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# Radii of Circles in Apollonius' Problem

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**Abstract**. The paper presents the relation for radii of the eight circles in Apollonius' problem for circles which are tangent to three given circles. Analogously, we derived the relations for radii of the $16$ spheres which are tangent to four given spheres, with coordinates of their centers and with their radii.

## 1. Introduction

It is well known that for three given circles generically there are eight different circles that are tangent to them. The problem of ruler and compass constructability of these eight circles is well-known. There are famous Apollonius' and Gergonne's solutions to this problem [3]. Special cases of the three given circles are considered and a number of other problems is known [2]. The first case is when we consider three sides of the original triangle as three circles with infinite radii. The incircle and three excircles of the original triangle are four solutions to Apollonius' problem. The second case is when we have three excircles as a starting point. Three sides of the original triangle are three solutions to Apollonius' problem with infinite radii [1]. The nine-point circle is tangent externally to the three excircles, by Feuerbach theorem, and a relatively new object - the Apollonius circle is tangent internally to three exircles (for some results about this circle see [4]-[7]). To these five circles we can add three Jenkins circles which are tangent to three excircles, by adding two of them externally and the third one internally.

## 2. Main result

Let us assume that the three given circles are $K_1(O_1, r_1)$, $K_2(O_2, r_2)$, $K_3(O_3, r_3)$, Figure 1, with distances between centers $O_2O_3 = a$, $O_3O_1 = b$, $O_1O_2 = c$ and with the area $(O_1O_2O_3) = \Delta$, which is different from $0$. The following theorem holds true:

**Theorem 1.** *Let us assume that the radii of the eight circles with centers $S_i$ given in Figure 1* ((a), (b), (c) *and* (d)) *are $p_j$, ($j = 1, 2, ..., 8$). Then*

$$\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{p_4} + \frac{1}{p_5} - \frac{1}{p_6} + \frac{1}{p_7} - \frac{1}{p_8} = 0. \qquad (1)$$

*Proof.* Let us introduce the angles $\varphi_1 = \angle O_2SO_3$, $\varphi_2 = \angle O_3SO_1$, $\varphi_3 = \angle O_1SO_2$, where $S = S_1$ for Figure 1 (a), so as to obtain

$$\cos^2\varphi_1 + \cos^2\varphi_2 + \cos^2\varphi_3 - 2\cos\varphi_1\cos\varphi_2\cos\varphi_3 = 1. \qquad (2)$$

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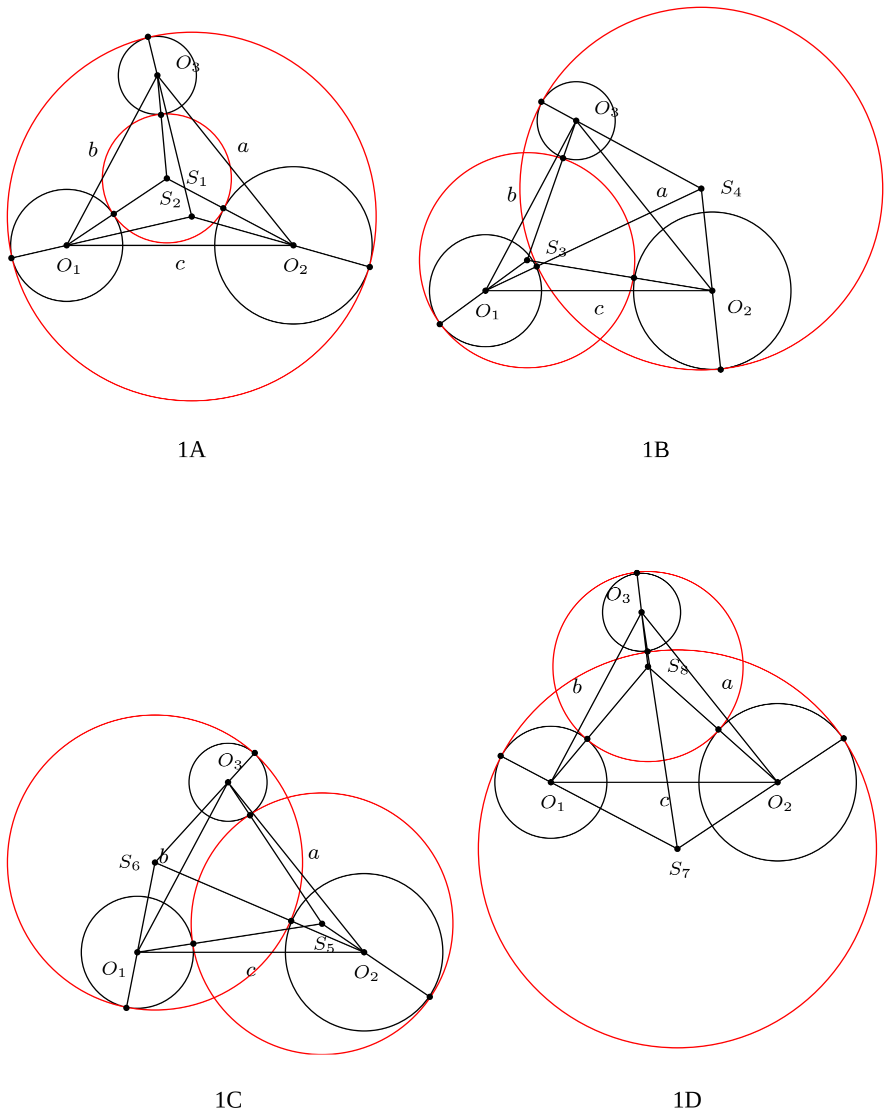


Figure 1. The eight different circles that are tangent to the three given circles.

