Forum Geometricorum
Volume 17 (2017) 373–381.

FORUM GEOM
ISSN 1534-1178

The Simson Triangle and Its Properties

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Abstract. Let ABC be a triangle and P_1, P_2, P_3 points on its circumscribed circle. The Simson triangle for P_1, P_2, P_3 is the triangle bounded by their Simson lines with respect to triangle ABC. We study some interesting properties of the Simson triangle.

1. Introduction

The following theorem is often called Simson's theorem. (see [2, p. 137, Theorem 192])

Theorem 1 (Wallace-Simson line)**.** *The feet of the perpendiculars to the sides of triangle from a point are collinear, if and only if the point is on the circumscribed circle of the triangle. This is Simson line* (*or Wallace-Simson line*)*.*

Definition (Simson triangle)**.** The Simson triangle is the triangle bounded by the Simson lines of three points on the circumscribed circle of a fixed triangle. It is degenerate if the three Simson lines are concurrent.

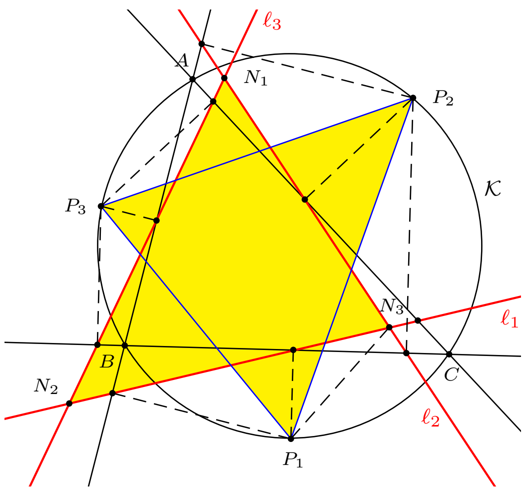

Figure 1

Theorem 2 (Orthopole, (see [2, p. 247, Theorem 406]))**.** *If perpendiculars are dropped on a line from the vertices of a triangle, the perpendiculars to the opposite sides from their feet are concurrent at a point called the orthopole of the line.*

Publication Date: June 28, 2017. Communicating Editor: Paul Yiu.

2. Notations

ABC is a triangle of reference.
The circumcircle $\mathcal{K}$ of $\triangle ABC$ has center O and radius R.
$P_1 \in \mathcal{K}, P_2 \in \mathcal{K}, P_3 \in \mathcal{K}$.
l_1, l_2, l_3 are the Simson lines of P_1, P_2, P_3 with respect to ABC.
$N_1N_2N_3$ is the Simson triangle bounded by these Simson lines:

$$N_1 = l_2 \cap l_3, \qquad N_2 = l_3 \cap l_1, \qquad N_3 = l_1 \cap l_2.$$

The circumcircle of $\triangle N_1N_2N_3$ is $\mathcal{K}_2$ with center O_2 and radius R_2.
The orthocenter of $\triangle ABC$ is H.
The orthocenter of $\triangle P_1P_2P_3$ is H_1.
The orthocenter of $\triangle N_1N_2N_3$ is H_2.
The center of nine-point circle for ABC is E.

We will use complex numbers in the proofs.
By u we shall denote the complex number, corresponding to point U.
Without loss of generality, we take the circumcircle $\mathcal{K}$ to be the unit circle. Then $R = 1$ and $O = 0$.
$a \cdot \bar{a} = b \cdot \bar{b} = c \cdot \bar{c} = p_1 \cdot \bar{p}_1 = p_2 \cdot \bar{p}_2 = p_3 \cdot \bar{p}_3 = 1$.
$h = a + b + c$; $e = 1/2(a + b + c)$; $h_1 = p_1 + p_2 + p_3$.

