Forum Geometricorum
Volume 17 (2017) 383–399.

FORUM GEOM
ISSN 1534-1178

The Periambic Constellation: Altitudes, Perpendicular Bisectors, and Other Radical Axes in a Triangle

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Abstract. Six circles may be constructed using a triangle's vertices as centers and its sides as radii. These circles determine twelve ordinary and three ideal radical axes, whose intersection points include the triangle's circumcenter and orthocenter, along with eight other ordinary points in interesting configurations. For instance, we show that the orthocenter, circumcenter, and two radical centers of the six circles form a parallelogram, and that six other radical centers (the intersection points of the altitudes and perpendicular bisectors) are the vertices of two congruent triangles which are inversely similar to the original. Underlying this "constellation" is a simple invariance property of three circles in which two are concentric.

1. Introduction

Consider a triangle ABC and the circles P_A, P_B, P_C having centers A, B, C and radii AB, BC, CA, respectively. Because these circles might whimsically be said to "walk around the triangle's perimeter," we call them *periambic circles*—*peri* suggesting "perimeter," and *ambic* echoing the Latin *ambire*, "go around." A second set of periambic circles Q_A, Q_B, Q_C, with centers A, B, C and radii AC, BA, CB, respectively, "walks" around triangle ABC in the opposite direction. We call P_A, P_B, P_C the *p-circles* and Q_A, Q_B, Q_C the *q-circles* (Figure 1).

The periambic circles give rise in pairs to $\binom{6}{2} = 15$ radical axes, many triples of which are concurrent in ordinary or ideal points. Among the ordinary points are the orthocenter and circumcenter of triangle ABC, along with eight others which may not yet be named in the literature. A rich geometric structure resides upon these ten points; underlying much of it are relatively simple results, including an invariance property of three circles in which two are concentric.

2. Definitions and notation

In triangle ABC, a, b, c are the sides opposite vertices A, B, C, respectively; R is the circumradius; and we define $\alpha = \angle BAC,\ \beta = \angle CBA$, and $\gamma = \angle ACB$. Positive angles are measured counterclockwise, and triangle ABC, labeled counterclockwise, has positive area Δ. If lines l_1 and l_2 intersect at V, we write $V = l_1 \wedge l_2$. Given distinct circles U_1 and U_2, we denote their radical axis by $\langle U_1, U_2\rangle$; note that $\langle U_1, U_2\rangle = \langle U_2, U_1\rangle$. The radical axes of three distinct circles are concurrent; we refer to this as the Radical Center Property.

Publication Date: November 28, 2017. Communicating Editor: Paul Yiu.
The author thanks Walt Tape for Mathematica help and for simplifying the proof of Theorem 4.

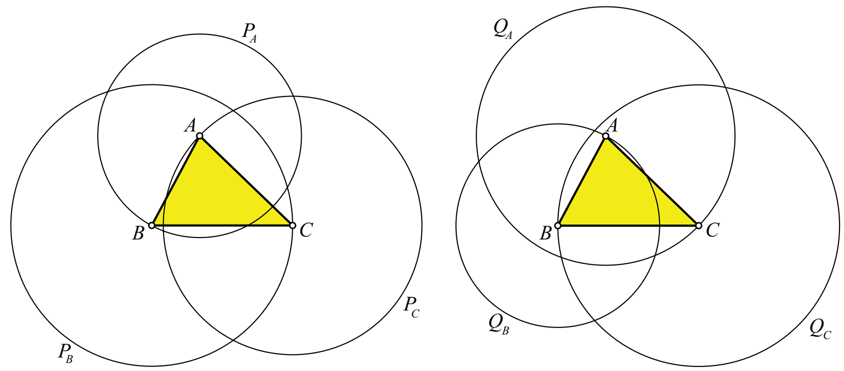

Figure 1. The p-circles P_A, P_B, P_C (left) and q-circles Q_A, Q_B, Q_C (right).

Definition 1 (p-lines and P)**.** *The p-lines of triangle ABC are*

$$p_a = \langle P_B, P_C \rangle$$
$$p_b = \langle P_C, P_A \rangle$$
$$p_c = \langle P_A, P_B \rangle.$$

The radical center of P_A, P_B, P_C is labeled P.

Definition 2 (q-lines and Q)**.** *The q-lines of triangle ABC are*

$$q_a = \langle Q_C, Q_B \rangle$$
$$q_b = \langle Q_A, Q_C \rangle$$
$$q_c = \langle Q_B, Q_A \rangle.$$

The radical center of Q_A, Q_B, Q_C is labeled Q.

We loosely characterize any set of points deriving exclusively from the p-circles as being of *gender* p; a set deriving from the q-circles is of *gender* q. While P and Q are clearly of genders p and q, respectively, there are other radical axes, concurrent in groups of three, defined by pairs of circles of opposite gender.

Definition 3. *The altitudes of triangle ABC are*

$$h_a = \langle P_C, Q_B \rangle$$
$$h_b = \langle P_A, Q_C \rangle$$
$$h_c = \langle P_B, Q_A \rangle \ ,$$

concurrent at the orthocenter H.

Definition 4. *The perpendicular bisectors of triangle ABC are*

$$o_a = \langle P_B, Q_C \rangle$$
$$o_b = \langle P_C, Q_A \rangle$$
$$o_c = \langle P_A, Q_B \rangle \ ,$$

concurrent at the circumcenter O.

Together, P, Q, H, and O comprise the *major periambic points* of triangle ABC.

A final set of points requiring names are the feet of the twelve just-defined radical axes on their respective sides of triangle ABC. The foot of a radical axis is labeled with the name of the axis, but with a capital rather than lowercase first letter; for instance, H_c is the foot of altitude h_c on side c. The twelve points P_a, H_a, O_a, Q_a, P_b, and so on will be called the *periambic feet*.

3. Constant-Distance Lemma

Given three circles of which two (but not all) are concentric, the radical axis of the concentric pair is the ideal line and the other two axes are parallel. While the radical center in this configuration is an ideal point, its ordinary parts possess a useful invariance property.