If we substitute

$$t_1 = \sin^2 \frac{\varphi_1}{2}, \qquad t_2 = \sin^2 \frac{\varphi_2}{2}, \qquad t_3 = \sin^2 \frac{\varphi_3}{2},$$

from relation (2) we have

$$t_1^2 + t_2^2 + t_3^2 - 2(t_1 t_2 + t_2 t_3 + t_3 t_1) + 4 t_1 t_2 t_3 = 0. \tag{3}$$

If we generally denote the center of the circle and radius by $S$ and $p$, then for the first unknown circle $L_1$ (see Figure 1 (a)) is $SO_1 = p + r_1$, $SO_2 = p + r_2$,

$SO_3 = p + r_3$ and

$$\cos\varphi_1 = \frac{(p+r_2)^2 + (p+r_3)^2 - a^2}{2(p+r_2)(p+r_3)} \implies t_1 = \frac{a^2 - (r_2 - r_3)^2}{4(p+r_2)(p+r_3)},$$

and analogously

$$t_2 = \frac{b^2 - (r_3 - r_1)^2}{4(p+r_3)(p+r_1)}, \qquad t_3 = \frac{c^2 - (r_1 - r_2)^2}{4(p+r_1)(p+r_2)}.$$

From relation (3) we now have relation (4):

$$\begin{aligned}
&(a^2 - (r_2 - r_3)^2)^2(p+r_1)^2 + (b^2 - (r_3 - r_1)^2)^2(p+r_2)^2 \\
&+ (c^2 - (r_1 - r_2)^2)^2(p+r_3)^2 \\
&- 2(a^2 - (r_2 - r_3)^2)(b^2 - (r_3 - r_1)^2)(p+r_1)(p+r_2) \\
&- 2(a^2 - (r_2 - r_3)^2)(c^2 - (r_1 - r_2)^2)(p+r_1)(p+r_3) \\
&- 2(b^2 - (r_3 - r_1)^2)(c^2 - (r_1 - r_2)^2)(p+r_2)(p+r_3) \\
&+ (a^2 - (r_2 - r_3)^2)(b^2 - (r_3 - r_1)^2)(c^2 - (r_1 - r_2)^2) \\
&= 0. \qquad (4)
\end{aligned}$$

or in another form

$$F_1(p,\ r_1,\ r_2,\ r_3) = 0. \qquad (5)$$

Equation (4) is of the second degree and is of the form

$$A_1 p^2 + B_1 p + C_1 = 0, \qquad (6)$$

where

$$\begin{aligned}
A_1 =\ & 4(a^2(r_1 - r_2)(r_1 - r_3) + b^2(r_2 - r_3)(r_2 - r_1) + c^2(r_3 - r_1)(r_3 - r_2)) \\
& - 16\Delta^2 \\
=\ & f(r_1, r_2, r_3), \qquad (7) \\
B_1 =\ & 2\{r_1(a^2 - (r_2 - r_3)^2)^2 + r_2(b^2 - (r_3 - r_1)^2)^2 + r_3(c^2 - (r_1 - r_2)^2)^2 \\
& - (a^2 - (r_2 - r_3)^2)(b^2 - (r_3 - r_1)^2)(r_1 + r_2) \\
& - (b^2 - (r_3 - r_1)^2)(c^2 - (r_1 - r_2)^2)(r_2 + r_3) \\
& - (c^2 - (r_1 - r_2)^2)(a^2 - (r_2 - r_3)^2)(r_3 + r_1)\} \\
=\ & g(r_1, r_2, r_3), \qquad (8) \\
C_1 =\ & r_1^2 a^4 + r_2^2 b^4 + r_3^2 c^4 + a^2 b^2 c^2 \\
& - a^2 b^2 (r_1^2 + r_2^2) - b^2 c^2 (r_2^2 + r_3^2) - c^2 a^2 (r_3^2 + r_1^2) \\
& + a^2 (r_1^2 - r_2^2)(r_1^2 - r_3^2) + b^2 (r_2^2 - r_3^2)(r_2^2 - r_1^2) \\
& + c^2 (r_3^2 - r_1^2)(r_3^2 - r_2^2) \\
=\ & h(r_1, r_2, r_3). \qquad (9)
\end{aligned}$$

For the second unknown circle $L_2$ (see Figure 1 (a)) we have $SO_1 = p - r_1$, $SO_2 = p-r_2$, $SO_3 = p-r_3$ and a corresponding equation in the form of equation (4), and

$$F_1(p,\ -r_1,\ -r_2,\ -r_3) = 0, \tag{10}$$

$$A_2p^2 + B_2p + C_2 = 0, \tag{11}$$

with

$$\begin{aligned} A_2 =&\ f_1(-r_1,\ -r_2,\ -r_3) = A_1, \\ B_2 =&\ g_1(-r_1,\ -r_2,\ -r_3) = B_1, \\ C_2 =&\ h_1(-r_1,\ -r_2,\ -r_3) = C_1, \end{aligned}$$

which implies that

$$A_1p_1^2 + B_1p_1 + C_1 = 0, \qquad A_1p_2^2 - B_1p_2 + C_1 = 0,$$

and

$$\frac{1}{p_1} - \frac{1}{p_2} = -\frac{B_1}{C_1}. \tag{12}$$

For the third circle $L_3$ (see Figure 1 (b)) we have $SO_1 = p-r_1$, $SO_2 = p+r_2$, $SO_3 = p + r_3$ and we get $F_1(p,\ -r_1,\ r_2,\ r_3) = 0$ with $A_3p^2 + B_3p + C_3 = 0$, $A_3 = f_1(-r_1,\ r_2,\ r_3)$, $B_3 = g_1(-r_1,\ r_2,\ r_3)$, $C_3 = h_1(-r_1\, r_2,\ r_3) = C_1$.

