tag{4$^*$}$$

$$\Delta\left(k_0\cdots k_{m-1}\,{i \atop j}\right)=\frac{-\,\Delta^*\left(k_m\cdots\widehat{i}\cdots\widehat{j}\cdots k_{n-1}\,{i \atop j}\right)}{\Delta^*\left(0\cdots n-1\right)}\cdot(\mathbf{p}_i\cdot\mathbf{p}^*_i)(\mathbf{p}_j\cdot\mathbf{p}^*_j)\prod_{r=0}^{m-1}(\mathbf{p}_{k_r}\cdot\mathbf{p}^*_{k_r})^2, \tag{5$^*$}$$

$$\mathbf{p}_k\cdot\mathbf{p}^*_k=\frac{\sqrt{\Delta^*\left(0\cdots n-1\right)}}{\sqrt{\Delta^*\left(0\cdots\widehat{k}\cdots n-1\right)}}. \tag{6$^*$}$$

**Lemma 7.** *We have*

$$\Delta\left(k_0\cdots k_m\right)=\Delta\left(k_0\cdots k_{m-1}\right)-\sum_{\ell=0}^{m-1}\Delta\begin{pmatrix}k_\ell\\ k_m\end{pmatrix}\Delta\left(k_0\cdots\widehat{k_\ell}\cdots k_{m-1}\,{k_m \atop k_\ell}\right), \tag{7}$$

*for* $m=0,\ldots,n-1$ *and pairwise distinct indices* $k_0,\ldots,k_m$. *Moreover, the condition* (b) *implies*

$$\Delta\left(k_0\cdots k_{m-1}\,{i \atop j}\right)=\Delta\left(k_0\cdots k_{m-2}\,{i \atop j}\right)-\Delta\begin{pmatrix}i\\ k_{m-1}\end{pmatrix}\Delta\left(k_0\cdots k_{m-2}\,{k_{m-1} \atop j}\right), \tag{8}$$

*for* $m=0,\ldots,n-2$ *and pairwise distinct indices* $k_0,\ldots,k_{m-1},i,j$.

*Proof.* We have

$$\begin{aligned}\Delta\left(k_0\cdots k_m\right)&=\sum_{\ell=0}^{m-1}(-1)^{m-\ell}\cdot\Delta\begin{pmatrix}k_\ell\\ k_m\end{pmatrix}\cdot\det M_\ell+1\cdot\Delta\left(k_m\right)\cdot\Delta\left(k_0\cdots k_{m-1}\right)\\ &=-\sum_{\ell=0}^{m-1}\Delta\begin{pmatrix}k_\ell\\ k_m\end{pmatrix}\Delta\left(k_0\cdots\widehat{k_\ell}\cdots k_{m-1}\,{k_m \atop k_\ell}\right)+\Delta\left(k_0\cdots k_{m-1}\right),\end{aligned}$$

where

$$M_\ell=\begin{pmatrix}\mathbf{p}_{k_0}\cdot\mathbf{p}_{k_0}&\cdots&\mathbf{p}_{k_0}\cdot\mathbf{p}_{k_{m-1}}\\ \vdots&&\vdots\\ \widehat{\mathbf{p}_{k_\ell}\cdot\mathbf{p}_{k_0}}&\cdots&\widehat{\mathbf{p}_{k_\ell}\cdot\mathbf{p}_{k_{m-1}}}\\ \vdots&&\vdots\\ \mathbf{p}_{k_m}\cdot\mathbf{p}_{k_0}&\cdots&\mathbf{p}_{k_m}\cdot\mathbf{p}_{k_{m-1}}\end{pmatrix}.$$

Moreover, if the condition (b) holds, we also have

$$
\begin{aligned}
&\Delta\begin{pmatrix} k_0 \cdots k_{m-1} & i \\ & j \end{pmatrix} \\
&= \sum_{\ell=0}^{m-2} (-1)^{m-1-\ell} \cdot \Delta\begin{pmatrix} k_\ell \\ k_{m-1} \end{pmatrix} \cdot \det M'_\ell \\
&\qquad + 1 \cdot \Delta\left(k_{m-1}\right) \cdot \Delta\begin{pmatrix} k_0 \cdots k_{m-2} & i \\ & j \end{pmatrix} \\
&\qquad + (-1) \cdot \Delta\begin{pmatrix} i \\ k_{m-1} \end{pmatrix} \cdot \Delta\begin{pmatrix} k_0 \cdots k_{m-2} & k_{m-1} \\ & j \end{pmatrix} \\
&= \sum_{\ell=0}^{m-2} 0 + \Delta\begin{pmatrix} k_0 \cdots k_{m-2} & i \\ & j \end{pmatrix} - \Delta\begin{pmatrix} i \\ k_{m-1} \end{pmatrix} \Delta\begin{pmatrix} k_0 \cdots k_{m-2} & k_{m-1} \\ & j \end{pmatrix},
\end{aligned}
$$

where the last equality is from the parallelism of the lower 2 rows of

$$
M'_\ell = \begin{pmatrix}
\mathbf{p}_{k_0} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_{m-2}} & \mathbf{p}_{k_0} \cdot \mathbf{p}_j \\
\vdots & & \vdots & \vdots \\
\widehat{\mathbf{p}_{k_\ell} \cdot \mathbf{p}_{k_0}} & \cdots & \widehat{\mathbf{p}_{k_\ell} \cdot \mathbf{p}_{k_{m-2}}} & \widehat{\mathbf{p}_{k_\ell} \cdot \mathbf{p}_j} \\
\vdots & & \vdots & \vdots \\
\mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_{k_{m-2}} & \mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_j \\
\mathbf{p}_i \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_i \cdot \mathbf{p}_{k_{m-2}} & \mathbf{p}_i \cdot \mathbf{p}_j
\end{pmatrix}.
$$

□

The equation (8) enables us to prove the following two lemmas and two corollaries by induction (the equation (7) is used in Appendix).

**Lemma 8.** *The condition* (b) *implies*

$$
\Delta\begin{pmatrix} k_0 \cdots k_{\ell-1} & i \\ & j \end{pmatrix} \Delta\begin{pmatrix} k'_0 \cdots k'_{m-1} & i' \\ & j' \end{pmatrix} = \Delta\begin{pmatrix} k_0 \cdots k_{\ell-1} & i' \\ & j \end{pmatrix} \Delta\begin{pmatrix} k'_0 \cdots k'_{m-1} & i \\ & j' \end{pmatrix},
$$

*for* $\ell, m = 0, \ldots, n-2$ *and indices* $k_0, \ldots, k_{\ell-1}, k'_0, \ldots, k'_{m-1}, i, j, i', j'$ *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

*Proof.* The conclusion above is called the type $(\ell, m)$. Without loss of generality, we can assume $\ell \geqq m$. The type $(0,0)$ is obvious, because, if $i \neq i'$ and $j \neq j'$ then it is (b), otherwise it is the identity. If $\ell > 0$, the type $(\ell, 0)$ is shown by:

$$
\begin{aligned}
&\Delta\begin{pmatrix} k_0 \cdots k_{\ell-1} & i \\ & j \end{pmatrix} \Delta\begin{pmatrix} i' \\ j' \end{pmatrix} - \Delta\begin{pmatrix} k_0 \cdots k_{\ell-1} & i' \\ & j \end{pmatrix} \Delta\begin{pmatrix} i \\ j' \end{pmatrix} \\
&= \left(\Delta\begin{pmatrix} k_0 \cdots k_{\ell-2} & i \\ & j \end{pmatrix} - \Delta\begin{pmatrix} i \\ k_{\ell-1} \end{pmatrix} \Delta\begin{pmatrix} k_0 \cdots k_{\ell-2} & k_{\ell-1} \\ & j \end{pmatrix}\right) \Delta\begin{pmatrix} i' \\ j' \end{pmatrix} \\
&\quad - \left(\Delta\begin{pmatrix} k_0 \cdots k_{\ell-2} & i' \\ & j \end{pmatrix} - \Delta\begin{pmatrix} i' \\ k_{\ell-1} \end{pmatrix} \Delta\begin{pmatrix} k_0 \cdots k_{\ell-2} & k_{\ell-1} \\ & j \end{pmatrix}\right) \Delta\begin{pmatrix} i \\ j' \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta\left(k_0\cdots k_{\ell-2}\,\begin{matrix} i \\ j \end{matrix}\right)\Delta\begin{pmatrix} i' \\ j' \end{pmatrix} - \Delta\left(k_0\cdots k_{\ell-2}\,\begin{matrix} i' \\ j \end{matrix}\right)\Delta\begin{pmatrix} i \\ j' \end{pmatrix} \\
&= 0,
\end{aligned}
$$