Lemma 3. *Let V and W be points on the unit circle. The orthogonal projection of a point P onto the line $\ell = VW$ is given by*

$$p_\ell = \frac{1}{2}(v + w + p - vw\bar{p}).$$

In particular, if P is also on the unit circle, then

$$p_\ell = \frac{1}{2}\left(v + w + p - \frac{vw}{p}\right).$$

Proof. Write the orthogonal projection as $p_\ell = (1-t)v + tw$ for some *real* number t. The vector $p - (1-t)v - tw$ is perpendicular to BC. This means that

$$(p - (1-t)v - tw)(\bar{v} - \bar{w}) + (\bar{p} - (1-t)\bar{v} - t\bar{w})(v - w) = 0.$$

Since v and w are on the unit circle, $\bar{v} = \frac{1}{v}$, $\bar{w} = \frac{1}{w}$. We have

$$(p - (1-t)v - tw)\left(\frac{1}{v} - \frac{1}{w}\right) + \left(\bar{p} - \frac{1-t}{v} - \frac{t}{w}\right)(v - w) = 0.$$

From this,

$$t = \frac{v - w - p + vw\bar{p}}{2(v - w)},$$

and the orthogonal projection is

$$p_\ell = (1-t)v + tw = \frac{1}{2}(v + w + p - vw\bar{p}).$$

If P is also on the unit circle, then $\bar{p} = \frac{1}{p}$, and $p_\ell = \frac{1}{2}\left(v + w + p - \frac{bc}{p}\right)$. $\square$

Proposition 4. *Let P be a point on the unit circumcircle of triangle ABC. The equation of its Simson line is*

$$2abc\bar{z} - 2pz + p^2 + (a+b+c)p - (bc+ca+ab) - \frac{abc}{p} = 0. \tag{1}$$

Proof. Let P be a point on the unit circumcircle. Its projections on the side lines of triangle ABC are, by Lemma 3,

$$p_a = \frac{1}{2}\left(b+c+p-\frac{bc}{p}\right),$$
$$p_b = \frac{1}{2}\left(c+a+p-\frac{ca}{p}\right),$$
$$p_c = \frac{1}{2}\left(a+b+p-\frac{ab}{p}\right).$$

The line joining p_b and p_c has equation

$$0 = \begin{vmatrix} z & p_b & p_c \\ \bar{z} & \bar{p}_b & \bar{p}_c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} z & \frac{1}{2}\left(c+a+p-\frac{ca}{p}\right) & \frac{1}{2}\left(a+b+p-\frac{ab}{p}\right) \\ \bar{z} & \frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{p}-\frac{p}{ca}\right) & \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{p}-\frac{p}{ab}\right) \\ 1 & 1 & 1 \end{vmatrix}$$
$$= \frac{(b-c)(p-a)(2abcp\bar{z} - 2p^2z + p^3 + (a+b+c)p^2 - (bc+ca+ab)p - abc)}{4abcp^2}$$
$$= \frac{(b-c)(p-a)(2abc\bar{z} - 2pz + p^2 + (a+b+c)p - (bc+ca+ab) - \frac{abc}{p})}{4abcp}.$$

Therefore, the equation of the Simson line of P is given by (1) above. □

Proposition 5. *The intersection of the Simson lines of two points P, $Q \in \mathcal{K}$ is the point with coordinates*

$$\frac{1}{2}\left(p+q+a+b+c+\frac{abc}{pq}\right). \tag{2}$$

Proof. Let ℓ_p, ℓ_q be the Simson lines of two points P, Q on the unit circumcircle of ABC. Their intersection is given by the solution of

$$2abc\bar{z} - 2pz + p^2 + (a+b+c)p - (bc+ca+ab) - \frac{abc}{p} = 0, \tag{3}$$
$$2abc\bar{z} - 2qz + q^2 + (a+b+c)q - (bc+ca+ab) - \frac{abc}{q} = 0. \tag{4}$$

Subtracting (4) from (3), we obtain

$$-2(p-q)z + (p^2-q^2) + (a+b+c)(p-q) - abc\left(\frac{1}{p}-\frac{1}{q}\right) = 0.$$

Dividing by $2(p-q)$, we obtain z as given in (2) above. □

3. Simson triangle

Theorem 6. *The Simson triangle $N_1N_2N_3$ is directly similar to triangle $P_1P_2P_3$ (see Figure 1).*