For the fourth circle $L_4$ (see Figure 1 (b)) we have $SO_1 = p+r_1$, $SO_2 = p-r_2$, $SO_3 = p - r_3$ and we get $F_1(p,\ r_1,\ -r_2,\ -r_3) = 0$ with $A_4p^2 + B_4p + C_4 = 0$, $A_4 = f_1(r_1,\ -r_2,\ -r_3) = A_3$, $B_4 = g_1(r_1,\ -r_2,\ -r_3) = -B_3$, $C_4 = h_1(r_1,-r_2,-r_3) = C_1$ and again we get

$$\frac{1}{p_3} - \frac{1}{p_4} = -\frac{B_3}{C_1}. \tag{13}$$

Analogously, we have

$$\frac{1}{p_5} - \frac{1}{p_6} = -\frac{B_5}{C_1}, \tag{14}$$

$$\frac{1}{p_7} - \frac{1}{p_8} = -\frac{B_7}{C_1}, \tag{15}$$

where

$$B_5 = g_1(r_1,\ -r_2,\ r_3), \qquad B_7 = g_1(r_1,\ r_2,-r_3).$$

Formula (1) follows from (12), (13), (14), (15) because

$$g_1(r_1,\ r_2,\ r_3) + g_1(-r_1,\ r_2,\ r_3) + g_1(r_1,\ -r_2,\ r_3) + g_1(r_1,\ r_2,\ -r_3) = 0.$$

□

*Remark* 1. If the index $j$ of the circle with radius $p_j$ is an even (odd) number, then $1/p_j$ (in formula (1)) has the sign $+$ ($-$).

*Remark* 2. In the first case of the three given circles mentioned in the introduction, we get the formula

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

where $r$, $r_1$, $r_2$, $r_3$ are the inradius and exradii of triangle $ABC$.

*Remark* 3. In the second case we get

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{q} = \frac{1}{m},$$

where $p_1$, $p_2$, $p_3$ are the radii of Jenkins circles, $q$ is the radius of Apollonius' circle and $m = R/2$ is the radius of Euler circle or nine-point circle.

This formula can be proved independently since

$$p_1 = \frac{a}{b+c}q, \quad p_2 = \frac{b}{c+a}q, \quad p_3 = \frac{c}{a+b}q.$$

*Remark* 4. In the same way, the same result can be proved for the three given circles, provided that two of them are inside of the third one.

## 3. Positions of $8$ circles

The radical circle of the three given circles $K_1(O_1, r_1)$, $K_2(O_2, r_2)$, $K_3(O_3, r_3)$, is the circle orthogonal to all of them, and pairs of circles $(L_1, L_2)$, $(L_3, L_4)$, $(L_5, L_6)$, $(L_7, L_8)$ – Figure 1, are inversive with respect to the radical circle. For this radical circle $K_0(S_0, r_0)$, Figure 2, the following holds true.

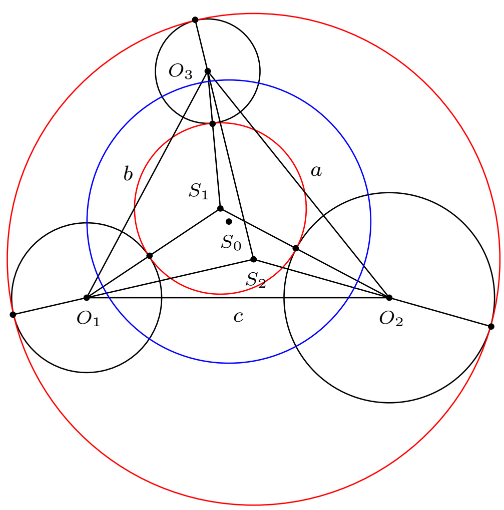


Figure 2. The radical circle of the three given circles with the circles $L_1$ and $L_2$, inversive to the radical circle, based on Figure 1 (a)

**Proposition 2.** (1) *The center $S_0(x_0 : y_0 : z_0)$ has barycentric coordinates with respect to triangle $O_1O_2O_3$*

$$\begin{aligned} x_0 = &\ a^2(b^2+c^2-a^2)+u_0, \\ y_0 = &\ b^2(c^2+a^2-b^2)+v_0, \\ z_0 = &\ c^2(a^2+b^2-c^2)+w_0, \end{aligned} \tag{16}$$

*where*

$$u_0 = (r_2^2+r_3^2-2r_1^2)a^2+(r^2-r_3^2)(b^2-c^2), \tag{17}$$

$$v_0 = (r_3^2+r_1^2-2r_2^2)b^2+(r^3-r_1^2)(c^2-a^2), \tag{18}$$

$$w_0 = (r_1^2+r_2^2-2r_3^2)a^2+(r^1-r_2^2)(a^2-b^2). \tag{19}$$

(2) *The radius of the radical circle is given by formula*

$$r_0^2 = \frac{C_1}{16\Delta^2}. \tag{20}$$

(3) *The coefficients $B_1$, $B_3$, $B_5$, $B_7$ are expressed in terms of the coordinates of $S_0$, i.e.,*

$$B_1 = -2(r_1x_0+r_2y_0+r_3z_0), \qquad B_3 = -2(r_1x_0-r_2y_0-r_3z_0), \tag{21}$$

$$B_5 = -2(-r_1x_0+r_2y_0-r_3z_0), \qquad B_7 = -2(-r_1x_0-r_2y_0+r_3z_0). \tag{22}$$