where the last equality is from the type $(\ell-1,0)$. If $\ell \geqq m > 0$, the type $(\ell, m)$ is shown by:

$$
\begin{aligned}
&\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i' \\ j' \end{matrix}\right) - \Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i' \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i \\ j' \end{matrix}\right) \\
&= \Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i \\ j \end{matrix}\right)\left(\Delta\left(k'_0\cdots k'_{m-2}\,\begin{matrix} i' \\ j' \end{matrix}\right) - \Delta\begin{pmatrix} i' \\ k'_{m-1} \end{pmatrix}\Delta\left(k'_0\cdots k'_{m-2}\,\begin{matrix} k'_{m-1} \\ j' \end{matrix}\right)\right) \\
&\qquad -\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i' \\ j \end{matrix}\right)\left(\Delta\left(k'_0\cdots k'_{m-2}\,\begin{matrix} i \\ j' \end{matrix}\right) - \Delta\begin{pmatrix} i \\ k'_{m-1} \end{pmatrix}\Delta\left(k'_0\cdots k'_{m-2}\,\begin{matrix} k'_{m-1} \\ j' \end{matrix}\right)\right) \\
&= \left(\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-2}\,\begin{matrix} i' \\ j' \end{matrix}\right) - \Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i' \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-2}\,\begin{matrix} i \\ j' \end{matrix}\right)\right) \\
&\qquad - \left(\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i \\ j \end{matrix}\right)\Delta\begin{pmatrix} i' \\ k'_{m-1} \end{pmatrix}\right. \\
&\qquad\qquad \left. -\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i' \\ j \end{matrix}\right)\Delta\begin{pmatrix} i \\ k'_{m-1} \end{pmatrix}\right)\Delta\left(k'_0\cdots k'_{m-2}\,\begin{matrix} k'_{m-1} \\ j' \end{matrix}\right) \\
&= 0,
\end{aligned}
$$

where the last equality is from the types $(\ell, m-1)$ and $(\ell, 0)$. □

**Lemma 9.** *The condition* (b) *implies*

$$
\Delta\left(k_0\cdots k_{\ell-1}k\,\begin{matrix} i \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i' \\ j' \end{matrix}\right) = \Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}k\,\begin{matrix} i' \\ j' \end{matrix}\right),
$$

*for* $\ell$, $m = 0,\ldots, n-3$ *and indices* $k_0,\ldots, k_{\ell-1}$, $k'_0,\ldots, k'_{m-1}$, $i$, $j$, $i'$, $j'$, $k$ *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

*Proof.* We have

$$
\begin{aligned}
&\Delta\left(k_0\cdots k_{\ell-1}k\,\begin{matrix} i \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i' \\ j' \end{matrix}\right) - \Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}k\,\begin{matrix} i' \\ j' \end{matrix}\right) \\
&= \left(\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i \\ j \end{matrix}\right) - \Delta\begin{pmatrix} i \\ k \end{pmatrix}\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} k \\ j \end{matrix}\right)\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i' \\ j' \end{matrix}\right) \\
&\qquad - \Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i \\ j \end{matrix}\right)\left(\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i' \\ j' \end{matrix}\right) - \Delta\begin{pmatrix} i' \\ k \end{pmatrix}\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} k \\ j' \end{matrix}\right)\right) \\
&= -\Delta\begin{pmatrix} i \\ k \end{pmatrix}\left(\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} k \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i' \\ j' \end{matrix}\right)\right. \\
&\qquad\qquad \left. - \Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i' \\ j \end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} k \\ j' \end{matrix}\right)\right) \\
&= 0,
\end{aligned}
$$

where the second and last equalities are the types $(\ell, 0)$ and $(\ell, m)$ of the previous lemma, respectively. □

**Corollary 10.** *The condition* (b) *also implies*

$$\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}=\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix},$$

*for* $\ell, m = 0,\ldots, n-3$ *and indices* $k_0,\ldots, k_{\ell-1}$, $k'_0,\ldots, k'_{m-1}$, $i$, $j$, $i'$, $j'$, $k$ *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

*Proof.* In particular, if $k \neq i'$, it is obvious from two previous lemmas. Otherwise, we have

$$\begin{aligned}
&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}i'\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}i'\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix}\\
=&\left(\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}i\\i'\end{pmatrix}\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}\right)\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}\\
&-\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}\left(\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}i\\i'\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}\right)\\
=&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix}\\
=&\,0.
\end{aligned}$$

□

**Corollary 11.** *The condition* (b) *also implies*

$$\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k'\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}=\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k'\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix},$$

*for* $\ell, m = 0,\ldots, n-3$ *and indices* $k_0,\ldots, k_{\ell-1}$, $k'_0,\ldots, k'_{m-1}$, $i$, $j$, $i'$, $j'$, $k$, $k'$ *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

*Proof.* If $k = k'$ then the conclusion is from Lemma 8, so assume $k \neq k'$. If $k \neq i'$, then we have

$$\begin{aligned}
&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k'\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}\\
=&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k'k\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}\\
=&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k'\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix},
\end{aligned}$$

where the first and last equalities are from Lemma 9 and Corollary 10, respectively. If $k' \neq i$, the proof is similar. Otherwise, i.e, if $k = i'$ and $k' = i$, then

$$\begin{aligned}
&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}i'\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}i\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}i\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}i'\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix}\\
=&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}i'\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\left(\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i'\\j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}i'\\i\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix}\right)\\
&-\left(\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i'\\j\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}i'\\i\end{pmatrix}\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}\,\begin{matrix}i\\j\end{matrix}\end{pmatrix}\right)\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}i'\,\begin{matrix}i\\j'\end{matrix}\end{pmatrix}
\end{aligned}$$

$$= \left(\Delta\begin{pmatrix} k_0\cdots k_{\ell-1} i' \, {i \atop j}\end{pmatrix}\Delta\begin{pmatrix} k'_0\cdots k'_{m-1}\, {i' \atop j'}\end{pmatrix} - \Delta\begin{pmatrix} k_0\cdots k_{\ell-1}\, {i' \atop j}\end{pmatrix}\Delta\begin{pmatrix} k'_0\cdots k'_{m-1} i'\, {i \atop j'}\end{pmatrix}\right)$$
$$- \Delta\begin{pmatrix} i' \\ i\end{pmatrix}\left(\Delta\begin{pmatrix} k_0\cdots k_{\ell-1} i' \, {i \atop j}\end{pmatrix}\Delta\begin{pmatrix} k'_0\cdots k'_{m-1}\, {i \atop j'}\end{pmatrix}\right.$$
$$\left. - \Delta\begin{pmatrix} k_0\cdots k_{\ell-1}\, {i \atop j}\end{pmatrix}\Delta\begin{pmatrix} k'_0\cdots k'_{m-1} i'\, {i \atop j'}\end{pmatrix}\right)$$
$$= 0,$$

where the last equality is from Lemma 9 and Corollary 10. □

Now, we prove the purposes of this section.

**Lemma 12.** *The condition* (b) *is equivalent to the condition* (b)*.

*Proof.* Assume that the condition (b) holds. Then, for pairwise distinct indices $i$, $i'$, $j$, $j'$, we have

$$\Delta^*\begin{pmatrix} i \\ j\end{pmatrix}\Delta^*\begin{pmatrix} i' \\ j'\end{pmatrix} = \frac{-\Delta\begin{pmatrix} 0\cdots\widehat{i}\cdots\widehat{j}\cdots n-1\, {i \atop j}\end{pmatrix}}{\Delta(0\cdots n-1)}(\mathbf{p}_i\cdot\mathbf{p}_i^*)(\mathbf{p}_j\cdot\mathbf{p}_j^*)$$
$$\cdot\frac{-\Delta\begin{pmatrix} 0\cdots\widehat{i'}\cdots\widehat{j'}\cdots n-1\, {i' \atop j'}\end{pmatrix}}{\Delta(0\cdots n-1)}(\mathbf{p}_{i'}\cdot\mathbf{p}_{i'}^*)(\mathbf{p}_{j'}\cdot\mathbf{p}_{j'}^*)$$
$$= \frac{-\Delta\begin{pmatrix} 0\cdots\widehat{i'}\cdots\widehat{j}\cdots n-1\, {i' \atop j}\end{pmatrix}}{\Delta(0\cdots n-1)}(\mathbf{p}_{i'}\cdot\mathbf{p}_{i'}^*)(\mathbf{p}_j\cdot\mathbf{p}_j^*)$$
$$\cdot\frac{-\Delta\begin{pmatrix} 0\cdots\widehat{i}\cdots\widehat{j'}\cdots n-1\, {i \atop j'}\end{pmatrix}}{\Delta(0\cdots n-1)}(\mathbf{p}_i\cdot\mathbf{p}_i^*)(\mathbf{p}_{j'}\cdot\mathbf{p}_{j'}^*)$$
$$= \Delta^*\begin{pmatrix} i' \\ j\end{pmatrix}\Delta^*\begin{pmatrix} i \\ j'\end{pmatrix},$$

where the first and last equalities are from (5) and the second equality is from Corollary 11. The proof of the converse is similar. □