Lemma 1 (Constant-Distance Lemma)**.** *Let U_1 and U_2 be fixed concentric circles with center K_1 and radii r_1 and r_2, respectively. For any circle U_3 of variable radius r_3 and fixed center $K_3 \neq K_1$, the distance between $\langle U_1, U_3\rangle$ and $\langle U_2, U_3\rangle$ is constant.*

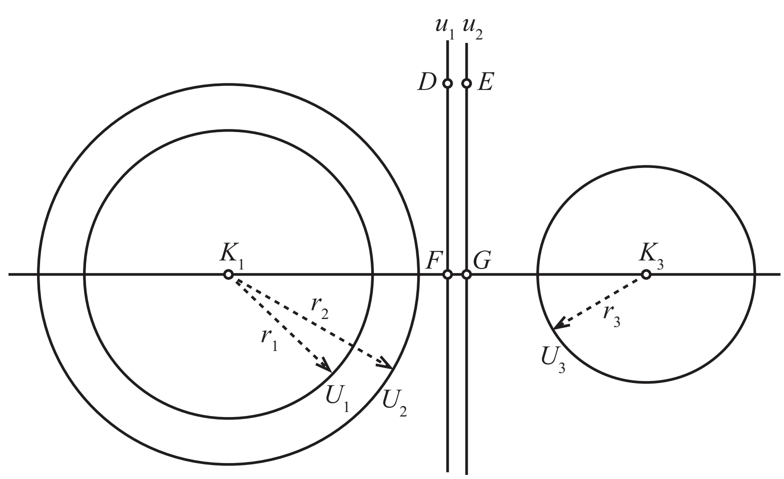

Figure 2. Fixed concentric circles U_1 and U_2, and variable circle U_3.

Proof. Let $u_1 = \langle U_1, U_3\rangle$ and $u_2 = \langle U_2, U_3\rangle$ meet K_1K_3 at F and G, respectively. Let D and E be points on u_1 and u_2, respectively, such that DE is parallel to K_1K_3 (Figure 2). The radical axis of two circles being the set of points having equal powers with respect to those circles, at D and E we have

$$DK_1^2 - r_1^2 = DK_3^2 - r_3^2$$
$$EK_1^2 - r_2^2 = EK_3^2 - r_3^2 \ .$$

By the Pythagorean relation in triangles DK_1F, EK_1G, DFK_3, and EGK_3, these become

$$K_1F^2 + FD^2 - r_1^2 = FK_3^2 + DF^2 - r_3^2 \tag{1}$$

$$K_1G^2 + GE^2 - r_2^2 = EG^2 + GK_3^2 - r_3^2. \tag{2}$$

Subtracting (2) from (1), cancelling the identical quantities FD^2 and GE^2, and rearranging, we obtain

$$r_2^2 - r_1^2 = (FK_3^2 - K_1F^2) + (K_1G^2 - GK_3^2) \ .$$

Factoring each difference of squares on the right, and simplifying using $FK_3 + K_1F = K_1G + GK_3 = K_1K_3$, it follows that the directed distance from u_1 to u_2 is

$$FG = \frac{r_2^2 - r_1^2}{2\ K_1K_3} \ ,$$

a formula comprising known constants. □

The Constant-Distance Lemma has immediate consequences for the periambic radical axes.

Proposition 2. *On a given side of triangle ABC, the directed distance from the p-line to the altitude is equal to the directed distance from the perpendicular bisector to the q-line. That is, for any $x \in \{a, b, c\}$, $P_xH_x = O_xQ_x$.*

Proof. Consider, for instance, the directed distances P_aH_a and O_aQ_a. Let P_B and Q_B be the Constant-Distance Lemma's fixed concentric circles U_1 and U_2, respectively, and let P_C and Q_C represent two positionings of the variable third circle U_3, so that $r_2 = c$, $r_1 = a$, and $K_1K_3 = a$. By the lemma, the distance between $p_a = \langle P_B, P_C\rangle$ and $h_a = \langle P_C, Q_B\rangle = \langle Q_B, P_C\rangle$ is $(c^2 - a^2)/2a$, which is also the distance between $o_a = \langle P_B, P_C\rangle$ and $q_a = \langle Q_B, Q_C\rangle = \langle Q_C, Q_B\rangle$. Thus $P_aH_a = O_aQ_a$, and similarly $P_bH_b = O_bQ_b$ and $P_cH_c = O_cQ_c$. □

4. Central parallelogram

We may now state the most obvious feature of the major periambic points.

Theorem 3. *In any triangle, $POQH$ is a parallelogram.*

Proof. Let $T = p_c \wedge o_b$, $W = q_b \wedge h_c$, $X = q_c \wedge h_b$, and $Y = p_b \wedge o_c$ (Figure 3). The quadrilaterals $QWHX$ and $OTPY$ not only have parallel corresponding sides, they are in fact congruent parallelograms, because their projections onto sides b and c have equal lengths, by Proposition 2. Thus their diagonals HQ and PO are congruent and parallel, and $POQH$ is a parallelogram. □

We call $POQH$ the *central parallelogram* of triangle ABC. It may be shown that for an isosceles triangle ABC the central parallelogram is a rhombus, for an isosceles right triangle a square, and for an equilateral triangle a point.

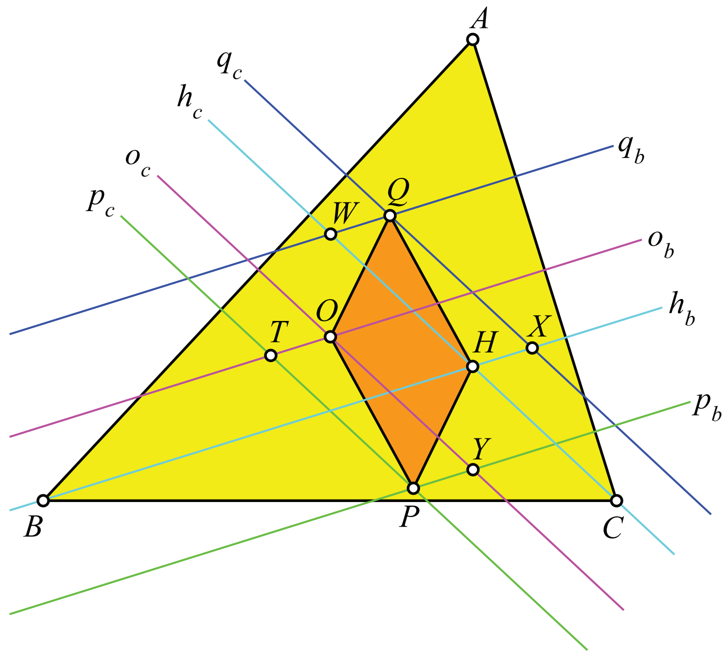

Figure 3. The central parallelogram $POQH$.