(4) *The line $S_0O_0$, where $O_0$ represents the circumcenter of triangle $O_1O_2O_3$, is orthogonal to the line $q:\ r_1^2x+r_2^2y+r_3^2z=0$. This line passes through the midpoints of segments $M_{11}M_{12}$, $M_{21}M_{22}$ and $M_{31}M_{32}$, where $M_{11}$ and $M_{12}$ are the inner and outer centers of similarity of circles $(K_2)$ and $(K_3)$, which can analogously be applied to the other points.*

For the eight solutions $L_j(S_j, p_j)$ of Apollonius' problem, with $S_j(x_j : y_j : z_j)$ we have

**Proposition 3.** (1) *The coordinates of the centers $S_j$ are as follows:*

$$\begin{aligned} x_1 &= 2p_1u_1+x_0, & y_1 &= 2p_1v_1+y_0, & z_1 &= 2p_1w_1+z_0, \\ x_2 &= -2p_2u_1+x_0, & y_2 &= -2p_2v_1+y_0, & z_2 &= -2p_2w_1+z_0, \end{aligned}$$

*where*

$$\begin{aligned} u_1 = &\ (r_2+r_3-2r_1)a^2+(r_2-r_3)(b^2-c^2) = u_1(r_1,r_2,r_3), \\ v_1 = &\ (r_3+r_1-2r_2)b^2+(r_3-r_1)(c^2-a^2) = v_1(r_1,r_2,r_3), \\ w_1 = &\ (r_1+r_2-2r_3)c^2+(r_1-r_2)(a^2-b^2) = w_1(r_1,r_2,r_3). \end{aligned}$$

$$\begin{aligned} x_3 &= 2p_3u_3+x_0, & y_3 &= 2p_3v_3+y_0, & z_3 &= 2p_3w_3+z_0, \\ x_4 &= -2p_4u_3+x_0, & y_4 &= -2p_4v_3+y_0, & z_4 &= -2p_4w_3+z_0, \end{aligned}$$

*where*

$$\begin{aligned} u_3(r_1,r_2,r_3) &= u_1(r_1,-r_2,-r_3), \\ v_3(r_1,r_2,r_3) &= v_1(r_1,-r_2,-r_3), \\ w_3(r_1,r_2,r_3) &= w_1(r_1,-r_2,-r_3). \end{aligned}$$

$$\begin{aligned} x_5 &= 2p_5u_5+x_0, & y_5 &= 2p_5v_5+y_0, & z_5 &= 2p_5w_5+z_0, \\ x_6 &= -2p_6u_5+x_0, & y_6 &= -2p_6v_5+y_0, & z_6 &= -2p_6w_5+z_0, \end{aligned}$$

*where*

$$u_5(r_1,r_2,r_3) = u_1(r_1,r_2,-r_3),$$
$$v_5(r_1,r_2,r_3) = v_1(r_1,r_2,-r_3),$$
$$w_5(r_1,r_2,r_3) = w_1(r_1,r_2,-r_3).$$

$$x_7 = 2p_7u_7 + x_0, \quad y_7 = 2p_7v_7 + y_0, \quad z_7 = 2p_7w_7 + z_0,$$
$$x_8 = -2p_8u_7 + x_0, \quad y_8 = -2p_8v_7 + y_0, \quad z_8 = -2p_8w_7 + z_0,$$

*where*

$$u_7(r_1,r_2,r_3) = u_1(r_1,-r_2,r_3),$$
$$v_7(r_1,r_2,r_3) = v_1(r_1,-r_2,r_3),$$
$$w_7(r_1,r_2,r_3) = w_1(r_1,-r_2,r_3).$$

(2) *There are collinear triplets of points* $(S_0,S_1,S_2)$, $(S_0,S_3,S_4)$, $(S_0,S_5,S_6)$ *and* $(S_0,S_7,S_8)$, *and*

$$S_0S_1 \perp q_1: \quad r_1x + r_2y + r_3z = 0,$$
$$S_0S_3 \perp q_3: \quad -r_1x + r_2y + r_3z = 0,$$
$$S_0S_5 \perp q_5: \quad r_1x - r_2y + r_3z = 0,$$
$$S_0S_7 \perp q_7: \quad r_1x + r_2y - r_3z = 0,$$

*where* $q_1$, $q_3$, $q_5$, $q_7$ *are the lines* $M_{12}M_{22}M_{32}$, $M_{12}M_{21}M_{31}$, $M_{11}M_{22}M_{31}$, *and* $M_{11}M_{21}M_{32}$ *respectively.*

(3)

$$\frac{1}{S_0S_1} - \frac{1}{S_2S_0} = \frac{2}{S_0V_1}, \tag{23}$$

*where* $U_1 = S_0S_1 \cap q_1$ *and* $V_1$ *and* $U_1$ *are inversive to each other with respect to the radical circle. Analogously, this is assumed for the other centers* $S_j$.