*Remark* 9. From the previous lemma, the condition (b) implies similar equations:

$$\Delta^*\begin{pmatrix} k_0\cdots k_{\ell-1}\, {i \atop j}\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m-1}\, {i' \atop j'}\end{pmatrix} = \Delta^*\begin{pmatrix} k_0\cdots k_{\ell-1}\, {i' \atop j}\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m-1}\, {i \atop j'}\end{pmatrix},$$
$$\Delta^*\begin{pmatrix} k_0\cdots k_{\ell-1}k\, {i \atop j}\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m-1}\, {i' \atop j'}\end{pmatrix} = \Delta^*\begin{pmatrix} k_0\cdots k_{\ell-1}\, {i \atop j}\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m-1}k\, {i' \atop j'}\end{pmatrix},$$
$$\Delta^*\begin{pmatrix} k_0\cdots k_{\ell-1}k\, {i \atop j}\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m-1}\, {i' \atop j'}\end{pmatrix} = \Delta^*\begin{pmatrix} k_0\cdots k_{\ell-1}\, {i' \atop j}\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m-1}k\, {i \atop j'}\end{pmatrix},$$
$$\Delta^*\begin{pmatrix} k_0\cdots k_{\ell-1}k\, {i \atop j}\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m-1}k'\, {i' \atop j'}\end{pmatrix} = \Delta^*\begin{pmatrix} k_0\cdots k_{\ell-1}k'\, {i' \atop j}\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m-1}k\, {i \atop j'}\end{pmatrix}.$$

**Corollary 13.** *The condition* (d) *is equivalent to the condition* (d)$^*$.

*Proof.* It is an natural consequence of Lemma 12 and the equivalence ($\gamma$) of Lemma 5. $\square$

## 4. $\mu_k$, $\nu_k$, $\mu_k^*$, and $\nu_k^*$

To represent orthocenters, we use four types of values: $\mu_k$, $\nu_k$, $\mu_k^*$, and $\nu_k^*$. To define them and to check their fundamental relations, we need two lemmas.

**Lemma 14.** *The condition* (b) *implies*

$$\begin{aligned}&\Delta\begin{pmatrix} k_0\cdots k_m \begin{matrix} i \\ j \end{matrix}\end{pmatrix}\Delta^*\begin{pmatrix} k_0\cdots k_m \begin{matrix} i' \\ j' \end{matrix}\end{pmatrix}\\ &=\Delta\begin{pmatrix} k_0\cdots k_{m-1} \begin{matrix} i \\ j \end{matrix}\end{pmatrix}\Delta^*\begin{pmatrix} k_0\cdots k_{m-1} \begin{matrix} i' \\ j' \end{matrix}\end{pmatrix}(\mathbf{p}_{k_m}\cdot\mathbf{p}_{k_m}^*)^2,\end{aligned}$$

*for* $m=0,\ldots,n-3$ *and indices* $k_0,\ldots,k_m$, $i$, $j$, $i'$, $j'$ *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

*Proof.* This is a natural consequence of (5) and

$$\begin{aligned}&\Delta\begin{pmatrix} k_0\cdots k_m \begin{matrix} i \\ j \end{matrix}\end{pmatrix}\Delta\begin{pmatrix} k_{m+1}\cdots\widehat{i'}\cdots\widehat{j'}\cdots k_{n-1} \begin{matrix} i' \\ j' \end{matrix}\end{pmatrix}\\ &=\Delta\begin{pmatrix} k_0\cdots k_{m-1} \begin{matrix} i \\ j \end{matrix}\end{pmatrix}\Delta\begin{pmatrix} k_m\cdots\widehat{i'}\cdots\widehat{j'}\cdots k_{n-1} \begin{matrix} i' \\ j' \end{matrix}\end{pmatrix},\end{aligned}$$

which is from Lemma 9, where $\{k_0,\ldots,k_m,k_{m+1},\ldots,k_{n-1}\}$ is a permutation of the set of all indices $\{0,\ldots,n-1\}$. $\square$

**Lemma 15.** *The condition* (b) (*which is equivalent to* (b)$^*$) *implies*

$$\begin{aligned}&\Delta\begin{pmatrix} i \\ k\end{pmatrix}\Delta\begin{pmatrix} k \\ j\end{pmatrix}\Delta\begin{pmatrix} i' \\ j'\end{pmatrix}=\Delta\begin{pmatrix} i' \\ k\end{pmatrix}\Delta\begin{pmatrix} k \\ j'\end{pmatrix}\Delta\begin{pmatrix} i \\ j\end{pmatrix},\\ &\Delta\begin{pmatrix} i \\ k\end{pmatrix}\Delta\begin{pmatrix} k \\ j\end{pmatrix}\Delta\begin{pmatrix} k \begin{matrix} i' \\ j' \end{matrix}\end{pmatrix}=\Delta\begin{pmatrix} i' \\ k\end{pmatrix}\Delta\begin{pmatrix} k \\ j'\end{pmatrix}\Delta\begin{pmatrix} k \begin{matrix} i \\ j \end{matrix}\end{pmatrix},\\ &\Delta^*\begin{pmatrix} i \\ k\end{pmatrix}\Delta^*\begin{pmatrix} k \\ j\end{pmatrix}\Delta^*\begin{pmatrix} i' \\ j'\end{pmatrix}=\Delta^*\begin{pmatrix} i' \\ k\end{pmatrix}\Delta^*\begin{pmatrix} k \\ j'\end{pmatrix}\Delta^*\begin{pmatrix} i \\ j\end{pmatrix},\\ &\Delta^*\begin{pmatrix} i \\ k\end{pmatrix}\Delta^*\begin{pmatrix} k \\ j\end{pmatrix}\Delta^*\begin{pmatrix} k \begin{matrix} i' \\ j' \end{matrix}\end{pmatrix}=\Delta^*\begin{pmatrix} i' \\ k\end{pmatrix}\Delta^*\begin{pmatrix} k \\ j'\end{pmatrix}\Delta^*\begin{pmatrix} k \begin{matrix} i \\ j \end{matrix}\end{pmatrix},\end{aligned}$$

*for indices* $i$, $j$, $i'$, $j'$, $k$ *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *or in an* $\Delta^*(\cdots)$ *are distinct.*

*Proof.* For the first equation, if $\{i,j\}\cap\{i',j'\}\neq\emptyset$ then it is obvious, for example, if $i=i'$ then $i\neq j'$, so we have

$$\Delta\begin{pmatrix} i \\ k\end{pmatrix}\Delta\begin{pmatrix} k \\ j\end{pmatrix}\Delta\begin{pmatrix} i \\ j'\end{pmatrix}=\Delta\begin{pmatrix} i \\ k\end{pmatrix}\Delta\begin{pmatrix} k \\ j'\end{pmatrix}\Delta\begin{pmatrix} i \\ j\end{pmatrix},$$

from (b). Otherwise, i.e., if $\{i, j\} \cap \{i', j'\} = \emptyset$, we have

$$\Delta\binom{i}{k}\Delta\binom{k}{j}\Delta\binom{i'}{j'} = \Delta\binom{i'}{k}\Delta\binom{k}{j}\Delta\binom{i}{j'} = \Delta\binom{i'}{k}\Delta\binom{k}{j'}\Delta\binom{i}{j},$$

where the both equalities are from (b). For the second equation, we have

$$\begin{aligned}\Delta\binom{i}{k}\Delta\binom{k}{j}\Delta\left(k\,{}^{i'}_{j'}\right) &= \Delta\binom{i}{k}\Delta\binom{k}{j}\left(\Delta\binom{i'}{j'} - \Delta\binom{i'}{k}\Delta\binom{k}{j'}\right)\\ &= \Delta\binom{i'}{k}\Delta\binom{k}{j'}\left(\Delta\binom{i}{j} - \Delta\binom{i}{k}\Delta\binom{k}{j}\right)\\ &= \Delta\binom{i'}{k}\Delta\binom{k}{j'}\Delta\left(k\,{}^{i}_{j}\right),\end{aligned}$$

where the second equality is from the first equation. Similarly, the third and last equations are from (b)*, so they are from (b). □

From the previous lemma, the following values are determined only by the index $k$.

