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Three Synthetic Proofs of the Butterfly Theorem

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Abstract. We give three synthetic proofs of the butterfly theorem, using Thales' theorem, the notions of isogonal conjugates, and spiral similarity respectively.

1. Introduction

Theorem (The Butterfly Theorem). *Through the midpoint M of a chord PQ of a circle, two other chords AB and CD are drawn such that A and D are on opposite sides of PQ . If AD and BC intersect PQ at X and Y respectively, then M is also the midpoint of XY.*

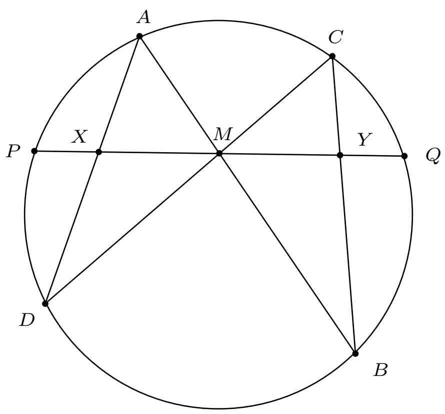

Figure 1.

2. The first proof: Thales' theorem

In Figure 2, the circumcircle of the triangle BOD intersects AB and CD again at E and F respectively, where O is the circumcenter of the cyclic quadrilateral $ACBD$. Since $\angle MFE = \angle DBE = \angle DCA$, $EF \parallel AC$. Let G be a point on OM such that $AG \parallel OE$. By Thales' Theorem, we have $\dfrac{MG}{MO} = \dfrac{MA}{ME} = \dfrac{MC}{MF}$. It follows that $CG \parallel OF$. Notice that $\angle GCB = \angle GCD + \angle DCB = \angle MFO + \angle DCB = \angle OBD + \angle DCB = 90^\circ$. Similarly, $\angle GAD = 90^\circ$. Since $GM \perp XY$, GX and GY are diameters of the circles (MAX) and (MCY) respectively. Therefore, $\angle MGX = \angle MAX = \angle MCY = \angle MGY$. The triangles MGX and MGY are congruent, and $MX = MY$. This proves the Butterfly Theorem.

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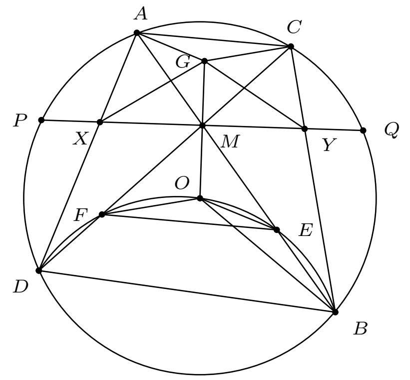

Figure 2.

3. The second proof: isogonal conjugates

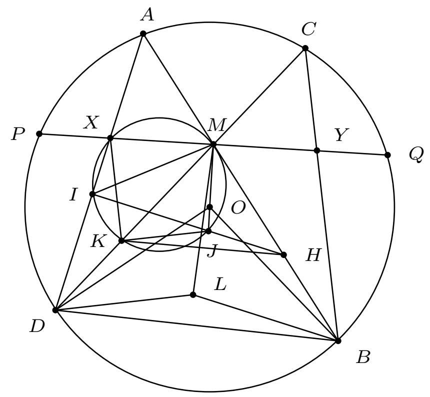

Figure 3.

Let O be the center of the circle. Denote by H and K the reflections of A and C through M. Since $MH \cdot MB = AM \cdot MB = CM \cdot MD = MK \cdot MD$, triangles MBD and MKH are oppositely similar (see Figure 3). Let L be the isogonal conjugate of O with respect to the triangle MBD. Denote by J the image of L under the dilative reflection that maps the triangle MBD onto the triangle MKH. HJ intersects AD at I. Since $\angle HMJ = \angle LMD = \angle HMO$, MJ passes through O. We have $\angle MHJ = \angle LDM = \angle BDO = \angle OBD = \angle MBL = \angle JKM = 90° - \angle DAB$. From this, together with the fact that M is the midpoint of AH, it follows that $\angle XIJ = \angle XMJ = 90°$, and $\angle JIM = \angle JKM = 90° - \angle DAB$. Hence, points M, J, K, I, and X lie on a circle whose diameter is JX, and $\angle JKM + \angle DCB = 90°$. Therefore, $KJ \perp KX$ and $KJ \perp BC$. It implies that $KX \parallel BC$. Triangles MXK and MYC are congruent, and $MX = MY$. This completes the second proof of the Butterfly Theorem.