## 4. Three-dimensional case

In this case we have four spheres and a maximum of 16 spheres, each of which being tangent to all of the four given spheres. Analogous relations for radii of these 16 spheres will be found subsequently. Let us assume that the four spheres are $\Phi_1(O_1,r_1)$, $\Phi_2(O_2,r_2)$, $\Phi_3(O_3,r_3)$, $\Phi_4(O_4,r_4)$. We can take the basic tetrahedron ABCD to be tetrahedron $O_1O_2O_3O_4$ with $O_1 = A(1:0:0:0)$, $O_2 = B(0:1:0:0)$, $O_3 = C(0:0:1:0)$, $O_4 = D(0:0:0:1)$ given in the barycentric coordinates with mutual distances $AB = c$, $AC = b$, $AD = d$, $BC = a$, $BD = e$, $CD = f$. An important role in further investigations is played by Cayley-Menger determinant $\Delta_0$ of tetrahedron $ABCD$ given as follows:

$$\Delta_0 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & d^2 \\ 1 & c^2 & 0 & a^2 & e^2 \\ 1 & b^2 & a^2 & 0 & f^2 \\ 1 & d^2 & e^2 & f^2 & 0 \end{vmatrix}. \tag{24}$$

The known result is that the volume $V$ of tetrahedron $ABCD$ is given by formula

$$\Delta_0 = 288V^2.$$

If we apply $\Delta_{ij}$ to denote the algebraic cofactor of element with row-column position $(i, j)$ in corresponding Cayley-Menger matrix, we obtain the following result.

**Proposition 4.** (1) *The center $O$ of the circumscribed sphere of tetrahedron $ABCD$ (or circumcenter) has barycentric coordinates*

$$O(\Delta_{12} : \Delta_{13} : \Delta_{14} : \Delta_{15}). \tag{25}$$

(2) *The circumradius $R$ of the upper circumsphere is given by*

$$R^2 = -\frac{\Delta_{11}}{2\Delta_0}. \tag{26}$$

(3) *If for point $P(x : y : z : t)$ we introduce two relevant expressions*

$$\tau = \tau(P) = x + y + z + t, \tag{27}$$

$$T = T(P) = a^2yz + b^2zx + c^2xy + d^2xt + e^2yt + f^2zt, \tag{28}$$

*then we have*

$$\tau(O) = \Delta_0, \qquad T(O) = \frac{1}{2}\Delta_0 \cdot \Delta_{11}. \tag{29}$$

Let us now introduce the radical sphere $\Phi_0(S_0, r_0)$, i.e., the sphere with property

$$S_0A^2 - r_1^2 = S_0B^2 - r_2^2 = S_0C^2 - r_3^2 = S_0D^2 - r_4^2 = r_0^2, \tag{30}$$

or the sphere which is orthogonal to the four given spheres $\Phi_1$, $\Phi_2$, $\Phi_3$, $\Phi_4$. Then we have

**Proposition 5.** (1) *This radical sphere corresponds to the equation*

$$T = \tau(r_1^2x + r_2^2y + r_3^2z + r_4^2t). \tag{31}$$

(2) *The center $S_0(x_0 : y_0 : z_0 : t_0)$ has coordinates*

$$x_0 = \Delta_{12} + r_1^2\Delta_{22} + r_2^2\Delta_{32} + r_3^2\Delta_{42} + r_4^2\Delta_{52}, \tag{32}$$

$$y_0 = \Delta_{13} + r_1^2\Delta_{23} + r_2^2\Delta_{33} + r_3^2\Delta_{43} + r_4^2\Delta_{53}, \tag{33}$$

$$z_0 = \Delta_{14} + r_1^2\Delta_{24} + r_2^2\Delta_{34} + r_3^2\Delta_{44} + r_4^2\Delta_{54}, \tag{34}$$

$$t_0 = \Delta_{15} + r_1^2\Delta_{25} + r_2^2\Delta_{35} + r_3^2\Delta_{45} + r_4^2\Delta_{55}. \tag{35}$$

(3) *For the radius $r_0$, the following formula holds.*

$$r_0^2 = R^2 - \left(r_1^2x(M) + r_2^2y(M) + r_3^2z(M) + r_4^2t(M)\right), \tag{36}$$

*where $M$ is the midpoint of the segment $S_0O$.*

In Figure 3 we introduce corresponding ordered quadruplets. An appropriate number $j$ in illustration (from (a) to (p)) denotes that sphere $L_j$ is tangent to the four given spheres. The plus sign in the second position (given in brackets in each figure) means that sphere with center $B$ is outside-externally tangent to sphere $L_j$, while the minus sign at the fourth position means that sphere with center $D$ is inside sphere $L_j$ – internally tangent, and similarly in other cases. For each of the 16 possible layouts, corresponding signs are given immediately under the figure,