**Definition 2.** On the assumption (b), for an index $k = 0, \ldots, n-1$, if there exists a pair of distinct indices $i$, $j \neq k$ with $\Delta\binom{i}{j} \neq 0$ (resp. $\Delta\left(k\,{}^{i}_{j}\right) \neq 0$, $\Delta^*\binom{i}{j} \neq 0$, $\Delta^*\left(k\,{}^{i}_{j}\right) \neq 0$), we define

$$\mu_k = \frac{\Delta\binom{i}{k}\Delta\binom{k}{j}}{\Delta\binom{i}{j}} \qquad \left(\text{resp. } \nu_k = -\frac{\Delta\binom{i}{k}\Delta\binom{k}{j}}{\Delta\left(k\,{}^{i}_{j}\right)},\right.$$

$$\left.\mu_k^* = \frac{\Delta^*\binom{i}{k}\Delta^*\binom{k}{j}}{\Delta^*\binom{i}{j}}, \qquad \nu_k^* = -\frac{\Delta^*\binom{i}{k}\Delta^*\binom{k}{j}}{\Delta^*\left(k\,{}^{i}_{j}\right)}\right),$$

which depends only the index $k$.

*Remark* 10. For distinct indices $k$ and $\ell$, if there exist $\mu_k$ and $\mu_\ell$, then, for some pairs of distinct indices $i$, $j$ and $i'$, $j'$ with $\ell \neq i \neq k \neq j$ and $i' \neq \ell \neq j' \neq k$, we have

$$\mu_k\mu_\ell = \frac{\Delta\binom{i}{k}\Delta\binom{k}{j}}{\Delta\binom{i}{j}}\frac{\Delta\binom{i'}{\ell}\Delta\binom{\ell}{j'}}{\Delta\binom{i'}{j'}}$$

$$= \frac{\Delta\binom{i}{\ell}\Delta\binom{k}{j}}{\Delta\binom{i}{j}}\frac{\Delta\binom{i'}{\ell}\Delta\binom{k}{j'}}{\Delta\binom{i'}{j'}} = \left(\Delta\binom{k}{\ell}\right)^2.$$

If there exist $\mu_k$ and $\nu_k$, then for some pairs of distinct indices $i$, $j \neq k$ and $i'$, $j' \neq k$, we have

$$(1-\mu_k)(1-\nu_k) = \left(1 - \frac{\Delta\binom{i}{k}\Delta\binom{k}{j}}{\Delta\binom{i}{j}}\right)\left(1 + \frac{\Delta\binom{i'}{k}\Delta\binom{k}{j'}}{\Delta\left(k\,{}^{i'}_{j'}\right)}\right)$$
$$= \frac{\Delta\left(k\,{}^{i}_{j}\right)}{\Delta\binom{i}{j}}\frac{\Delta\binom{i'}{j'}}{\Delta\left(k\,{}^{i'}_{j'}\right)} = 1,$$

where the last equality is from Lemma 9 (We also have

$$(1-\mu_k^*)(1-\nu_k^*) = 1$$

if $\mu_k^*$ and $\nu_k^*$ exist. It is obvious that the existence of $\mu_k$ (resp. $\nu_k$, $\mu_k^*$, and $\nu_k^*$) and $\mu_k \neq 1$ (resp. $\nu_k \neq 1$, $\mu_k^* \neq 1$, and $\nu_k^* \neq 1$) implies the existence of $\nu_k$ (resp. $\mu_k$, $\nu_k^*$, and $\mu_k^*$)).

The values $\mu_k^*$ (resp. $\mu_k$) and $\nu_k$ (resp. $\nu_k^*$) exist simultaneously and satisfy an equation.

**Lemma 16.** *Assume that the condition* (b) *holds. Then, for an index* $k = 0,\ldots,n-1$, *there exists* $\mu_k^*$ (*resp.* $\mu_k$) *if and only if there exists* $\nu_k$ (*resp.* $\nu_k^*$). *Moreover, if there exists* $\nu_k$ (*resp.* $\nu_k^*$), *we have*

$$1-\mu_k^* = (1-\nu_k)(\mathbf{p}_k\,.\,\mathbf{p}_k^*)^2 \qquad (\textit{resp. } 1-\mu_k = (1-\nu_k^*)(\mathbf{p}_k\,.\,\mathbf{p}_k^*)^2).$$

*Proof.* Suppose that there does not exist $\mu_k^*$, i.e., for an arbitrary pair of distinct indices $i$, $j \neq k$,

$$\mathbf{p}_i^* \cdot \mathbf{p}_j^* = \Delta^*\binom{i}{j} = 0 \tag{9}$$

holds. Then, $\{\mathbf{p}_0^*,\ldots,\mathbf{p}_{k-1}^*,\mathbf{p}_k,\mathbf{p}_{k+1}^*,\ldots,\mathbf{p}_{n-1}^*\}$ is an orthonormal basis. Hence, we have

$$\mathbf{p}_k^* = (\mathbf{p}_k\cdot\mathbf{p}_k^*)\mathbf{p}_k + \sum_{\ell=0}^{\substack{\ell\neq k\\ n-1}}\Delta^*\binom{\ell}{k}\mathbf{p}_\ell^*, \tag{10}$$

$$\mathbf{p}_\ell = (\mathbf{p}_\ell\cdot\mathbf{p}_\ell^*)\mathbf{p}_\ell^* + \Delta\binom{\ell}{k}\mathbf{p}_k \qquad \text{for an index } {}^\forall\ell\neq k. \tag{11}$$

So, we have

$$\Delta\binom{i}{j} = \mathbf{p}_i\cdot\mathbf{p}_j = \Delta\binom{i}{k}\Delta\binom{j}{k}. \tag{12}$$

Therefore, we also have $\Delta\begin{pmatrix} & i \\ k & \\ & j\end{pmatrix} = 0$, which means that there does not exist $\nu_k$. Similarly, we can prove the nonentity of $\mu_k$ implies the nonentity of $\nu_k^*$. Conversely, suppose that there exists $\mu_k$, i.e., there exists a pair of distinct indices $i$, $j \neq k$ with $\Delta\begin{pmatrix} i \\ j\end{pmatrix} \neq 0$. If there also exists $\mu_k^*$, i.e., if there exists a pair of distinct indices $i'$, $j' \neq k$ with $\Delta^*\begin{pmatrix} i' \\ j'\end{pmatrix} \neq 0$, we have $\Delta^*\begin{pmatrix} & i' \\ k & \\ & j'\end{pmatrix} \neq 0$ from

$$\Delta\begin{pmatrix} & i \\ k & \\ & j\end{pmatrix}\Delta^*\begin{pmatrix} & i' \\ k & \\ & j'\end{pmatrix} = \Delta\begin{pmatrix} i \\ j\end{pmatrix}\Delta^*\begin{pmatrix} i' \\ j'\end{pmatrix}(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2,$$

which is from Lemma 14. Hence there exists $\nu_k^*$. Otherwise, i.e., if there does not exist $\mu_k^*$, we have

$$\begin{aligned}\Delta\begin{pmatrix} i \\ j\end{pmatrix} = \Delta\begin{pmatrix} i \\ k\end{pmatrix}\Delta\begin{pmatrix} j \\ k\end{pmatrix} &= \left(-\Delta^*\begin{pmatrix} i \\ k\end{pmatrix}\frac{\mathbf{p}_i \cdot \mathbf{p}_i^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\right)\left(-\Delta^*\begin{pmatrix} j \\ k\end{pmatrix}\frac{\mathbf{p}_j \cdot \mathbf{p}_j^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\right) \\ &= \frac{(\mathbf{p}_i \cdot \mathbf{p}_i^*)(\mathbf{p}_j \cdot \mathbf{p}_j^*)}{(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2}\left(\Delta^*\begin{pmatrix} i \\ k\end{pmatrix}\Delta^*\begin{pmatrix} j \\ k\end{pmatrix} - \Delta^*\begin{pmatrix} i \\ j\end{pmatrix}\right) \\ &= -\frac{(\mathbf{p}_i \cdot \mathbf{p}_i^*)(\mathbf{p}_j \cdot \mathbf{p}_j^*)}{(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2}\Delta^*\begin{pmatrix} & i \\ k & \\ & j\end{pmatrix},\end{aligned}$$

where the first and third equality is from (12) and (9), respectively, and the second equality is from