5. The minor periambic points

So far only two points, P and Q, have been defined as radical centers of triples of periambic circles. However, the number of triples of distinct periambic circles is the number of 3-letter words on the symbols P and Q, or 2^3. The eight radical axes thus defined and the names of their radical centers are listed in Table 1.

Description	**Triple of circles**	**Triple of radical axes**	**Radical center**
3 p-circles	$\{P_A, P_B, P_C\}$	$\{p_c, p_a, p_b\}$	P
2 p-circles, 1 q-circle	$\{Q_A, P_B, P_C\}$	$\{h_c, p_a, o_b\}$	A_p
	$\{P_A, Q_B, P_C\}$	$\{o_c, h_a, p_b\}$	B_p
	$\{P_A, P_B, Q_C\}$	$\{p_c, o_a, h_b\}$	C_p
1 p-circle, 2 q-circles	$\{P_A, Q_B, Q_C\}$	$\{o_c, q_a, h_b\}$	A_q
	$\{Q_A, P_B, Q_C\}$	$\{h_c, o_a, q_b\}$	B_q
	$\{Q_A, Q_B, P_C\}$	$\{q_c, h_a, o_b\}$	C_q
3 q-circles	$\{Q_A, Q_B, Q_C\}$	$\{q_c, q_a, q_b\}$	Q

Table 1. The eight triples of periambic circles and their radical centers.

The orthocenter and circumcenter are not products of the Radical Center Property, since the radical axes which define each point require six circles. We show that *triples of pairs* of opposite-gender periambic circles may determine two ordinary and four ideal points. Let P_A, P_B, and P_C be the first member of the first, second, and third pair, respectively. This reduces the problem to counting the permutations of Q_A, Q_B, and Q_C as second members in each pair, which is $3! = 6$. However, if in any pair of circles the subscripts match, then that pair is concentric, and their radical axis is an ideal line. To enumerate the ordinary points we must count, not the permutations of the q-circles, but their derangements, which is $3!(1 - 1/1! + 1/2! - 1/3!) = 2$. This analysis does not say anything about concurrency, but of course we know that the two cases produce H and O.

We call A_p, B_p, C_p, A_q, B_q, and C_q the *minor periambic points,* and triangle $A_pB_pC_p$ and triangle $A_qB_qC_q$ the *minor periambic triangles* (Figure 4). From

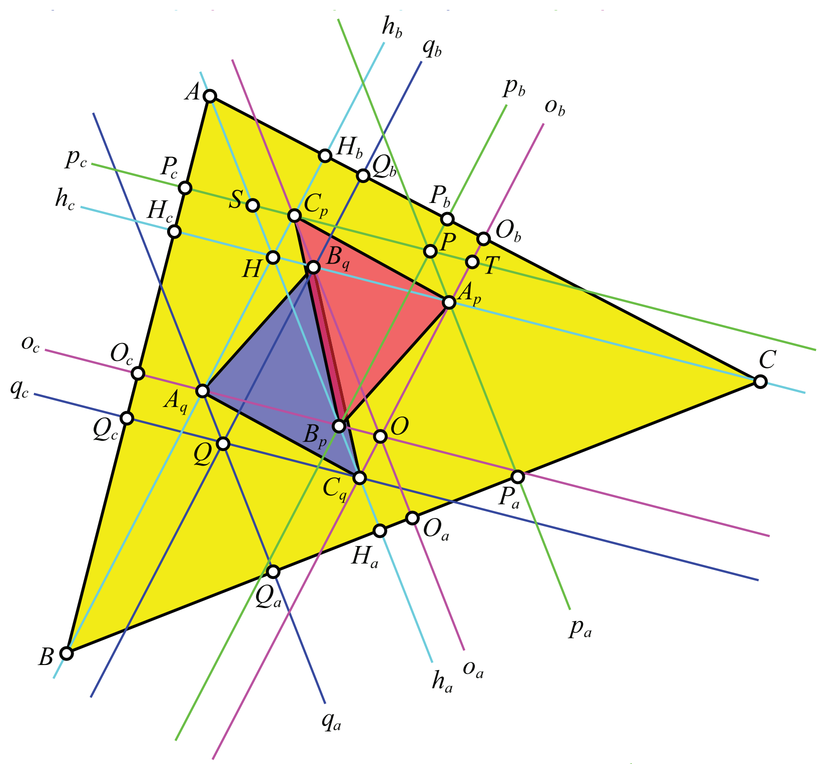

Figure 4. The minor periambic triangles $A_pB_pC_p$ and $A_qB_qC_q$.

Table 1 we see that the minor periambic points are entirely determined by intersections of non-parallel altitudes and perpendicular bisectors in triangle ABC. From the many aspects of their arrangement suggested by Figure 4, we begin with the following.

Theorem 4. *The minor periambic triangles are inversely similar to triangle* ABC.

Proof. Our "brute force" proof comprises lengthy mundane arithmetic; most of the coordinates and equalities below were computed in Mathematica [4]. Without loss of generality, place triangle ABC in the Cartesian plane with $A = (0,0)$, $B = (1,0)$, and $C = (c_1, c_2)$, where $c_2 > 0$. The altitudes and perpendicular bisectors are

$$\begin{aligned}
h_a &: y = \frac{1-c_1}{c_2}x & o_a &: y = \frac{1-c_1}{c_2}x + \frac{c_1^2+c_2^2-1}{2c_2} \\
h_b &: y = -\frac{c_1}{c_2}x + \frac{c_1}{c_2} & o_b &: y = -\frac{c_1}{c_2}x + \frac{c_1^2+c_2^2}{2c_2} \\
h_c &: x = c_1 & o_c &: x = \frac{1}{2}
\end{aligned}$$

from which it follows that

$$\begin{aligned}
A_p &= h_c \wedge o_b = \left(c_1, \frac{c_2^2-c_1^2}{2c_2}\right) \\
B_p &= h_a \wedge o_c = \left(\frac{1}{2}, \frac{1-c_1}{2c_2}\right) \\
C_p &= h_b \wedge o_a = \left(\frac{1+2c_1-c_1^2-c_2^2}{2}, \frac{c_1-2c_1^2+c_1^3+c_1c_2^2}{2c_2}\right) \ .
\end{aligned}$$