Proof. For three points P_1, P_2, P_3 on $\mathcal{K}$, by Proposition 5, the pairwise intersections of their Simson lines are

$$n_1 = \frac{1}{2}\left(p_2 + p_3 + a + b + c + \frac{abc}{p_2p_3}\right),$$
$$n_2 = \frac{1}{2}\left(p_3 + p_1 + a + b + c + \frac{abc}{p_3p_1}\right),$$
$$n_3 = \frac{1}{2}\left(p_1 + p_2 + a + b + c + \frac{abc}{p_1p_2}\right),$$

the vertices of the Simson triangle. Note that

$$\begin{aligned} n_2 - n_3 &= \frac{1}{2}\left(p_3 + p_1 + a + b + c + \frac{abc}{p_3p_1}\right) - \frac{1}{2}\left(p_1 + p_2 + a + b + c + \frac{abc}{p_1p_2}\right) \\ &= \frac{1}{2}\left(p_3 - p_2 + \frac{abc(p_2 - p_3)}{p_1p_2p_3}\right) \\ &= \frac{abc - p_1p_2p_3}{2p_1p_2p_3}(p_2 - p_3). \end{aligned}$$

Since the factor $k := \frac{abc - p_1p_2p_3}{2p_1p_2p_3}$ is symmetric in p_1, p_2, p_3, we conclude that

$$N_2N_3 = k \cdot P_2P_3, \qquad N_3N_1 = k \cdot P_3P_1, \qquad N_1N_2 = k \cdot P_1P_2,$$

and the triangles $N_1N_2N_3$ and $P_1P_2P_3$ are directly similar. □

Corollary 7. *The Simson triangle $N_1N_2N_3$ has circumradius $\frac{|abc - p_1p_2p_3|}{2}$, and circumcenter at the midpoint of the segment joining the orthocenters H of ABC and H_1 of $P_1P_2P_3$ (see Figure 2).*

Proof. Since the Simson triangle is similar to $P_1P_2P_3$ with similarity factor $k = \frac{|abc - p_1p_2p_3|}{2}$, it clearly has circumradius k.

The orthocenters of triangles ABC and $P_1P_2P_3$ are the points

$$h = a + b + c \qquad \text{and} \qquad h_1 = p_1 + p_2 + p_3.$$

With these, we rewrite

$$n_1 = \frac{h_1 + h}{2} + \frac{1}{2}\left(\frac{abc}{p_2p_3} - p_1\right) = \frac{h_1 + h}{2} + \frac{1}{2}\left(\frac{abc - p_1p_2p_3}{p_2p_3}\right).$$

Therefore,

$$\left|n_1 - \frac{h_1 + h}{2}\right| = \frac{1}{2}\left|\frac{abc - p_1p_2p_3}{p_2p_3}\right| = k,$$

since $|p_2| = |p_3| = 1$. The same relation holds if n_1 is replaced by n_2 and n_3. This shows that the Simson triangle has circumcenter $\frac{h_1+h}{2}$, which is the midpoint of H_1 and H. It also confirms independently that the circumradius is k. □

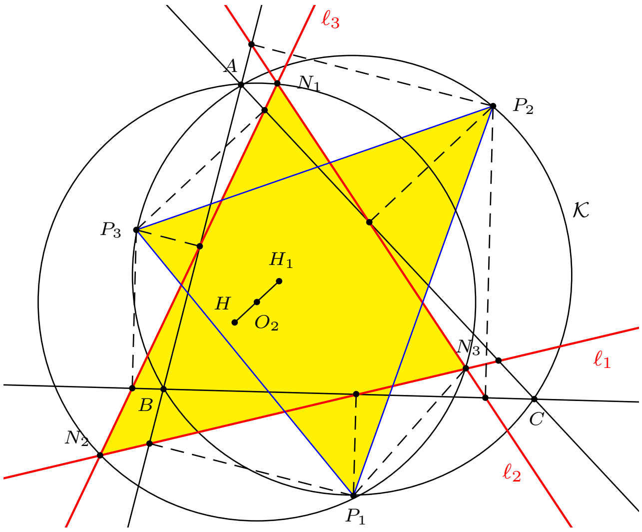

Figure 2

4. The Simson triangle and orthopoles

Lemma 8. *Let V and W be points on the unit circumcircle of triangle ABC, A_1 the orthogonal projection of A onto VW. The perpendicular from A_1 to BC has equation*

$$\bar{z} - \frac{z}{bc} + \frac{a-(v+w)}{2vw} + \frac{(a^2-bc)+a(v+w)-vw}{2abc} = 0. \tag{5}$$