$$0 = \mathbf{p}_i \cdot \mathbf{p}_k^* = (\mathbf{p}_k \cdot \mathbf{p}_k^*)\Delta\begin{pmatrix} i \\ k\end{pmatrix} + (\mathbf{p}_i \cdot \mathbf{p}_i^*)\Delta^*\begin{pmatrix} i \\ k\end{pmatrix},$$

$$0 = \mathbf{p}_j \cdot \mathbf{p}_k^* = (\mathbf{p}_k \cdot \mathbf{p}_k^*)\Delta\begin{pmatrix} j \\ k\end{pmatrix} + (\mathbf{p}_j \cdot \mathbf{p}_j^*)\Delta^*\begin{pmatrix} j \\ k\end{pmatrix},$$

whose last equalities are from (10) or (11). So we have $\Delta^*\begin{pmatrix} & i \\ k & \\ & j\end{pmatrix} \neq 0$, hence there exists $\nu_k^*$. Similarly, we can prove that the existence of $\mu_k^*$ implies the existence of $\nu_k$. Moreover, if there exists $\nu_k$, there also exists $\mu_k^*$, so, there exist pairs of distinct indices $i$, $j \neq k$ and $i'$, $j' \neq k$ with $\Delta\begin{pmatrix} & i \\ k & \\ & j\end{pmatrix} \neq 0 \neq \Delta^*\begin{pmatrix} i' \\ j'\end{pmatrix}$. Hence, we have

$$1 - \mu_k^* = \frac{\Delta^*\begin{pmatrix} & i' \\ k & \\ & j'\end{pmatrix}}{\Delta^*\begin{pmatrix} i' \\ j'\end{pmatrix}} = \frac{\Delta\begin{pmatrix} i \\ j\end{pmatrix}}{\Delta\begin{pmatrix} & i \\ k & \\ & j\end{pmatrix}}(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2 = (1 - \nu_k)(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2,$$

where the second equality is from Lemma 14. Similarly, the existence of $\nu_k^*$ implies the existence of $\mu_k$ and

$$1 - \mu_k = (1 - \nu_k^*)(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2.$$

□

## 5. Main results

We define the vector $\tilde{\mathbf{h}}_k$ (resp. $\tilde{\mathbf{h}}_k^*$) whose normalization $\mathbf{h}_k$ (resp. $\mathbf{h}_k^*$) is an orthocenter (see Lemma 20).

**Definition 3.** On the assumption (b), for an index $k = 0,\ldots, n-1$, if there exists $\nu_k$ (resp. $\nu_k^*$), then we define

$$\tilde{\mathbf{h}}_k = \mathbf{p}_k^* - (1-\nu_k)(\mathbf{p}_k \cdot \mathbf{p}_k^*)\mathbf{p}_k = \mathbf{p}_k^* - \frac{1-\mu_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\mathbf{p}_k$$
$$\Big(\text{resp. } \tilde{\mathbf{h}}_k^* = \mathbf{p}_k - (1-\nu_k^*)(\mathbf{p}_k \cdot \mathbf{p}_k^*)\mathbf{p}_k^* = \mathbf{p}_k - \frac{1-\mu_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\mathbf{p}_k^*\Big),$$

which is in $\mathbb{R}^n$. Moreover, if $\tilde{\mathbf{h}}_k \neq \mathbf{0}$ (resp. $\tilde{\mathbf{h}}_k^* \neq \mathbf{0}$), then we define

$$\mathbf{h}_k = \frac{\tilde{\mathbf{h}}_k}{|\tilde{\mathbf{h}}_k|} \qquad \Big(\text{resp. } \mathbf{h}_k^* = \frac{\tilde{\mathbf{h}}_k^*}{|\tilde{\mathbf{h}}_k^*|}\Big),$$

which is in $\mathbb{S}^{n-1}$.

The equations of the following lemma are relations of $\tilde{\mathbf{h}}_0,\ldots, \tilde{\mathbf{h}}_{n-1}, \tilde{\mathbf{h}}_0^*,\ldots, \tilde{\mathbf{h}}_{n-1}^*$, which means that existing ones of normalizations of them, $\mathbf{h}_0,\ldots, \mathbf{h}_{n-1}, \mathbf{h}_0^*,\ldots, \mathbf{h}_{n-1}^*$, are consistent, with the exception of the sign.

**Lemma 17.** *Assume that the condition* (b) *holds. Then, for pairwise distinct indices $k$, $\ell$, $i = 0,\ldots, n-1$, if there exist $\nu_k^*$ (resp. $\nu_k$) and $\nu_\ell^*$ (resp. $\nu_\ell$), we have*

$$\Delta\binom{i}{k}\tilde{\mathbf{h}}_\ell^* = \Delta\binom{i}{\ell}\tilde{\mathbf{h}}_k^* \qquad \Big(\textit{resp. } \Delta^*\binom{i}{k}\tilde{\mathbf{h}}_\ell = \Delta^*\binom{i}{\ell}\tilde{\mathbf{h}}_k\Big),$$

*if there exist $\nu_k$ (resp. $\nu_k^*$) and $\nu_\ell^*$ (resp. $\nu_\ell$), we have*

$$\nu_k\tilde{\mathbf{h}}_\ell^* = \Delta\binom{k}{\ell}\frac{\tilde{\mathbf{h}}_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*} \qquad \Big(\textit{resp. } \nu_k^*\tilde{\mathbf{h}}_\ell = \Delta^*\binom{k}{\ell}\frac{\tilde{\mathbf{h}}_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\Big),$$

*if there exist $\nu_k$ and $\nu_k^*$, we have*

$$\nu_k\tilde{\mathbf{h}}_k^* = \mu_k\frac{\tilde{\mathbf{h}}_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*} \qquad \Big(\textit{resp. } \nu_k^*\tilde{\mathbf{h}}_k = \mu_k^*\frac{\tilde{\mathbf{h}}_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\Big).$$

*Proof.* We have the following equations by comparing the inner product of $\mathbf{p}_0,\ldots, \mathbf{p}_{n-1}$ and both sides:

$$\Delta\binom{i}{k}\left(\mathbf{p}_\ell - \frac{1-\mu_\ell}{\mathbf{p}_\ell \cdot \mathbf{p}_\ell^*}\mathbf{p}_\ell^*\right) = \Delta\binom{i}{\ell}\left(\mathbf{p}_k - \frac{1-\mu_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\mathbf{p}_k^*\right),$$
$$\nu_k\left(\mathbf{p}_\ell - \frac{1-\mu_\ell}{\mathbf{p}_\ell \cdot \mathbf{p}_\ell^*}\mathbf{p}_\ell^*\right) = \Delta\binom{k}{\ell}\left(\frac{\mathbf{p}_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*} - (1-\nu_k)\mathbf{p}_k\right),$$
$$\nu_k\left(\mathbf{p}_k - \frac{1-\mu_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\mathbf{p}_k^*\right) = \mu_k\left(\frac{\mathbf{p}_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*} - (1-\nu_k)\mathbf{p}_k\right).$$

□

The following three lemmas show the implication (d) $\Rightarrow$ (c).

**Lemma 18.** *Assume* (d). *Then there exists* $\nu_k$ *(resp.* $\nu_k^*$*), and*

$$\tilde{\mathbf{h}}_k \neq \mathbf{0} \qquad (\textit{resp.}\ \tilde{\mathbf{h}}_k^* \neq \mathbf{0}),$$

*for some index* $k = 0, \ldots, n-1$.