The squares of the side lengths of triangle ABC are

$$\begin{aligned}
(AB)^2 &= 1 \\
(BC)^2 &= 1-2c_1+c_1^2+c_2^2 \\
(CA)^2 &= c_1^2+c_2^2 \ .
\end{aligned}$$

while the squares of the side lengths of triangle $A_pB_pC_p$ work out to be

$$\begin{aligned}
(A_pB_p)^2 &= -\frac{1}{4} - \frac{c_1}{2} + \frac{c_1^2}{2} + \frac{1}{4c_2^2} - \frac{c_1}{2c_2^2} + \frac{3c_1^2}{4c_2^2} - \frac{c_1^3}{2c_2^2} + \frac{c_1^4}{4c_2^2} + \frac{c_2^2}{4} \\
(B_pC_p)^2 &= -\frac{c_1}{2} + 2c_1^2 - 2c_1^3 + \frac{3c_1^4}{4} + \frac{1}{4c_2^2} - \frac{c_1}{c_2^2} + \frac{2c_1^2}{c_2^2} - \frac{5c_1^3}{2c_2^2} + \frac{2c_1^4}{c_2^2} - \frac{c_1^5}{c_2^2} \\
&\quad + \frac{c_1^6}{4c_2^2} - c_1c_2^2 + \frac{3c_1^2c_2^2}{4} + \frac{c_2^4}{4} \\
(C_pA_p)^2 &= \frac{1}{4} - \frac{c_1}{2} + \frac{c_1^2}{2} - c_1^3 + \frac{3c_1^4}{4} + \frac{c_1^2}{4c_2^2} - \frac{c_1^3}{2c_2^2} + \frac{3c_1^4}{4c_2^2} - \frac{c_1^5}{2c_2^2} + \frac{c_1^6}{4c_2^2} - \frac{c_2^2}{4} \\
&\quad - \frac{c_1c_2^2}{2} + \frac{3c_1^2c_2^2}{4} + \frac{c_2^4}{4} \ .
\end{aligned}$$

Computer calculations reveal that

$$\left(\frac{A_pB_p}{AB}\right)^2 = \left(\frac{B_pC_p}{BC}\right)^2 = \left(\frac{C_pA_p}{CA}\right)^2 = \rho^2 \ ,$$

where

$$\rho^2 = -\frac{1}{4} - \frac{c_1}{2} + \frac{c_1^2}{2} + \frac{1}{4c_2^2} - \frac{c_1}{2c_2^2} + \frac{3c_1^2}{4c_2^2} - \frac{c_1^3}{2c_2^2} + \frac{c_1^4}{4c_2^2} + \frac{c_2^2}{4} \ ,$$

hence triangles ABC and $A_pB_pC_p$ are similar.

To show that the similarity is inverse rather than direct, note first that the area Δ of triangle ABC is $c_2/2$. Taking A_p, B_p, and C_p to be vector endpoints, we compute the area Δ_p of triangle $A_pB_pC_p$ as

$$\begin{aligned} \Delta_p =& \frac{1}{2}\begin{vmatrix} \vec{B}_p - \vec{A}_p \\ \vec{C}_p - \vec{A}_p \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \frac{1}{2} - c_1 & \frac{1 - c_1 + c_1^2 - c_2^2}{2c_2} \\ \frac{1 - c_1^2 - c_2^2}{2} & \frac{c_1 - c_1^2 + c_1^3 + c_1c_2^2 - c_2^2}{2c_2} \end{vmatrix} \\ =& \frac{1}{2}\left[\frac{c_2}{4} + \frac{c_1c_2}{2} - \frac{c_1^2c_2}{2} - \frac{1}{4c_2} + \frac{c_1}{2c_2} - \frac{3c_1^2}{4c_2} + \frac{c_1^3}{2c_2} - \frac{c_1^4}{4c_2} - \frac{c_2^3}{4}\right] \qquad (3) \\ =& -\rho^2\Delta \ . \end{aligned}$$

Since Δ_p is negative, we conclude that triangles $A_pB_pC_p$ and ABC are inversely similar. Much the same sequence of calculations shows that triangles $A_qB_qC_q$ and ABC are inversely similar as well. □

Proposition 5. *The minor periambic triangles are congruent, and radially symmetric around the midpoint of the Euler segment HO.*

Proof sketch. The congruence and radial symmetry arise because HA_pOA_q is a parallelogram whose diagonal A_pA_q is bisected by HO; the same is true for B_pB_q and C_pC_q in HB_pOB_q and HC_pOC_q, respectively. □

Proposition 5 implies that results proved about P, A_p, B_p, and C_p are automatically true for Q, A_q, B_q, and C_q, a fact which applies to the next result.

Proposition 6. *P, A_p, B_p, and C_p are concyclic.*

Proof. By construction, $\angle C_pPA_p = \alpha + \gamma$, while $\angle A_pB_pC_p = \beta$ by Theorem 4. With supplementary opposite angles, quadrilateral $PA_pB_pC_p$ is cyclic. □

6. A second look at area

The computer-calculated expression (3) for a minor periambic triangle's area could be described as unenlightening. We now derive a slightly less forbidding area formula having more obvious references to the configuration's geometry. The process begins with a lemma giving directed distances between parallel altitudes and perpendicular bisectors.