Proof. By Lemma 3, the coordinates a_1 of A_1 and its complex conjugate are

$$\begin{aligned} a_1 &= \frac{1}{2}\left(v+w+a-\frac{vw}{a}\right), \\ \bar{a}_1 &= \frac{-a^2+a(v+w)+vw}{2avw}. \end{aligned}$$

By Lemma 3 again, the coordinates a_2 of the orthogonal projection A_2 of A_1 onto BC, together with its complex conjugate, are

$$\begin{aligned} a_2 &= \frac{1}{2}(b+c+a_1-bc\bar{a}_1) \\ &= \frac{a^2bc-abc(v+w)+(a^2-bc+2ca+2ab)vw+avw(v+w)-v^2w^2}{4avw}, \\ \bar{a}_2 &= \frac{-a^2bc+abc(v+w)-(a^2-bc-2ca-2ab)vw-avw(v+w)+v^2w^2}{4abcvw}. \end{aligned}$$

The line A_1A_2 contains a point with coordinates z if and only if

$$\begin{aligned} 0 &= \begin{vmatrix} z & a_1 & a_2 \\ \bar{z} & \bar{a}_1 & \bar{a}_2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{f(a,b,c,v,w)g(a,b,c,v,w,z)}{8a^2bcv^2w^2}, \end{aligned}$$

where

$$\begin{aligned} f(a,b,c,v,w) &:= a^2bc - abc(v+w) - (a^2+bc-2ca-2ab)vw \\ &\quad - avw(v+w) + v^2w^2, \\ g(a,b,c,v,w,z) &:= 2abcvw\bar{z} - 2avwz + a^2bc - abc(v+w) \\ &\quad + (a^2-bc)vw + avw(v+w) - v^2w^2. \end{aligned}$$

Therefore, the equation of the perpendicular is $g(a,b,c,v,w,z) = 0$. Dividing by $2abcvw$, we obtain the equation (5). □

Now we consider the construction in Lemma 8 beginning with all three vertices of $\triangle ABC$. This results in the three lines

$$\begin{aligned} \bar{z} - \frac{z}{bc} + \frac{a-(v+w)}{2vw} + \frac{(a^2-bc)+a(v+w)-vw}{2abc} &= 0, \\ \bar{z} - \frac{z}{ca} + \frac{b-(v+w)}{2vw} + \frac{(b^2-ca)+b(v+w)-vw}{2abc} &= 0, \\ \bar{z} - \frac{z}{ab} + \frac{c-(v+w)}{2vw} + \frac{(c^2-ab)+c(v+w)-vw}{2abc} &= 0. \end{aligned}$$

The intersection of the last two lines is given by

$$\begin{aligned} &-\frac{z}{ca} + \frac{z}{ab} + \frac{b-(v+w)}{2vw} - \frac{c-(v+w)}{2vw} \\ &\quad + \frac{(b^2-ca)+b(v+w)-vw}{2abc} - \frac{(c^2-ab)+c(v+w)-vw}{2abc} = 0, \\ &-\frac{(b-c)z}{abc} + \frac{b-c}{2vw} + \frac{(b-c)(a+b+c+v+w)}{2abc} = 0, \end{aligned}$$

Multiplying by $\frac{abc}{b-c}$, we obtain

$$z = \frac{1}{2}\left(a+b+c+v+w+\frac{abc}{vw}\right).$$

Note that this is symmetric in a, b, c. This means that the three perpendiculars form A_1 to BC, B_1 to CA, and C_1 to AB are concurrent. The point of concurrency is the orthopole N of the line VW. By Proposition 5, this is also the same as the intersection of the Simson lines of V and W with respect to $\triangle ABC$ (see Figure 3 and [1, p.289, Theorem 697]). Applying this to the three side lines of the triangle $P_1P_2P_3$ for three points P_1, P_2, $P_3 \in \mathcal{K}$, we obtain the following theorem.