*Proof.* The condition (d) is equivalent to the condition (d)$^*$, so, we can assume (3)$^*$ for some pairwise distinct indices $i$, $k$, $\ell$. From the former and latter inequalities of (3)$^*$, we have $\mathbf{p}_k \neq \mathbf{p}_k^*$ and the existence of $\mu_k^*$ and $\tilde{\mathbf{h}}_k$, respectively. The linear independence of $\mathbf{p}_k$ and $\mathbf{p}_k^*$ implies $\tilde{\mathbf{h}}_k \neq \mathbf{0}$. □

**Lemma 19.** *Assume* (b) *and the existence of* $\nu_k$ (*resp.* $\nu_k^*$) *and* $\tilde{\mathbf{h}}_k \neq \mathbf{0}$ (*resp.* $\tilde{\mathbf{h}}_k^* \neq \mathbf{0}$) *for some index* $k = 0, \ldots, n-1$. *Then, orthocenters exist only at most two points and they are antipodal each other if there exist two orthocenters.*

*Proof.* From the existence of $\mu_k^*$, there exist pair of distinct indices $i$, $j \neq k$ such that $\Delta^* \begin{pmatrix} i \\ j \end{pmatrix} \neq 0$. We have $\mathbf{p}_i \neq \mathbf{p}_i^*$ (resp. $\mathbf{p}_j \neq \mathbf{p}_j^*$), so, there exists the unique great circular $C_i$ (resp. $C_j$) passing through $\mathbf{p}_i$ and $\mathbf{p}_i^*$ (resp. $\mathbf{p}_j$ and $\mathbf{p}_j^*$). If there exist orthocenters, they are in the intersection of $C_i$ and $C_j$, so, it is enough to show $C_i \neq C_j$. So, assume $C_i = C_j$. Then, $\{\mathbf{p}_i, \mathbf{p}_j\}$ is a basis of the 2-dim Euclidean space $L$ spanned by $C_i$, so $\mathbf{p}_k^*$ is perpendicular to $L$. $\mathbf{p}_i^*$ and $\mathbf{p}_j^*$ are in $L$, so we have

$$\Delta^* \begin{pmatrix} i \\ k \end{pmatrix} = 0 = \Delta^* \begin{pmatrix} j \\ k \end{pmatrix},$$

which implies $\mu_k^* = 0$. From

$$\mathbf{0} \neq \tilde{\mathbf{h}}_k = \mathbf{p}_k^* - \frac{1 - \mu_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*} \mathbf{p}_k = \mathbf{p}_k^* - \frac{\mathbf{p}_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*},$$

we also have $\mathbf{p}_k \neq \mathbf{p}_k^*$. Hence there exists an index $\ell \neq k$ such that $\Delta^* \begin{pmatrix} \ell \\ k \end{pmatrix} \neq 0$. Inequalities $i \neq \ell \neq j$ implies

$$0 \neq \Delta^* \begin{pmatrix} i \\ j \end{pmatrix} \Delta^* \begin{pmatrix} \ell \\ k \end{pmatrix} = \Delta^* \begin{pmatrix} i \\ k \end{pmatrix} \Delta^* \begin{pmatrix} \ell \\ j \end{pmatrix} = 0 \cdot \Delta^* \begin{pmatrix} \ell \\ j \end{pmatrix},$$

which is a contradiction. □

**Lemma 20.** *Assume* (b) *and the existence of* $\nu_k$ (*resp.* $\nu_k^*$) *and* $\tilde{\mathbf{h}}_k \neq \mathbf{0}$ (*resp.* $\tilde{\mathbf{h}}_k^* \neq \mathbf{0}$( *for some index* $k = 0, \ldots, n-1$. *Then*

$$\pm\, \mathbf{h}_k \qquad (\textit{resp.}\ \ \pm\, \mathbf{h}_k^*)$$

*are an pair of orthocenters.*

*Proof.* It is enough to show that either $\mathbf{p}_i = \mathbf{p}_i^*$ holds or $\tilde{\mathbf{h}}_k$ is a linear combination of $\mathbf{p}_i$ and $\mathbf{p}_i^*$ for each index $i \neq k$. The proof is divided into 4 cases. First, suppose

$\Delta^* \begin{pmatrix} i \\ j \end{pmatrix} \neq 0$ for some index $j \neq i$, $k$ and the existence of $\mu_i^*$. Then we have

$$\tilde{\mathbf{h}}_k = \frac{\Delta^* \begin{pmatrix} k \\ j \end{pmatrix}}{\Delta^* \begin{pmatrix} i \\ j \end{pmatrix}} \tilde{\mathbf{h}}_i = \frac{\Delta^* \begin{pmatrix} k \\ j \end{pmatrix}}{\Delta^* \begin{pmatrix} i \\ j \end{pmatrix}} \left( \mathbf{p}_i^* - \frac{1 - \mu_i^*}{\mathbf{p}_i \cdot \mathbf{p}_i^*} \mathbf{p}_i \right).$$

Secondly, suppose $\Delta^* \begin{pmatrix} i \\ j \end{pmatrix} \neq 0$ for some index $j \neq i$, $k$ and the nonentity of $\mu_i^*$. Then the set $\{\mathbf{p}_0^*, \ldots, \mathbf{p}_{i-1}^*, \mathbf{p}_i, \mathbf{p}_{i+1}^*, \ldots, \mathbf{p}_{n-1}^*\}$ is an orthonormal basis, so, we have

$$\mu_k^* = \frac{\Delta^* \begin{pmatrix} i \\ k \end{pmatrix} \Delta^* \begin{pmatrix} k \\ j \end{pmatrix}}{\Delta^* \begin{pmatrix} i \\ j \end{pmatrix}} = \frac{\Delta^* \begin{pmatrix} i \\ k \end{pmatrix} \cdot 0}{\Delta^* \begin{pmatrix} i \\ j \end{pmatrix}},$$

$$\mathbf{p}_k = (\mathbf{p}_k \cdot \mathbf{p}_k^*)\mathbf{p}_k^* + \Delta \begin{pmatrix} k \\ i \end{pmatrix} \mathbf{p}_i,$$

hence, we also have

$$\tilde{\mathbf{h}}_k = \mathbf{p}_k^* - \frac{1 - \mu_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*} \mathbf{p}_k = \mathbf{p}_k^* - \frac{\mathbf{p}_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*} = -\frac{\Delta \begin{pmatrix} k \\ i \end{pmatrix}}{\mathbf{p}_k \cdot \mathbf{p}_k^*} \mathbf{p}_i.$$

Thirdly, suppose $\Delta^* \begin{pmatrix} i \\ j' \end{pmatrix} = 0$ for an arbitrary index $j' \neq i$, $k$ and $\Delta^* \begin{pmatrix} k \\ j \end{pmatrix} \neq 0$ for some index $j \neq i$, $k$. Then there exists $\mu_i^*$ and

$$\tilde{\mathbf{h}}_i = \frac{\Delta^* \begin{pmatrix} i \\ j \end{pmatrix}}{\Delta^* \begin{pmatrix} k \\ j \end{pmatrix}} \tilde{\mathbf{h}}_k = \frac{0}{\Delta^* \begin{pmatrix} k \\ j \end{pmatrix}} \tilde{\mathbf{h}}_k,$$

so, $\mathbf{p}_i = \mathbf{p}_i^*$. At last, suppose $\Delta^* \begin{pmatrix} i \\ j' \end{pmatrix} = 0 = \Delta^* \begin{pmatrix} k \\ j' \end{pmatrix}$ for an arbitrary index $j' \neq i$, $k$. Then vectors $\mathbf{p}_k$, $\mathbf{p}_k^*$, $\mathbf{p}_i$, $\mathbf{p}_i^*$ are in a 2-dim Euclidean space, so, either $\mathbf{p}_i = \mathbf{p}_i^*$ holds, or $\tilde{\mathbf{h}}_k$, a linear combination of $\mathbf{p}_k$ and $\mathbf{p}_k^*$, is a linear combination of $\mathbf{p}_i$ and $\mathbf{p}_i^*$. □

Now, we can prove the conclusions of this paper.

*Proof of Theorems 1 and 2.* The equivalences (b) ⇔ (b)* and (d) ⇔ (d)* are already shown by Lemma 12 and Corollary 13, respectively. And the implications (a) ⇒ (b) and (d) ⇒ (c) are shown by Lemmas 3 and 4, and Lemmas 18, 19, and 20, respectively. So we only have to show the converses (b) ⇒ (a) and

¬(d) ⇒ ¬(c), where ¬ means the negation. To show them, because of the implications (c) ⇒ (a) and ¬(b) ⇒ ¬(a), it is enough to show (b) ∧ ¬(d) ⇒ (a) ∧ ¬(c), which is already mentioned in Remark 6. □

