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Two Interesting Integer Parameters of Integer-sided Triangles

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Abstract. When a triangle is described in terms of the segments into which its sides are divided by an inscribed circle, it permits determination of all integer sided triangles for which the area is an integer multiple of the perimeter. It is not possible to have integer-sided triangles with R/r an integer, where R and r are the radii of the circumcircle and incircle respectively, for right triangles, isosceles triangles, and triangles whose sides are in arithmetic progression. The exception is the equilateral triangle for which $R/r = 2$.

First consider integer-sided triangles for which the area A is an integer multiple m of the perimeter P.

$$A = mP. \tag{1}$$

For the particular case of right triangles (Figure 1), equation (1) becomes:

$$1/2ab = m(a + b + c). \tag{2}$$

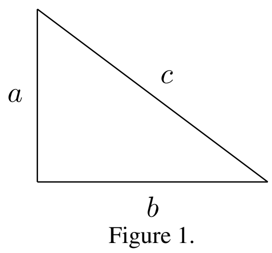

Figure 1.

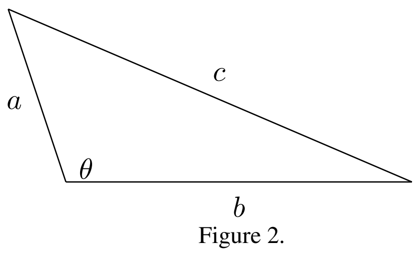

Figure 2.

Rearrangement gives

$$ab - 2m(a + b) = 2mc.$$

Squaring both sides of the equation, noting that $c^2 = a^2 + b^2$, gives

$$a^2b^2 + 4m^2(a + b)^2 - 4mab(a + b) = 4m^2(a^2 + b^2),$$

which can be simplified to

$$ab + 8m^2 - 4m(a + b) = 0.$$

This last equation can be factored to give

$$(a - 4m)(b - 4m) = 8m^2. \tag{3}$$

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The factors of $8m^2$ yield all right triangles satisfying equation (2). This solution was given by Goehl [1].

For a general triangle [2, 3, 4, 5, 6], with $a \leq b \leq c$, equation (2) is generalized to:

$$\frac{1}{2}ab\sin\theta = m(a+b+c). \tag{4}$$

Rearrangement gives:

$$ab\sin\theta - 2m(a+b) = 2mc.$$

Squaring both sides of the equation, noting that $c^2 = a^2 + b^2 - 2ab\cos\theta$, gives

$$a^2b^2\sin^2\theta + 4m^2(a+b)^2 - 4mab(a+b)\sin\theta = 4m^2(a^2+b^2-2ab\cos\theta)$$

which can be simplified to

$$ab\sin^2\theta + 8m^2 - 4m(a+b)\sin\theta = -8m^2ab\cos\theta.$$

This last equation can be factored to give

$$(a\sin\theta - 4m)(b\sin\theta - 4m) = 8m^2(1-\cos\theta). \tag{5}$$

Note that, of course, equation (5) reduces to equation (3) when $\theta = 90°$.

In order to determine allowed values of θ, consider the circle inscribed in the triangle. Now

$$\begin{aligned} a &= \alpha + \beta, \\ b &= \alpha + \gamma, \\ c &= \beta + \gamma. \end{aligned}$$

Since $a \leq b \leq c$, $\alpha \leq \beta \leq \gamma$.

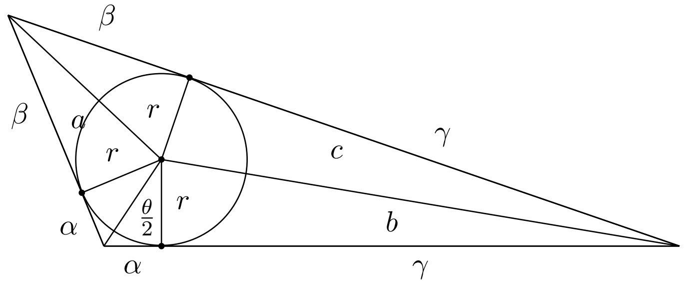

Figure 3.

From Heron's formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi perimeter and $P = 2s = 2(\alpha+\beta+\gamma)$.