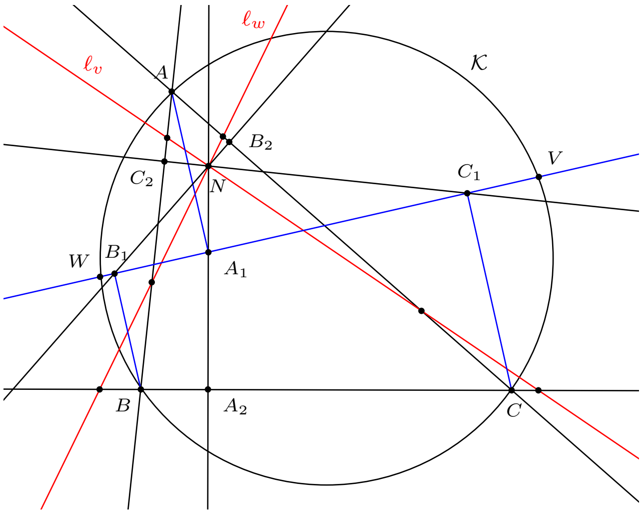

Figure 3

Theorem 9. *For three points P_1, P_2, P_3 on the circumcircle $\mathcal{K}$ of $\triangle ABC$, the orthopoles of the lines P_2P_3, P_3P_1, P_1P_2 coincide with the vertices N_1, N_2, N_3 of the Simson triangle bounded by the Simson lines of P_1, P_2, P_3 with respect to $\triangle ABC$.*

5. Examples

Example 1. *Let $\triangle P_1P_2P_3$ be an equilateral triangle. The circumcenter of the Simson triangle coincides with the center of nine-point circle for $\triangle ABC$.*

Proof. If $P_1P_2P_3$ is equilateral, its orthocenter coincides with the circumcenter. This means that $p_1 + p_2 + p_3 = 0$. The circumcenter of the Simson triangle is $\frac{1}{2}(a+b+c+p_1+p_2+p_3) = \frac{1}{2}(a+b+c)$, the center E of the nine-point circle of $\triangle ABC$. □

Example 2. *Let $P_1 \in \mathcal{K}$ and let P_1E meet the circle $\mathcal{K}$ again in P_3 (E is the center of nine-point circle for $\triangle ABC$). Let $P_2 \in \mathcal{K}$ and $EP_2 \perp P_1P_3$. Then the circumcenter of the Simson triangle lies on the circle with center H (the orthocenter) and radius $\frac{1}{2}R$ (see Figure 4).*

Proof. Let P_4 be another point of $\mathcal{K}$ on the line P_2E.

$$\begin{aligned} P_1P_3 \perp P_2P_4 &\Rightarrow (p_1 - p_3)\left(\frac{1}{p_2} - \frac{1}{p_4}\right) + \left(\frac{1}{p_1} - \frac{1}{p_3}\right)(p_2 - p_4) = 0 \\ &\Rightarrow (p_1 - p_3)(p_2 - p_4)(p_1p_3 + P_2p_4) = 0. \end{aligned}$$