## 6. Appendix.

For an orthocentric simplex, by repeating to replace a vertex with one of its orthocenters, new vertex does not appear except original vertices, their antipodals, and orthocenters. To prove it, we consider a generic case, that is, we assume that neither $\Delta\begin{pmatrix} i \\ j \end{pmatrix}$, $\Delta\begin{pmatrix} k & i \\ & j \end{pmatrix}$, $\Delta^*\begin{pmatrix} i \\ j \end{pmatrix}$, nor $\Delta^*\begin{pmatrix} k & i \\ & j \end{pmatrix}$ is equal to $0$ for pairwise distinct $i$, $j$, $k = 0, \ldots, n-1$, for an orthocentric simplex $S$. Then, for $k = 0, \ldots, n-1$, all of $\mu_k$, $\nu_k$, $\mu_k^*$, $\nu_k^*$ exist, neither of them is equal to $0$, and neither of them is equal to $1$. The simplex replacing the last vertex with the orthocenter $\mathbf{h}_{n-1}$ is denoted by $\mathcal{M}(S)$, that is, vertices of $S' = \mathcal{M}(S)$ are

$$\mathbf{p}_k'^* = \mathbf{p}_k^*, \qquad \mathbf{p}_{n-1}'^* = \mathbf{h}_{n-1}, \tag{13}$$

and vertices of $S'^*$ are

$$\mathbf{p}_k' = \frac{\mathbf{p}_k - \dfrac{\Delta\begin{pmatrix} k \\ n-1 \end{pmatrix}}{\mu_{n-1}}\mathbf{p}_{n-1}}{\sqrt{-\mu_k\left(\dfrac{1}{\nu_k} + \dfrac{1}{\nu_{n-1}}\right)}}, \qquad \mathbf{p}_{n-1}' = \operatorname{sgn}\nu_{n-1} \cdot \mathbf{p}_{n-1}, \tag{14}$$

for $k = 0, \ldots, n-2$. Notice that $S'$ is also orthocentric (see Lemma 3). Then, we have

$$\begin{aligned} \nu_k' &= \frac{1}{1 - \dfrac{1}{\mu_k'}} = \frac{1}{1 - \dfrac{\Delta'\begin{pmatrix} i \\ j \end{pmatrix}}{\Delta'\begin{pmatrix} i \\ k \end{pmatrix}\Delta'\begin{pmatrix} k \\ j \end{pmatrix}}} = \frac{1}{1 - \dfrac{\Delta\begin{pmatrix} i \\ j \end{pmatrix}\nu_{n-1}\mu_k\left(\dfrac{1}{\nu_k} + \dfrac{1}{\nu_{n-1}}\right)}{\Delta\begin{pmatrix} i \\ k \end{pmatrix}\Delta\begin{pmatrix} k \\ j \end{pmatrix}}} \\ &= \frac{1}{1 - \nu_{n-1}\left(\dfrac{1}{\nu_k} + \dfrac{1}{\nu_{n-1}}\right)} = -\frac{\nu_k}{\nu_{n-1}}, \end{aligned} \tag{15}$$

for $k = 0, \ldots, n-2$, where the third equality is from

$$\Delta'\begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{p}_i' \cdot \mathbf{p}_j' = \frac{\Delta\begin{pmatrix} i \\ j \end{pmatrix} - \dfrac{2\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}\Delta\begin{pmatrix} j \\ n-1 \end{pmatrix}}{\mu_{n-1}} + \dfrac{\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}\Delta\begin{pmatrix} j \\ n-1 \end{pmatrix}}{\mu_{n-1}^2}}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i} + \dfrac{1}{\nu_{n-1}}\right)}\sqrt{-\mu_j\left(\dfrac{1}{\nu_j} + \dfrac{1}{\nu_{n-1}}\right)}}$$

$$= \frac{\Delta\binom{i}{j}(1-2+\dfrac{1}{\mu_{n-1}})}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i}+\dfrac{1}{\nu_{n-1}}\right)}\sqrt{-\mu_j\left(\dfrac{1}{\nu_j}+\dfrac{1}{\nu_{n-1}}\right)}}$$

$$= \frac{-\Delta\binom{i}{j}/\nu_{n-1}}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i}+\dfrac{1}{\nu_{n-1}}\right)}\sqrt{-\mu_j\left(\dfrac{1}{\nu_j}+\dfrac{1}{\nu_{n-1}}\right)}}, \tag{16}$$

for an arbitrary pair of distinct $i$, $j = 0, \ldots, n-2$. We also have

$$\nu'_{n-1} = \frac{1}{1-\dfrac{1}{\mu'_{n-1}}} = \frac{1}{1-\dfrac{\Delta'\binom{i}{j}}{\Delta'\binom{i}{n-1}\Delta'\binom{n-1}{j}}} = \frac{1}{1+\dfrac{\Delta\binom{i}{j}\nu_{n-1}}{\Delta\binom{i}{n-1}\Delta\binom{n-1}{j}}}$$

$$= \frac{1}{1+\dfrac{\nu_{n-1}}{\mu_{n-1}}} = \frac{1}{\nu_{n-1}}, \tag{17}$$

where the third equality is from (16) and

$$\Delta'\binom{i}{n-1} = \operatorname{sgn}\nu_{n-1}\cdot\frac{\Delta\binom{i}{n-1} - \dfrac{\Delta\binom{i}{n-1}}{\mu_{n-1}}}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i}+\dfrac{1}{\nu_{n-1}}\right)}} = \operatorname{sgn}\nu_{n-1}\cdot\frac{\Delta\binom{i}{n-1}/\nu_{n-1}}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i}+\dfrac{1}{\nu_{n-1}}\right)}},$$

for arbitrary $i = 0, \ldots, n-2$. For $k = 0, \ldots, n-2$, it is obvious that

$$\nu'^{*}_k = \nu^{*}_k, \tag{18}$$

because

$$\Delta'^{*}\binom{i}{j} = \mathbf{p}'^{*}_i \cdot \mathbf{p}'^{*}_j = \mathbf{p}^{*}_i \cdot \mathbf{p}^{*}_j = \Delta^{*}\binom{i}{j},$$

for an arbitrary pair of distinct $i$, $j = 0, \ldots, n-2$. To calculate $\nu'^{*}_{n-1}$, we need preparation. From (8), for $m = 0, \ldots, n-2$ and distinct $i$, $j = m, \ldots, n-1$, we can prove

$$\Delta\left(0\cdots m-1\,{}^{i}_{j}\right) = \frac{\Delta\binom{i}{j}}{(1-\nu_0)\cdots(1-\nu_{m-1})} \tag{19}$$

$$\left(\text{resp.}\quad \Delta^{*}\left(0\cdots m-1\,{}^{i}_{j}\right) = \frac{\Delta^{*}\binom{i}{j}}{(1-\nu^{*}_0)\cdots(1-\nu^{*}_{m-1})}\right), \tag{19}^{*}$$

by induction of $m$. From (7) and (19), for $m = 0, \ldots, n-1$, we can also prove

$$\Delta\,(0\cdots m) = \frac{1-\nu_0-\cdots-\nu_m}{(1-\nu_0)\cdots(1-\nu_m)} \tag{20}$$

$$\Big(\text{resp.}\quad \Delta^*\,(0\cdots m) = \frac{1-\nu_0^*-\cdots-\nu_m^*}{(1-\nu_0^*)\cdots(1-\nu_m^*)}\Big), \tag{20$^*$}$$

by induction. Hence, we have

$$(\mathbf{p}_{n-1}\cdot\mathbf{p}_{n-1}^*)^2 = \frac{\Delta^*\,(0\cdots n-1)}{\Delta^*\,(0\cdots n-2)} = \frac{1-\nu_0^*-\cdots-\nu_{n-2}^*-\nu_{n-1}^*}{(1-\nu_0^*-\cdots-\nu_{n-2}^*)(1-\nu_{n-1}^*)}, \tag{21}$$

and

$$\begin{aligned}|\tilde{\mathbf{h}}_{n-1}|^2 &= \left|\mathbf{p}_{n-1}^* - \frac{\mathbf{p}_{n-1}}{(1-\nu_{n-1}^*)\mathbf{p}_{n-1}\cdot\mathbf{p}_{n-1}^*}\right|^2\\ &= 1-\frac{2}{1-\nu_{n-1}^*}+\frac{1}{(1-\nu_{n-1}^*)^2(\mathbf{p}_{n-1}\cdot\mathbf{p}_{n-1}^*)^2}\\ &= \frac{\nu_{n-1}^*(\nu_0^*+\cdots+\nu_{n-2}^*+\nu_{n-1}^*)}{(1-\nu_{n-1}^*)(1-\nu_0^*-\cdots-\nu_{n-2}^*-\nu_{n-1}^*)}.\end{aligned} \tag{22}$$