The problem $A = mP$ is equivalent to $\sqrt{(\alpha+\beta+\gamma)(\gamma)(\beta)(\alpha(} = m \cdot 2(\alpha + \beta + \gamma)$. Simplification gives [5]

$$\alpha\beta\gamma = 4m^2(\alpha+\beta+\gamma). \tag{6}$$

Clearly there are no equilateral triangles as solutions because if $\alpha = \beta = \gamma$, equation (6) implies that $\alpha^3 = 4m^2(3\alpha)$ or $\alpha^2 = 12m^2$. So α is not an integer. Note that the radius of the incircle r is equal to $2m$ [2, 6]. The combined area of the 6 small right triangles in the figure that have r as one side is equal to the area of the triangle:

$$A = \frac{1}{2}(r\alpha + r\alpha + r\beta + r\beta + r\gamma + r\gamma) = r(\alpha+\beta+\gamma) = \frac{rP}{2} = mP,$$

so $r = 2m$.

From equation (6), it follows that $\alpha^2\beta\gamma = \alpha r^2(\alpha+\beta+\gamma)$ and factorization yields

$$(\alpha\beta - r^2)(\alpha\gamma - r^2) = r^2(\alpha^2 + r^2). \tag{7}$$

This result also follows from identifying the angle in equation (5).

From the above figure, $\sin\frac{\theta}{2} = \frac{r}{\sqrt{\alpha^2+r^2}}$ and $\cos\frac{\theta}{2} = \frac{\alpha}{\sqrt{\alpha^2+r^2}}$. Now, $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2r\alpha}{\alpha^2+r^2}$ and $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{\alpha^2-r^2}{\alpha^2+r^2}$. Substituting these values into equation (5) yields:

$$\left(a\left(\frac{2r\alpha}{\alpha^2+r^2}\right) - 2r\right)\left(b\left(\frac{2r\alpha}{\alpha^2+r^2}\right) - 4m\right) = 2r^2\left(1 - \frac{\alpha^2-r^2}{\alpha^2+r^2}\right).$$

Simplifying:

$$(a\alpha - (\alpha^2+r^2))(b\alpha - (\alpha^2+r^2)) = r^2(\alpha^2+r^2).$$

Substituting $a = \alpha+\beta$, and $b = \alpha+\gamma$ yields:

$$(\alpha\beta - r^2)(\alpha\gamma - r^2) = r^2(\alpha^2 + r^2).$$

Since c is the largest side, θ is the largest angle. The largest value of the smallest parameter, α, results when $\alpha = \beta = \gamma$ and $\theta = 60°$. So $\left(\tan\frac{\theta}{2}\right)_{\min} = \frac{r}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\frac{r}{\alpha} \geq \frac{1}{\sqrt{3}}$. Therefore allowed values for α are given by $1 \leq \alpha < \sqrt{3}r = 2\sqrt{3}m$.

From the figure, $\tan\frac{\theta}{2} = \frac{r}{\alpha}$ and $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2r\alpha}{\alpha^2-r^2}$.

Of course, $m = 1$ corresponds to area = perimeter.

For $m = 1$, $r = 2m = 2$. So $1 \leq \alpha < 2\sqrt{3} \approx 3.46$, and the values for α are 1, 2, and 3. Equation (7) becomes $(\alpha\beta - 4)(\alpha\gamma - 4) = 4(\alpha^2 + 4)$.

For $\alpha = 1$, $\tan\theta = \frac{4}{1-4} = -\frac{4}{3}$, $\theta = 126.8698976°$, and $(\beta-4)(\gamma-4) = 4(1+4)$. The three factorizations of 20, namely, $(4)(5)$, $(2)(10)$, and $(1)(20)$, all yield triangles:

$$\alpha,\ \beta,\ \gamma = \begin{cases} 1,\ 8,\ 9 \\ 1,\ 6,\ 14 \\ 1,\ 5, 24 \end{cases} \qquad \text{and} \qquad a,\ b,\ c = \begin{cases} 9,\ 10,\ 17 \\ 7,\ 15,\ 20 \\ 6,\ 25, 29 \end{cases}.$$

For $\alpha = 2$, $\theta = 90°$, and $(\beta-2)(\gamma-2) = 4+4$. Both factorizations of 8, (2)(4) and (1)(8), yield triangles:

$$\alpha,\ \beta,\ \gamma = \begin{cases} 2,\ 4,\ 6 \\ 2,\ 3,\ 10 \end{cases} \qquad \text{and} \qquad a,\ b,\ c = \begin{cases} 6,\ 8,\ 10 \\ 6,\ 12, 13 \end{cases}.$$

For $\alpha = 3$, $\tan\theta = \frac{12}{9-4} = \frac{12}{5}$, $\theta = 67.38013505°$, and $(3\beta - 4)(3\gamma - 4) = 4(9 + 4)$. The three factorizations of 52, namely, $(4)(13)$, $(2)(26)$, and $(1)(52)$, yield no new solutions because $\alpha < \beta$.