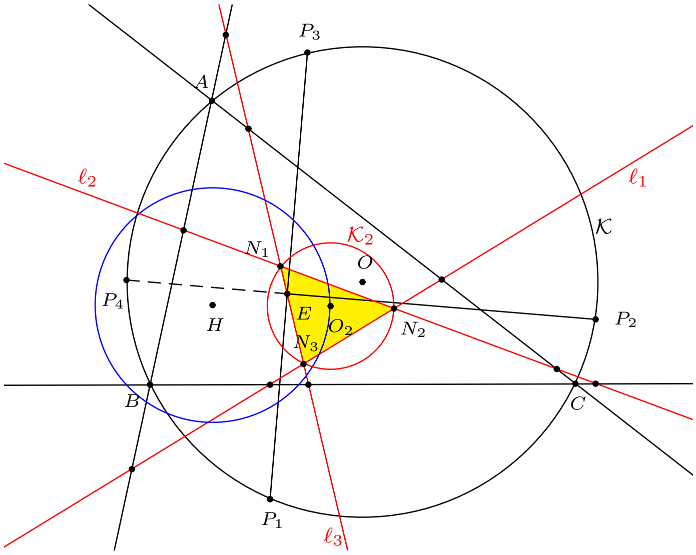

Figure 4

Therefore, $p_1p_3 + p_2p_4 = 0$ (see [3, p. 45]).

By Lemma 3 or [3, p. 45],

$$E \in P_1P_3 \Rightarrow p_1 + p_3 = e + p_1p_3\bar{e},$$
$$E \in P_2P_4 \Rightarrow p_2 + p_4 = e + p_2p_4\bar{e}.$$

Therefore,

$$p_1 + p_2 + p_3 + p_4 = 2e + p_1p_3 + p_2p_4 = 2e = a + b + c.$$

By Theorem 7, the circumcenter O_2 of the Simson triangle of $P_1P_2P_2$ has coordinates

$$o_2 = \frac{1}{2}(a + b + c + p_1 + p_2 + p_3) = a + b + c - \frac{p_4}{2},$$

and $|o_2 - h| = \left|\frac{p_4}{2}\right| = \frac{1}{2}$. Hence, O_2 lies on a circle with radius $\frac{1}{2}R$ and center H. □

Example 3. *Let A, B, C, A', B', C' be points on a circle $\mathcal{K}$. Construct the Simson triangle for A', B', C' with respect to $\triangle ABC$ and the Simson triangle for A, B, C with respect to $\triangle A'B'C'$. The six vertices of these two Simson triangles lie on a circle (see Figure 5).*

Proof. Let $N_1N_2N_3$ be the Simson triangle for A', B', C' with respect to $\triangle ABC$, and $N_1'N_2'N_3'$ that of A, B, C with respect to $\triangle A'B'C'$. By Theorem 7, the circumcircles of $N_1N_2N_3$ and $N_1'N_2'N_3'$ both have radius $\frac{|abc - a'b'c'|}{2}$, and center $\frac{1}{2}(a + b + c + a' + b' + c')$. Therefore the two circumcircles coincide. □

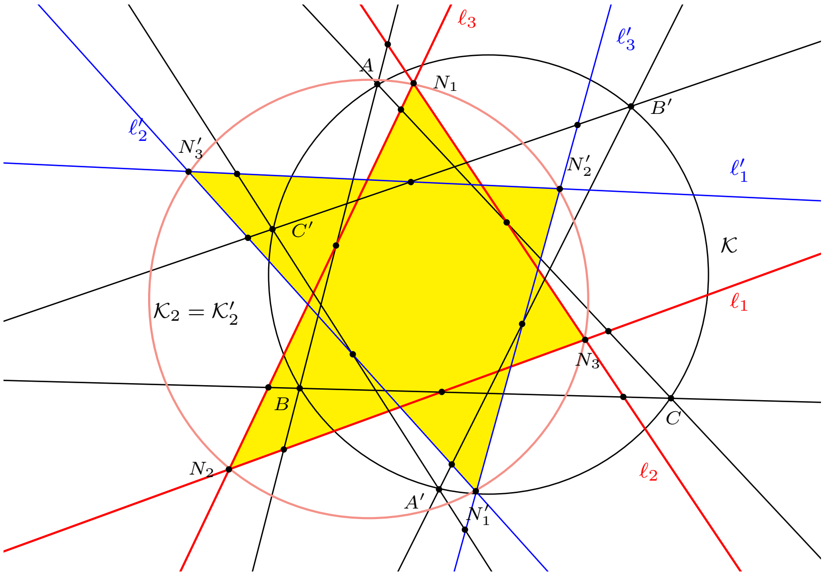

Figure 5

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