Lemma 7. *In directed distances and angles,*

$$\begin{aligned} O_aH_a &= R\sin(\gamma - \beta) \\ O_bH_b &= R\sin(\alpha - \gamma) \\ O_cH_c &= R\sin(\beta - \alpha) \ . \end{aligned}$$

Proof. To prove the first of these, multiply the identity $\sin(\gamma-\beta)=\sin\gamma\cos\beta-\sin\beta\cos\gamma$ by the successive terms in

$$R=\frac{c}{2\sin\gamma}=\frac{b}{2\sin\beta}$$

to obtain

$$\begin{aligned}R\sin(\gamma-\beta)&=\frac{c}{2\sin\gamma}\sin\gamma\cos\beta-\frac{b}{2\sin\beta}\sin\beta\cos\gamma\\&=\tfrac{1}{2}(c\cos\beta-b\cos\gamma)\ .\end{aligned}$$

In triangles ABH_a and AH_aC we have

$$c\cos\beta=BH_a\ ,\quad\text{and}\quad b\cos\gamma=H_aC\ ,$$

respectively. Thus, since $BH_a+H_aC=BC$ and $O_aH_a+H_aC=O_aC$, it follows that

$$\begin{aligned}R\sin(\gamma-\beta)&=\tfrac{1}{2}(BH_a-H_aC)\\&=\tfrac{1}{2}\big((BC-H_aC)-H_aC\big)\\&=\frac{BC}{2}-H_aC\\&=O_aC-H_aC\\&=O_aH_a\ .\end{aligned}$$

The proofs for O_bH_b and O_cH_c are similar. □

Proposition 8. *In a non-degenerate triangle ABC, the minor periambic triangles each have areas equal to*

$$\frac{R^2}{2}\left[\frac{\sin(\alpha-\gamma)\sin(\beta-\alpha)}{\sin\alpha}+\frac{\sin(\beta-\alpha)\sin(\gamma-\beta)}{\sin\beta}+\frac{\sin(\gamma-\beta)\sin(\alpha-\gamma)}{\sin\gamma}\right]\ . \quad (4)$$

Proof. Without loss of generality, position triangle ABC with its circumcenter at the pole of a polar coordinate system, and with BC parallel to the axis $\theta=0$ (Figure 5). Let L and N be the feet of the perpendiculars from O to h_c and h_b, respectively, and let M be the foot of the perpendicular from B_p to o_a. To find the coordinates $(r_1,\theta_1),\ (r_2,\theta_2)$, and (r_3,θ_3) of A_p, B_p, and C_p, respectively, apply Lemma 7 in triangle OA_pL to get

$$r_1=OA_p=\frac{OL}{\sin\alpha}=\frac{O_cH_c}{\sin\alpha}=\frac{R\sin(\beta-\alpha)}{\sin\alpha}\ .$$

In triangle OMB_p one has

$$r_2=OB_p=\frac{MB_p}{\sin\beta}=\frac{O_aH_a}{\sin\beta}=\frac{R\sin(\gamma-\beta)}{\sin\beta}\ ,$$

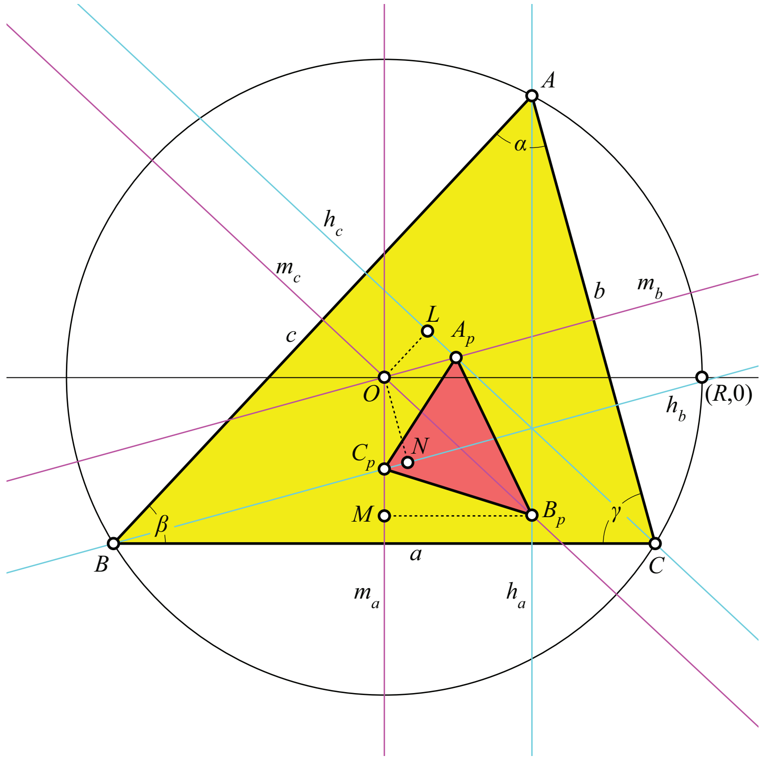

Figure 5. Polar coordinates for calculating the area of triangle $A_pB_pC_p$.

and in triangle OC_pN

$$r_3 = OC_p = \frac{ON}{\sin\gamma} = \frac{O_bH_b}{\sin\gamma} = \frac{R\sin(\alpha-\gamma)}{\sin\gamma}\ . \tag{5}$$

By construction, $\theta_1 = \alpha + \beta + \pi/2$, $\theta_2 = \beta - \pi/2$, and $\theta_3 = \pi/2$. (Note that $r_3 < 0$ in Equation (5) if $\alpha - \gamma < 0$.) In [1] the area of the triangle with vertices (r_1, θ_1), (r_2, θ_2), (r_3, θ_3) is given as

$$\tfrac{1}{2}\left[r_1r_2\sin(\theta_2-\theta_1) + r_2r_3\sin(\theta_3-\theta_2) + r_3r_1\sin(\theta_1-\theta_3)\right]\ ,$$

and we therefore obtain

$$\begin{aligned}\Delta_p = \tfrac{1}{2}\Bigg[&R^2\frac{\sin(\beta-\alpha)}{\sin\alpha}\frac{\sin(\gamma-\beta)}{\sin\beta}\sin(-\alpha-\pi)\\ &+ R^2\frac{\sin(\gamma-\beta)}{\sin\beta}\frac{\sin(\alpha-\gamma)}{\sin\gamma}\sin(\pi-\beta)\\ &+ R^2\frac{\sin(\alpha-\gamma)}{\sin\gamma}\frac{\sin(\beta-\alpha)}{\sin\alpha}\sin(\pi-\gamma)\Bigg]\ ,\end{aligned}$$

which simplifies to (4). Triangle $A_qB_qC_q$ has the same area as triangle $A_pB_pC_p$ by Proposition 5. □

Because the minor periambic triangles are inversely similar to triangle ABC, the formula (4) necessarily yields a non-positive value. This can be confirmed directly, if somewhat laboriously; one can for instance reduce (4) to a function of two variables, whose partial derivatives can be analyzed for local maxima, all of which are less than or equal to zero.