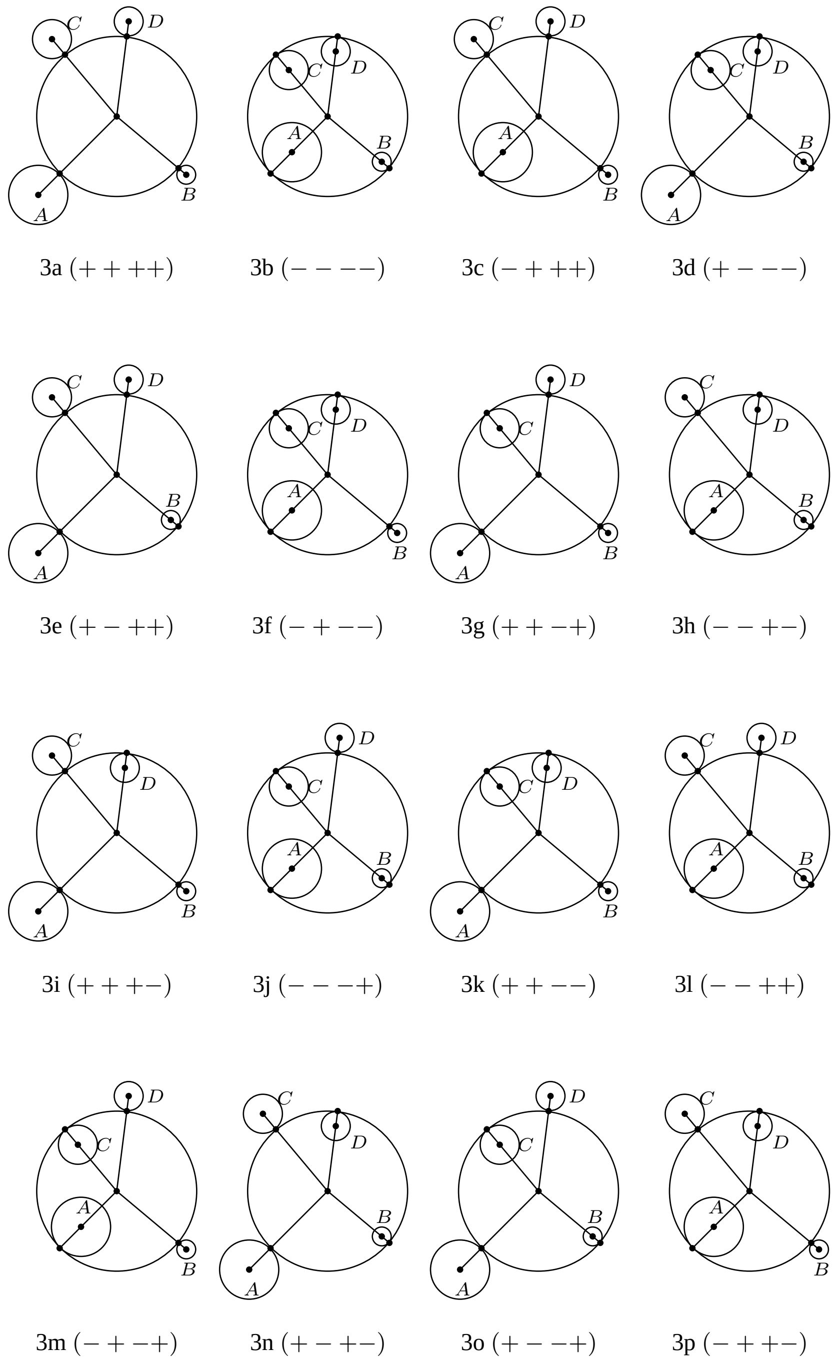


Figure 3. The 16 possible quadruplets- spheres each of which being tangent to all of the four given spheres

related to the respective spheres with centers $A$, $B$, $C$ and $D$, depending on their position in relation to sphere $L_j$.

First of all, we will investigate sphere $L_1$. If we generally denote the center of the sphere and the radius by S and p, then for the first unknown sphere $L_1$ (see Figure 3) is

$$SO_1 = p + r_1, \quad SO_2 = p + r_2, \quad SO_3 = p + r_3, \quad SO_4 = p + r_4,$$

and equations of spheres $\Phi_1'(A, p+r_1)$, $\Phi_2'(B, p+r_2)$, $\Phi_3'(C, p+r_3)$, $\Phi_4'(D, p+r_4)$ are

$$T = \tau\{-(p+r_1)^2x + (c^2 - (p+r_1)^2)y + (b^2 - (p+r_1)^2)z + (d^2 - (p+r_1)^2)t\}, \tag{37}$$

$$T = \tau\{(c^2 - (p+r_2)^2)x - (p+r_2)^2y + (a^2 - (p+r_2)^2)z + (e^2 - (p+r_2)^2)t\}, \tag{38}$$

$$T = \tau\{(b^2 - (p+r_3)^2)x + (a^2 - (p+r_3)^2)y - (p+r_3)^2z + (f^2 - (p+r_3)^2)t\}, \tag{39}$$

$$T = \tau\{(d^2 - (p+r_4)^2)x + (e^2 - (p+r_4)^2)y + (f^2 - (p+r_4)^2)z - (p+r_4)^2t\}. \tag{40}$$

These equations determine the point $S_1(x_1 : y_1 : z_1 : t_1)$ which is the center of the sphere $L_1$ and radius $p = p_1$ of that sphere. For them we have

$$x_1 = x_0 + 2p(r_1\Delta_{22} + r_2\Delta_{32} + r_3\Delta_{42} + r_4\Delta_{52}), \tag{41}$$

$$y_1 = y_0 + 2p(r_1\Delta_{23} + r_2\Delta_{33} + r_3\Delta_{43} + r_4\Delta_{53}), \tag{42}$$

$$z_1 = z_0 + 2p(r_1\Delta_{24} + r_2\Delta_{34} + r_3\Delta_{44} + r_4\Delta_{54}), \tag{43}$$

$$t_1 = t_0 + 2p(r_1\Delta_{25} + r_2\Delta_{35} + r_3\Delta_{45} + r_4\Delta_{55}), \tag{44}$$