Now, we can calculate $\nu_{n-1}'^*$:

$$\nu_{n-1}'^* = \frac{1}{1-\dfrac{1}{\mu_{n-1}'^*}} = \frac{1}{1-\dfrac{\Delta'^*\begin{pmatrix} i\\ j\end{pmatrix}}{\Delta'^*\begin{pmatrix} i\\ n-1\end{pmatrix}\Delta'^*\begin{pmatrix} j\\ n-1\end{pmatrix}}} = \frac{1}{1-\dfrac{\Delta^*\begin{pmatrix} i\\ j\end{pmatrix}|\tilde{\mathbf{h}}_{n-1}|^2}{\Delta^*\begin{pmatrix} i\\ n-1\end{pmatrix}\Delta^*\begin{pmatrix} j\\ n-1\end{pmatrix}}}$$

$$= \frac{1}{1-\dfrac{|\tilde{\mathbf{h}}_{n-1}|^2}{\mu_{n-1}^*}} = 1-\nu_0^*-\cdots-\nu_{n-2}^*-\nu_{n-1}^*, \tag{23}$$

where the third equality is from $\Delta'^*\begin{pmatrix} i\\ j\end{pmatrix} = \Delta^*\begin{pmatrix} i\\ j\end{pmatrix}$ and

$$\Delta'^*\begin{pmatrix} i\\ n-1\end{pmatrix} = \mathbf{p}_i'^*\cdot\mathbf{p}_{n-1}'^* = \frac{\mathbf{p}_i^*\cdot\tilde{\mathbf{h}}_{n-1}}{|\tilde{\mathbf{h}}_{n-1}|} = \frac{\mathbf{p}_i^*\cdot\mathbf{p}_{n-1}^*}{|\tilde{\mathbf{h}}_{n-1}|} = \frac{\Delta^*\begin{pmatrix} i\\ n-1\end{pmatrix}}{|\tilde{\mathbf{h}}_{n-1}|},$$

for an arbitrary pair of distinct $i, j = 0,\ldots, n-2$. To get the purpose of this section, it is enough to consider two following cases. In the case where $S' = \mathcal{M}(S)$ and $S'' = \mathcal{M}(S')$ holds, $S''$ is consistent with $S$. In particular, we have

$$\mathbf{p}_k''^* = \mathbf{p}_k'^* = \mathbf{p}_k^*,$$

for $k = 0,\ldots, n-2$, and

$$\mathbf{p}_{n-1}''^* = \mathbf{h}_{n-1}' = \mathbf{p}_{n-1}^*,$$

where the last equality is from

$$\begin{aligned}
\tilde{\mathbf{h}}'_{n-1} &= \mathbf{p}'^{*}_{n-1} - (1-\nu'_{n-1})(\mathbf{p}'_{n-1}\cdot\mathbf{p}'^{*}_{n-1})\mathbf{p}'_{n-1} \\
&= \frac{\tilde{\mathbf{h}}_{n-1} - \left(1-\dfrac{1}{\nu_{n-1}}\right)(\mathbf{p}_{n-1}\cdot\tilde{\mathbf{h}}_{n-1})\mathbf{p}_{n-1}}{|\tilde{\mathbf{h}}_{n-1}|} \\
&= \frac{\left(\mathbf{p}^{*}_{n-1} - (1-\nu_{n-1})\,(\mathbf{p}_{n-1}\cdot\mathbf{p}^{*}_{n-1})\mathbf{p}_{n-1}\right) - \left(1-\dfrac{1}{\nu_{n-1}}\right)\nu_{n-1}(\mathbf{p}_{n-1}\cdot\mathbf{p}^{*}_{n-1})\mathbf{p}_{n-1}}{|\tilde{\mathbf{h}}_{n-1}|} \\
&= \frac{\mathbf{p}^{*}_{n-1}}{|\tilde{\mathbf{h}}_{n-1}|}.
\end{aligned}$$

On the other hand, in the case where $S' = \mathcal{M}(S)$ holds, $S''$ is the simplex replacing last two vertices of $S'$ each other, i.e.,

$$\mathbf{p}''^{*}_{k} = \mathbf{p}'^{*}_{k} \quad \text{for } k = 0,\ldots,n-3, \qquad \mathbf{p}''^{*}_{n-2} = \mathbf{p}'^{*}_{n-1}, \qquad \mathbf{p}''^{*}_{n-1} = \mathbf{p}'^{*}_{n-2},$$

and $S''' = \mathcal{M}(S'')$ holds, $S'''$ is the simplex with vertices

$$\mathbf{p}'''^{*}_{k} = \mathbf{p}''^{*}_{k} = \mathbf{p}'^{*}_{k} = \mathbf{p}^{*}_{k},$$

for $k = 0,\ldots, n-3$,

$$\mathbf{p}'''^{*}_{n-2} = \mathbf{p}''^{*}_{n-2} = \mathbf{p}'^{*}_{n-1} = \mathbf{h}_{n-1},$$

and

$$\mathbf{p}'''^{*}_{n-1} = \mathbf{h}''_{n-1} = \operatorname{sgn}\left(\frac{-\Delta\begin{pmatrix} n-2 \\ n-1\end{pmatrix}}{(1-\nu^{*}_{n-2})\nu_{n-1}}\right)\mathbf{p}^{*}_{n-1},$$

where the last equality is from

$$\begin{aligned}
\tilde{\mathbf{h}}''_{n-1} &= \mathbf{p}''^{*}_{n-1} - \frac{\mathbf{p}''_{n-1}}{(1-\nu'''^{*}_{n-1})\mathbf{p}''_{n-1}\cdot\mathbf{p}''^{*}_{n-1}} \\
&= \mathbf{p}'^{*}_{n-2} - \frac{\mathbf{p}'_{n-2}}{(1-\nu'^{*}_{n-2})\mathbf{p}'_{n-2}\cdot\mathbf{p}'^{*}_{n-2}} \\
&= \mathbf{p}^{*}_{n-2} - \frac{\mathbf{p}_{n-2} + \dfrac{1-\nu_{n-1}}{\nu_{n-1}}\Delta\begin{pmatrix} n-2 \\ n-1\end{pmatrix}\mathbf{p}_{n-1}}{(1-\nu^{*}_{n-2})\left(\mathbf{p}_{n-2} + \dfrac{1-\nu_{n-1}}{\nu_{n-1}}\Delta\begin{pmatrix} n-2 \\ n-1\end{pmatrix}\mathbf{p}_{n-1}\right)\cdot\mathbf{p}^{*}_{n-2}} \\
&= \mathbf{p}^{*}_{n-2} - \frac{\mathbf{p}_{n-2} + \dfrac{1-\nu_{n-1}}{\nu_{n-1}}\Delta\begin{pmatrix} n-2 \\ n-1\end{pmatrix}\mathbf{p}_{n-1}}{(1-\nu^{*}_{n-2})\mathbf{p}_{n-2}\cdot\mathbf{p}^{*}_{n-2}} \\
&= -\frac{\tilde{\mathbf{h}}^{*}_{n-2} + \dfrac{1-\nu_{n-1}}{\nu_{n-1}}\Delta\begin{pmatrix} n-2 \\ n-1\end{pmatrix}\mathbf{p}_{n-1}}{(1-\nu^{*}_{n-2})\mathbf{p}_{n-2}\cdot\mathbf{p}^{*}_{n-2}}
\end{aligned}$$

$$= -\frac{\dfrac{\Delta\dbinom{n-2}{n-1}}{\nu_{n-1}}\left(\dfrac{\tilde{\mathbf{h}}_{n-1}}{\mathbf{p}_{n-1}\cdot\mathbf{p}^*_{n-1}}+(1-\nu_{n-1})\mathbf{p}_{n-1}\right)}{(1-\nu^*_{n-2})\mathbf{p}_{n-2}\cdot\mathbf{p}^*_{n-2}}$$

$$= -\frac{\dfrac{\Delta\dbinom{n-2}{n-1}}{\nu_{n-1}}\cdot\dfrac{\mathbf{p}^*_{n-1}}{\mathbf{p}_{n-1}\cdot\mathbf{p}^*_{n-1}}}{(1-\nu^*_{n-2})\mathbf{p}_{n-2}\cdot\mathbf{p}^*_{n-2}}.$$

From two cases above, if we denote the orthocenters of $S$ by $\pm\mathbf{p}^*_n$, then orthocenters of the simplex with vertices $\mathbf{p}^*_0,\ldots,\widehat{\mathbf{p}^*_k},\ldots,\mathbf{p}^*_n$ are $\pm\mathbf{p}^*_k$, for $k=0,\ldots,n-1$ (notice that neither $\mu_i$, $\nu_i$, $\mu^*_i$, nor $\nu^*_i$ changes if a vertex is replaced with its antipodal for arbitrary $i=0,\ldots,n-1$). Moreover, for $k=0,\ldots,n$, from (18), $\nu^*_k$ does not change if $S$ is replaced with the simplex with vertices $\mathbf{p}^*_0,\ldots,\widehat{\mathbf{p}^*_i},\ldots,\mathbf{p}^*_n$ for some $i=0,\ldots,\widehat{k},\ldots,n-1$. If we denote $\nu'^*_{n-1}$ for $S'=\mathcal{M}(S)$ by $\nu^*_n$, we have

$$\nu^*_0+\cdots+\nu^*_{n-1}+\nu^*_n=1, \tag{24}$$

from (23) (see the equations (2) of [1] and (4) of [2]).

## References


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