By definition and construction, triangle ABC has positive area, and we have proved that its minor periambic triangles have negative areas. However, the sign of the central parallelogram's area seems ambiguous at first sight. A consequence of the next result is that the central parallelogram, too, should be viewed as having negative area.

Proposition 9. *The area of the central parallelogram is equal to the sum of the areas of the minor periambic triangles.*

Proof. This is an instance of Theorem 107 in [3]: *Two triangles whose vertices lie on the sides of a given triangle at equal distances from their midpoints are equal in area.* In our case, triangles HPO and $A_pB_pC_p$ are inscribed in triangle TC_qS, where $S = h_a \wedge p_c$ and $T = o_b \wedge p_c$ (Figure 4). Because $P_aO_a = H_aQ_a$, we see that $PS = C_pT$, and so on; thus the two inscribed triangles have equal areas, and the area of triangle HPO is half the area of $PHQO$. □

7. Centroids

Centroids are at the heart of several interesting relationships between the minor periambic triangles and the central parallelogram. Investigation of these properties is aided by the following lemma concerning the distances between various periambic feet.

Lemma 10. *In absolute distances,*

$$P_aH_a \sin\alpha = O_bH_b \sin\beta$$
$$P_bH_b \sin\beta = O_cH_c \sin\gamma$$
$$P_cH_c \sin\gamma = O_aH_a \sin\alpha,$$

and also

$$P_aO_a \sin\alpha = O_cH_c \sin\gamma$$
$$P_bO_b \sin\beta = O_aH_a \sin\alpha$$
$$P_cO_c \sin\gamma = O_bH_b \sin\beta\ .$$

Proof. Each identity is derived by calculating a triangle side length in two ways. Referring to Figure 4, triangle A_pC_qH is clearly similar to triangle ABC; furthermore, the altitude from A_p to HC_q has length P_aH_a. Thus $P_aH_a/HA_p = \sin\beta$, or $HA_p = P_aH_a \sin\beta$. But also, the altitude from H to A_pC_q has length O_bH_b, so $HA_p = O_bH_b \sin\alpha$. This proves the first identity. The others are derived similarly, respectively using triangle A_qB_pH and sides A_qH and B_pA_q; triangle C_pHS and sides C_pH and HS; triangle A_pOB_q and sides OB_q and B_qA_p; triangle OC_qB_p and sides OC_q and C_qB_p; and triangle TOC_p and sides C_pT and TO. □

Proposition 11. *The centroids of triangles $A_pB_pC_p$ and HPO coincide.*

Proof. First, assume that triangle ABC is not isosceles. Theorem 276 in [3] states: *If the vertices of one triangle lie on the sides of a second, and divide them in a fixed ratio, the triangles have the same centroid.* We show that A_p, B_p, and C_p divide the sides of triangle TSC_q (Figure 4) in just such a fixed ratio. We calculate ratios in two ways using different altitudes in a triangle and absolute distances. By construction, triangles TSC_q and ABC are similar; thus, in triangle B_pC_pS,

$$C_pS = \frac{O_aH_a}{\sin\beta} \quad \text{and} \quad SB_p = \frac{P_cO_c}{\sin\beta}\ . \tag{6}$$

In triangle A_pTC_p,

$$TC_p = \frac{O_bH_b}{\sin\alpha} \quad \text{and} \quad A_pT = \frac{P_cH_c}{\sin\alpha}\ , \tag{7}$$

and in triangle $A_pB_pC_q$,

$$C_qA_p = \frac{P_aH_a}{\sin\gamma} \quad \text{and} \quad B_pC_q = \frac{P_bO_b}{\sin\gamma}\ . \tag{8}$$

The assumption that no two vertex angles in triangle ABC are equal allows us to form ratios incorporating non-zero segment lengths from Equations (6), (7), and (8). Applying substitutions chosen from the identities in Lemma 10, we find that

$$\begin{aligned} \frac{TC_p}{C_pS} &= \frac{O_bH_b\sin\beta}{O_aH_a\sin\alpha} \\ \frac{C_qA_p}{A_pT} &= \frac{P_aH_a\sin\alpha}{P_cH_c\sin\gamma} = \frac{O_bH_b\sin\beta}{O_aH_a\sin\alpha} \\ \frac{SB_p}{B_pC_q} &= \frac{P_cO_c\sin\gamma}{P_bO_b\sin\beta} = \frac{O_bH_b\sin\beta}{O_aH_a\sin\alpha}\ . \end{aligned} \tag{9}$$

It follows by [3, Theorem 276] that $A_pB_pC_p$ and TSC_q share the same centroid. Because C_pP, A_pC_q, and B_pH are each bisected by the midpoints of ST, TC_q, and C_qS, respectively, P, O, and H divide those sides in a constant ratio, namely the inverse of the ratio determined by A_p, B_p, and C_p. Thus triangles TSC_q, $A_pB_pC_p$, and HPO share the same centroid.

Now, if triangle ABC is isosceles (but not equilateral), there are three cases to consider:

(1) If $\alpha = \beta$, then the triangle is symmetric around the coincident lines $o_c = h_c$. Thus $O_aH_a = O_bH_b$, and the ratio $O_aH_a \sin\alpha / O_bH_b \sin\beta$ in Equation (9) is equal to 1. This means that triangle $A_pB_pC_p$ is the medial triangle of triangle TSC_q, and shares its centroid.

(2) If $\beta = \gamma$, we have o_a coincident with h_a, $O_aH_a = 0$, and $O_aH_a \sin\alpha / O_bH_b \sin\beta = 0$. Thus S is coincident with C_p, T with A_p, and C_q with B_p. Consequently, triangle $A_pB_pC_p$ is coincident with triangle TSC_q, with a trivially shared centroid.

(3) If $\gamma = \alpha$, then o_b is coincident with h_b and $O_bH_b = 0$. The reciprocal of the ratio in (9), namely $O_aH_a \sin\alpha / O_bH_b \sin\beta$, is therefore equal to

0, and thus S is coincident with B_p, T with C_p, and C_q with A_p. As in the previous case, triangles $A_pB_pC_p$ and TSC_q and their centroids are coincident.

Finally, if triangle ABC is equilateral, then the altitude, perpendicular bisector, and p- and q-lines perpendicular to a given side are coincident, and triangles $A_pB_pC_p$ and TSC_q are reduced to a pair of coincident points.