and these formulae lead us to the equation for $p = p_1$

$$F_1(p,\ r_1,\ r_2,\ r_3,\ r_4) = A_1p^2 + B_1p + C_1 = 0, \tag{45}$$

where

$$\begin{aligned} A_1 = {} & 2r_1(r_1\Delta_{22} + r_2\Delta_{32} + r_3\Delta_{42} + r_4\Delta_{52}) \\ & + 2r_2(r_1\Delta_{23} + r_2\Delta_{33} + r_3\Delta_{43} + r_4\Delta_{53}) \\ & + 2r_3(r_1\Delta_{24} + r_2\Delta_{34} + r_3\Delta_{44} + r_4\Delta_{54}) \\ & + 2r_4(r_1\Delta_{25} + r_2\Delta_{35} + r_3\Delta_{45} + r_4\Delta_{55}) + \Delta_0 \\ = {} & f(r_1, r_2, r_3, r_4), \end{aligned} \tag{46}$$

$$\begin{aligned} B_1 = {} & 2r_1(\Delta_{12} + r_1^2\Delta_{22} + r_2^2\Delta_{32} + r_3^2\Delta_{42} + r_4^2\Delta_{52}) \\ & + 2r_2(\Delta_{13} + r_1^2\Delta_{23} + r_2^2\Delta_{33} + r_3^2\Delta_{43} + r_4^2\Delta_{53}) \\ & + 2r_3(\Delta_{14} + r_1^2\Delta_{24} + r_2^2\Delta_{34} + r_3^2\Delta_{44} + r_4^2\Delta_{54}) \\ & + 2r_4(\Delta_{15} + r_1^2\Delta_{25} + r_2^2\Delta_{35} + r_3^2\Delta_{45} + r_4^2\Delta_{55}) \\ = {} & 2(r_1x_0 + r_2y_0 + r_3z_0 + r_4t_0) \\ = {} & g(r_1, r_2, r_3, r_4), \end{aligned} \tag{47}$$

$$\begin{aligned} C_1 = \ & \frac{1}{2}\left\{r_1^2(\Delta_{12} + r_1^2\Delta_{22} + r_2^2\Delta_{32} + r_3^2\Delta_{42} + r_4^2\Delta_{52})\right. \\ & + r_2^2(\Delta_{13} + r_1^2\Delta_{23} + r_2^2\Delta_{33} + r_3^2\Delta_{43} + r_4^2\Delta_{53}) \\ & + r_3^2(\Delta_{14} + r_1^2\Delta_{24} + r_2^2\Delta_{34} + r_3^2\Delta_{44} + r_4^2\Delta_{54}) \\ & + r_4^2(\Delta_{15} + r_1^2\Delta_{25} + r_2^2\Delta_{35} + r_3^2\Delta_{45} + r_4^2\Delta_{55}) \\ & \left. + \ (\Delta_{11} + r_1^2\Delta_{12} + r_2^2\Delta_{13} + r_3^2\Delta_{14} + r_4^2\Delta_{15})\right\} \\ = \ & h(r_1, r_2, r_3, r_4), \end{aligned} \tag{48}$$

For the second unknown sphere $L_2$ (see Figure 3) we have $SO_1 = p - r_1$, $SO_2 = p - r_2$, $SO_3 = p - r_3$, $SO_4 = p - r_4$ as illustrated by the equation

$$F_1(p,\ -r_1,\ -r_2,\ -r_3,\ -r_4) \equiv A_2p^2 + B_2p + C_2 \equiv A_1p^2 - B_1p + C_1 = 0. \tag{49}$$

Now we get

$$\frac{1}{p_1} - \frac{1}{p_2} = -\frac{B_1}{C_1}. \tag{50}$$

This means that to the difference $1/p_1 - 1/p_2$ we can relate the ordered quadruple $(+,+,+,+)$ related to $(r_1, r_2, r_3, r_4)$ since $B_1$ is linear with respect to all $r_j$. Since $C_1$ is the same for all 16 combinations $(\varepsilon_1 r_1,\ \varepsilon_2 r_2,\ \varepsilon_3 r_3,\ \varepsilon_4 r_4)$ for all $\varepsilon \in \{-1,\ 1\}$. When combining the signs of the ordered quadruplets, we obtain the following results given in the next theorem.

**Theorem 6.** *Let us assume that the radii of the sixteen spheres given in Figure 3 are* $p_j$ $(j = 1, 2, ..., 16)$ *and the volume* $V$ *is different from* $0$. *Then*

$$\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_3} - \frac{1}{p_4}\right) - \left(\frac{1}{p_5} - \frac{1}{p_6}\right) - \left(\frac{1}{p_{11}} - \frac{1}{p_{12}}\right) = 0, \tag{51}$$

$$\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_3} - \frac{1}{p_4}\right) - \left(\frac{1}{p_7} - \frac{1}{p_8}\right) - \left(\frac{1}{p_{13}} - \frac{1}{p_{14}}\right) = 0, \tag{52}$$

$$\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_3} - \frac{1}{p_4}\right) - \left(\frac{1}{p_9} - \frac{1}{p_{10}}\right) - \left(\frac{1}{p_{15}} - \frac{1}{p_{16}}\right) = 0, \tag{53}$$

$$\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_5} - \frac{1}{p_6}\right) - \left(\frac{1}{p_7} - \frac{1}{p_8}\right) + \left(\frac{1}{p_{15}} - \frac{1}{p_{16}}\right) = 0, \tag{54}$$

$$\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_5} - \frac{1}{p_6}\right) - \left(\frac{1}{p_9} - \frac{1}{p_{10}}\right) + \left(\frac{1}{p_{13}} - \frac{1}{p_{14}}\right) = 0, \tag{55}$$