4. The third proof: spiral similarity

Lemma. *The diagonals of a quadrilateral $ACBD$ that is inscribed in a circle (O) intersect at R. Let S be the center of the spiral similarity that maps AC onto DB . Then, SR is the common angle bisector of the angles ASB and CSD, and passes through O.*

Proof. Since S is the center of the spiral similarity that maps AC onto DB, the triangles SAD and SCB are directly similar. Therefore, the triangles SAC and SDB are also directly similar. It means that S is the center of the spiral similarity that maps AD onto CB (see Figure 4).

Since the triangles RCB and RAD are similar, $\dfrac{RC}{RA} = \dfrac{CB}{AD} = \dfrac{SC}{SA}$. Similarly, $\dfrac{RA}{RD} = \dfrac{AC}{DB} = \dfrac{SA}{SD}$. Hence, $\dfrac{RC}{RD} = \dfrac{SC}{SD}$, and SR is the angle bisector of $\angle CSD$. Similarly, SR is also the angle bisector of $\angle ASB$. Denote by (a, b) the directed angle from the line a to the line b (See [5, pp. 11–15]). Since the quadrilateral $ACBD$ is cyclic, we have

$$\begin{aligned}
(SC,\ SD) &= (SC,\ SA) + (SA,\ SD) \\
&= (CB,\ AD) + (AC,\ DB) \\
&= (BC,\ BD) + (DB,\ DA) + (CA,\ CB) + (BC, BD) \\
&= 2(BC,\ BD) \\
&= (OC,\ OD).
\end{aligned}$$

It follows that S, C, D and O are concyclic. From $OC = OD$, SO is the bisector of angle CSD. We deduce that SR passes through O. $\square$

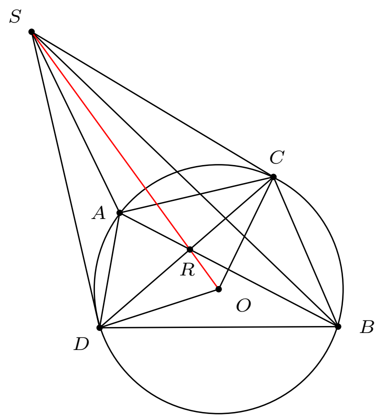

Figure 4

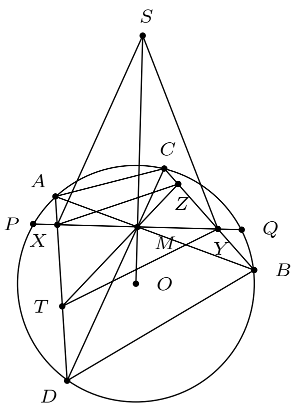

Figure 5

Proof of the butterfly theorem. Let Z and T be the points on the segments BC and AD respectively so that $\frac{ZC}{ZB} = \frac{XA}{XD}$ and $\frac{TA}{TD} = \frac{YC}{YB}$. As the triangles MAD and MCB are similar, $\angle CMZ = \angle AMX = \angle BMY = \angle DMT$ and $\angle MXT = \angle MZY$. Hence, ZT passes through M and the quadrilateral $XZYT$ is cyclic (see Figure 5). Let S be the center of the spiral similarity mapping AD onto CB. This spiral similarity maps X into Z, T into Y and XT onto ZY. Let (O) be the circumcircle of the cyclic quadrilateral $ACBD$. Then, by the Lemma, SM passes through O. Since M is the midpoint of PQ, $SM \perp XY$. Applying the Lemma to the cyclic quadrilateral $XZYT$, SM is the bisector of angle XSY. Combining with $SM \perp XY$, it follows that $MX = MY$ by symmetry. This completes the third proof of the Butterfly Theorem.

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