A similar proof applies for triangles $A_qB_qC_q$ and OQH. □

Proposition 12. *The centroids of the minor periambic triangles lie on and trisect* PQ.

Proof. It has long been known (c.f. [2, Exercise 98]) that one of a parallelogram's diagonals divides it into two triangles whose centroids trisect the other diagonal. This means that the centroids of triangle PHO and triangle QOH trisect PQ, and the result follows directly by Proposition 11. □

8. Trilinear coordinates

$P:$		$Q:$	
x	$\frac{b^2-2a^2}{a}\cot\beta+\frac{a^2}{c}\csc\beta$	x	$\frac{c^2-2a^2}{a}\cot\gamma+\frac{a^2}{b}\csc\gamma$
y	$\frac{c^2-2b^2}{b}\cot\gamma+\frac{b^2}{a}\csc\gamma$	y	$\frac{a^2-2b^2}{b}\cot\alpha+\frac{b^2}{c}\csc\alpha$
z	$\frac{a^2-2c^2}{c}\cot\alpha+\frac{c^2}{b}\csc\alpha$	z	$\frac{b^2-2c^2}{c}\cot\beta+\frac{c^2}{a}\csc\beta$
$A_p:$		$A_q:$	
x	$\cos\beta$	x	$\cos\gamma$
y	$\cos\alpha$	y	$-\cos 2\alpha$
z	$-\cos 2\alpha$	z	$\cos\alpha$
$B_p:$		$B_q:$	
x	$-\cos 2\beta$	x	$\cos\beta$
y	$\cos\gamma$	y	$\cos\alpha$
z	$\cos\beta$	z	$-\cos 2\beta$
$C_p:$		$C_q:$	
x	$\cos\gamma$	x	$-\cos 2\gamma$
y	$-\cos 2\gamma$	y	$\cos\gamma$
z	$\cos\alpha$	z	$\cos\beta$

Table 2. Relative trilinear coordinates $x : y : z$
for P, Q, and the minor periambic points.

Table 2 shows relative trilinear coordinates for P, Q, and the minor periambic points. Trilinears of the minor periambic points are easily derived from the altitudes and perpendicular bisectors that define them. For instance, C_p is the intersection of the perpendicular bisector o_a

$$x \sin(\beta - \gamma) + y \sin \beta - z \sin \gamma = 0$$

(given in [6, Exercise 33]), and altitude h_b

$$x \cos \alpha - z \cos \gamma = 0 \ .$$

To calculate trilinear coordinates for P, we use the following information about the feet of the p-lines.

Lemma 13. *In directed distances,*

$$\begin{aligned} P_bA &= \frac{c^2}{2b} \ , & P_aC &= \frac{b^2}{2a} \ , & P_cB &= \frac{a^2}{2c} \ , \\ CP_b &= \frac{2b^2 - c^2}{2b} \ , & BP_a &= \frac{2a^2 - b^2}{2a} \ , & AP_c &= \frac{2c^2 - a^2}{2c} \ . \end{aligned} \tag{10}$$

Proof. We prove the identity for P_bA. P_C passes through A; let G be the other intersection point of P_C with CA. Observe that $p_b = \langle P_C, P_A \rangle$ is the inversion of P_A in P_C, thus $GA \cdot P_bA = c^2$. Since $GA = 2b$, we have $P_bA = c^2/2b$. Corresponding arguments produce the other equalities in row 1 of (10), and note that these quantities are always positive. Row 2 is obtained by subtracting the terms in row 1 from the respective sides of triangle ABC. □

Returning to P and its trilinears, assume first that triangle ABC is not a right triangle. Let $X_1 = p_a \wedge CA$, $X_2 = p_b \wedge AB$, and $X_3 = p_c \wedge BC$. From triangle P_aCX_1 and Lemma 13 we have

$$CX_1 = \frac{P_aC}{\cos \gamma} \ , \tag{11}$$

while the inversely similar right triangle PP_bX_1 yields

$$P_bX_1 = P_bP \tan \gamma \ . \tag{12}$$

In directed distances, $CP_b + P_bX_1 = CX_1$. From Lemma 13 and Equations (11) and (12), it follows that

$$\begin{aligned} P_bX_1 &= -CP_b + CX_1 \\ P_bP \tan \gamma &= \frac{c^2 - 2b^2}{2b} + \frac{P_aC}{\cos \gamma} \\ P_bP &= \frac{c^2 - 2b^2}{2b} \cot \gamma + \frac{b^2}{2a} \csc \gamma \ . \end{aligned} \tag{13}$$

This gives the distance from side b to P. The factor $1/2$ has been divided out of the relative coordinates for P in Table 2, since it occurs for P_aP and P_cP in similar derivations using X_2 and X_3.

If γ is a right angle, then P_bP is parallel to BC, and directly we have $P_bP = P_aC = b^2/2a$ in accordance with Equation (13).

The relative trilinears for Q are developed similarly.

9. Construction using isotomic points

The p- and q-circles were defined with radii equal to the side lengths of triangle ABC; for instance, P_B and Q_C each have radius a. Suppose B_a and C_a are isotomic conjugates on side a; that is, the segment B_aC_a is bisected by O_a [5]. Construct C_b and A_b on b such that C_aC_b and A_bB_a are parallel to c, and draw A_c and B_c on c so that A_bA_c and C_bB_c are parallel to a. Then C_b and A_b are isotomic conjugates on b, and A_c and B_c are isotomic conjugates on c. Furthermore, $A_bA/CA = B_cB/AB = C_aC/BC$ in this construction. We call the set of points $\{B_a, C_a, C_b, A_b, A_c, B_c\}$ a *proportional isotomic hexad.*

We define *isoperiambic circles* P'_x and Q'_x, $x \in \{A, B, C\}$, using centers and radii established above, as shown in Table 3. Radical axes p'_y, q'_y, and h'_y, $y \in \{a, b, c\}$, may then be defined by replacing P_x with P'_x and Q_x with Q'_x in Definitions 1–3. The points P', Q', and H' are the points of concurrency of the p'_y, q'_y, and h'_y, respectively; but note that the perpendicular bisectors and O remain unchanged when constructed with these new circles.