$$\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_7} - \frac{1}{p_8}\right) - \left(\frac{1}{p_9} - \frac{1}{p_{10}}\right) + \left(\frac{1}{p_{11}} - \frac{1}{p_{12}}\right) = 0, \tag{56}$$

$$\begin{aligned} & \left(\frac{1}{p_1} - \frac{1}{p_2}\right) + \left(\frac{1}{p_3} - \frac{1}{p_4}\right) - \left(\frac{1}{p_5} - \frac{1}{p_6}\right) - \left(\frac{1}{p_7} - \frac{1}{p_8}\right) \\ & - \left(\frac{1}{p_9} - \frac{1}{p_{10}}\right) + \left(\frac{1}{p_{11}} - \frac{1}{p_{12}}\right) + \left(\frac{1}{p_{13}} - \frac{1}{p_{14}}\right) + \left(\frac{1}{p_{15}} - \frac{1}{p_{16}}\right) = 0, \end{aligned} \tag{57}$$

$$\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\left(\frac{1}{p_3}-\frac{1}{p_4}\right)+\left(\frac{1}{p_5}-\frac{1}{p_6}\right)-\left(\frac{1}{p_7}-\frac{1}{p_8}\right) -\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)-\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)-\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{58}$$

$$\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_5}-\frac{1}{p_6}\right)+\left(\frac{1}{p_7}-\frac{1}{p_8}\right) -\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)-\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)+\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)-\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{59}$$

$$\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_5}-\frac{1}{p_6}\right)-\left(\frac{1}{p_7}-\frac{1}{p_8}\right) +\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)-\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)-\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)+\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0. \tag{60}$$

**Corollary 7.** *From the above formulae we can also obtain*

$$\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_5}-\frac{1}{p_6}\right)+\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)+\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{61}$$

$$\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_7}-\frac{1}{p_8}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)+\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{62}$$

$$\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)+\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)=0, \tag{63}$$

$$\left(\frac{1}{p_5}-\frac{1}{p_6}\right)-\left(\frac{1}{p_7}-\frac{1}{p_8}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)-\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)=0, \tag{64}$$

$$\left(\frac{1}{p_5}-\frac{1}{p_6}\right)-\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)-\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{65}$$

$$\left(\frac{1}{p_7}-\frac{1}{p_8}\right)-\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)+\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)-\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0. \tag{66}$$

These formulae and formulae listed in Theorem 6 are all possible formulae of this type.

**Corollary 8.** *For the radius* $r_0$ *of the radical circle,*

$$r_0^2=-\frac{C_1}{\Delta_0}. \tag{67}$$

*Proof.* From formula (37) we have

$$T_1=\tau_1\left[(c^2y_1+b^2z_1+d^2t_1)=(p_1+r_1)^2\tau_1\right].$$

Since

$$\tau_1=\tau(S_1)=\tau(S_0)=\Delta_0,$$

for the coefficient $C_1'$ without $o_1$, we have

$$C_1' = T_0 - \Delta_0(c^2 y_0 b^2 z_0 + d^2 t_0) + r_1^2 \Delta_0^2.$$

Now the desired formula follows from

$$\begin{aligned} C_1' =& \ \Delta_0 \cdot C_1, \\ r_0^2 =& \ \frac{1}{\tau_0}\left((c^2 y_0 + b^2 z_0 + d^2 t_0) - r_1^2 \tau_0 - \frac{T_0}{\tau_0}\right) \\ =& \ \frac{1}{\Delta_0}\left((c^2 y_0 + b^2 z_0 + d^2 t_0) - r_1^2 \Delta_0 - \frac{T_0}{\Delta_0}\right). \end{aligned}$$

□

*Remark* 5. If in the two-dimensional case, we introduce the determinant

$$\Delta_0 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix},$$

then we have $\Delta_0 = -16\Delta^2$. Consequently, $r_0^2 = -C_1/\Delta_0$ in both cases.

**Corollary 9.** *The centers $S_1$ and $S_2$ have coordinates*

$$\begin{array}{llll} x_1 = x_0 + 2p_1 u_1, & y_1 = y_0 + 2p_1 v_1, & z_1 = z_0 + 2p_1 w_1, & t_1 = t_0 + 2p_1 \eta_1, \\ x_2 = x_0 - 2p_2 u_1, & y_2 = y_0 - 2p_2 v_1, & z_2 = z_0 - 2p_2 w_1, & t_2 = t_0 - 2p_2 \eta_1, \end{array}$$

*where*

$$\begin{aligned} u_1 &= r_1 \Delta_{22} + r_2 \Delta_{32} + r_3 \Delta_{42} + r_4 \Delta_{52}, \\ v_1 &= r_1 \Delta_{23} + r_2 \Delta_{33} + r_3 \Delta_{43} + r_4 \Delta_{53}, \\ w_1 &= r_1 \Delta_{24} + r_2 \Delta_{34} + r_3 \Delta_{44} + r_4 \Delta_{54}, \\ \eta_1 &= r_1 \Delta_{25} + r_2 \Delta_{35} + r_3 \Delta_{45} + r_4 \Delta_{55}, \end{aligned}$$

*with the property*

$$u_1 + v_1 + w_1 + \eta_1 = 0.$$

Analogously to the planar case we can obtain coordinates of centers for all 16 spheres. There are eight planes, and each of them passes through six of the twelve inner or outer centers of mutual similarity of the given four spheres. As earlier, point $S_0$ with two centers is perpendicular to one of these eight planes.

Theorem 1 can be proved by the same technique used in the proof of Theorem 6.

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