Circle	Center	Radius
P'_A	A	AB_c
P'_B	B	BC_a
P'_C	C	CA_b
Q'_A	A	AC_b
Q'_B	B	BA_c
Q'_C	C	CB_a

Table 3. Definitions of the isoperiambic circles on pairs of isotomic conjugates (B_a, C_a), (C_b, A_b), and (A_c, B_c).

Not surprisingly, the radical axes of the isoperiambic circles determine a parallelogram and pair of triangles (Figure 6). It can be shown that the "isoperiambic constellation" is identical to the periambic constellation of triangle ABC, but dilated around O.

Theorem 14. *Let B_a be a point on side a of triangle ABC. Construct the proportional isotomic hexad $\{B_a, C_a, C_b, A_b, A_c, B_c\}$ and radical axes p'_y, q'_y, and h'_y, $y \in \{a, b, c\}$, as described above. Let $A' = OA \wedge h'_a$, $B' = OB \wedge h'_b$, and $C' = OC \wedge h'_c$. Then the radical axes of the isoperiambic circles of triangle ABC are the radical axes of the periambic circles of triangle $A'B'C'$.*

Proof sketch. Let $d_1 = BB_a = CC_a$, $d_2 = BB_c = AA_c$, and $d_3 = AA_b = CC_b$. By construction, triangle AA_bA_c and triangle CC_aC_B are similar to triangle ABC,

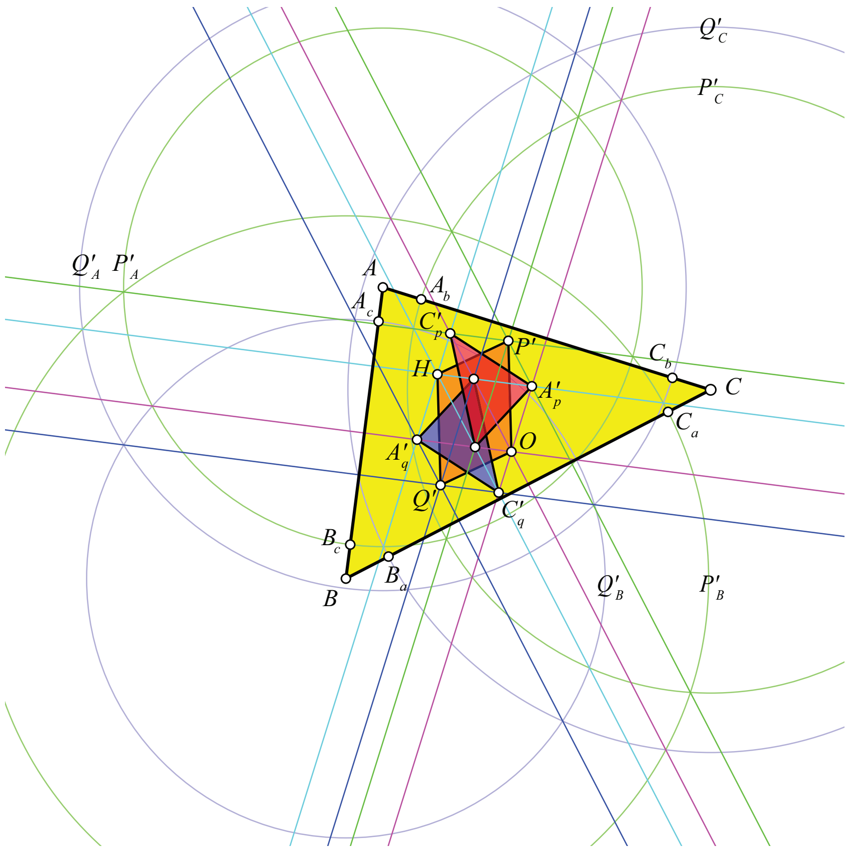

Figure 6. The isoperiambic constellation constructed on the proportional isotomic hexad $\{B_a, C_a, C_b, A_b, A_c, B_c\}$. Color key: green, p'-lines; blue, q'-lines; cyan, "altitudes" h'; magenta, perpendicular bisectors.

so

$$d_2 = \frac{cd_1}{a} \, ,$$
$$d_3 = \frac{bd_1}{a} \, .$$

H'_a lies on $\langle P'_C, Q'_B \rangle$, so its equal powers with respect to these circles yields

$$BH'^{\,2}_a - BA_c^2 = CH'^{\,2}_a - CA_b^2 \tag{14}$$
$$BH'^{\,2}_a - (c - d_2)^2 = CH'^{\,2}_a - (b - d_3)^2 \tag{15}$$
$$BH'^{\,2}_a - c^2 \left(1 - \frac{d_1}{a}\right)^2 = CH'^{\,2}_a - b^2 \left(1 - \frac{d_1}{a}\right)^2 \, . \tag{16}$$

Set $\kappa = \left(1 - \frac{d_1}{a}\right)^2$. Substituting $BH'_a = BO_a + O_aH'_a$, $CH'_a = CO_a - O_aH'_a$, and $BO_a = CO_a$ in Equation (14), one obtains

$$O_aH'_a = \kappa\frac{c^2 - b^2}{2a} = \kappa O_aH_a \ . \tag{17}$$

Repeating this argument on sides b and c, one shows that, in the isoperiambic construction, the distance between each parallel perpendicular bisector and altitude in triangle ABC is dilated by a factor of κ around O, which remains invariant. One can show that triangle $A'B'C'$ is a dilation of triangle ABC from the same center and with the same scaling factor. For instance, let Q be the foot of the perpendicular to H'_a from O. In triangle H_bBC we have $\angle CBH_b = \pi - \gamma$; therefore $\angle LOO_b = \pi - \gamma$, and $\angle QOA' = \pi + \beta - \gamma$. From Equation (17), Lemma 7, and the relation

$$\frac{OQ}{OA'} = \cos\angle QOA' = -\sin(\beta - \gamma) = \sin(\gamma - \beta) \ ,$$

we see that

$$OA' = \frac{OQ}{\sin(\gamma - \beta)} = \frac{O_aH'_a}{\sin(\gamma - \beta)} = \frac{\kappa R\sin(\gamma - \beta)}{\sin(\gamma - \beta)} = \kappa R = \kappa OA \ .$$

Proceeding in this way, it may be shown that the radical axes of the periambic circles of triangle $A'B'C'$ are merely those of the isoperiambic circles of triangle ABC, dilated by a factor of κ around O. Details are left to the interested reader. □

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