The results for $m = 1$ and $m = 2$ are shown in Table 1.

A	β	γ	α	B	c	$a+b+c$	m	θ
1	5	24	6	25	29	60	1	126.8698976
1	6	14	7	15	20	42	1	126.8698976
1	8	9	9	10	17	36	1	126.8698976
2	3	10	5	12	13	30	1	90
2	4	6	6	8	10	24	1	90
1	17	288	18	289	305	612	2	151.9275131
1	18	152	19	153	170	342	2	151.9275131
1	20	84	21	85	104	210	2	151.9275131
1	24	50	25	51	74	150	2	151.9275131
1	32	33	33	34	65	132	2	151.9275131
2	9	88	11	90	97	198	2	126.8698976
2	10	48	12	50	58	120	2	126.8698976
2	12	28	14	30	40	84	2	126.8698976
2	13	24	15	26	37	78	2	126.8698976
2	16	18	18	20	34	72	2	126.8698976
3	6	72	9	75	78	162	2	106.2602047
3	7	32	10	35	39	84	2	106.2602047
3	8	22	11	25	30	66	2	106.2602047
3	12	12	15	15	24	54	2	106.2602047
4	5	36	9	40	41	90	2	90
4	6	20	10	24	26	60	2	90
4	8	12	12	16	20	48	2	90
6	7	8	13	14	15	42	2	67.38013505

Table 1.

An interesting special case is that of isosceles triangles. For isosceles triangles, let $\beta = \gamma$, then $\alpha\beta^2 = 4m^2(\alpha + 2\beta)$. The result is the quadratic equation: $\alpha\beta^2 - 8m^2\beta - 4m^2\alpha = 0$. The solutions are:

$$\beta = \frac{2m}{\alpha}\left(2m \pm \sqrt{4m^2 + \alpha^2}\right). \tag{8}$$

The positive value is chosen. The restriction, $\alpha \leq \beta$, cannot be assumed.

For $m = 1$, $\beta = \frac{2}{\alpha}\left(2 + \sqrt{4 + \alpha^2}\right)$. To get integer solutions, $4 + \alpha^2$ must be the square of an integer, I^2.

So $I^2 - \alpha^2 = 4$, or $(I + \alpha)(I - \alpha) = 4$. The two possible factorizations, $(2)(2)$ and $(4)(1)$, result in no solutions. The first factorization gives $\alpha = 0$ while the second gives $\alpha = 2.5$.

For $m = 2$, $\beta = \frac{4}{\alpha}\left(4 + \sqrt{16 + \alpha^2}\right)$. To get integer solutions, $16 + \alpha^2$ must be the square of an integer, I^2.

So $I^2 - \alpha^2 = 16$, or $(I + \alpha)(I - \alpha) = 16$. The three possible factorizations, $(4)(4)$, $(8)(2)$, and $(16)(1)$, result in one solution. These factorizations give values for α of 0, 3, and 7.5. The one integer value for α also gives an integer value for β: 12.

The results for values of m from 2 through 6 are shown in Table 2. Note that there are no solutions for $m = 1$ and $m = 7$.

m	α	β	γ	a	b	c
2	3	12	12	15	15	24
3	8	12	12	20	20	24
4	6	24	24	30	30	48
5	15	15	24	39	39	30
6	5	60	60	65	65	120
6	9	36	36	45	45	72
6	16	24	24	40	40	48

Table 2.

Triangles with sides in arithmetic progression

Another interesting special case is that of triangles with sides in arithmetic progression. The problem has been studied before for consecutive integers [7]. Now consider the general case for an arithmetic progression $\alpha = \beta - \delta$ and $\gamma = \beta + \delta$. The sides become: $a = \alpha + \beta = 2\beta - \delta$, $b = \alpha + \gamma = 2\beta$, and $c = \beta + \gamma = 2\beta + \delta$. Now equation (6) becomes

$$(\beta - \delta)\beta(\beta + \delta) = 4m^2(\beta - \delta + \beta + \beta + \delta)$$

or

$$(\beta - \delta)(\beta + \delta) = 12m^2.$$

To find all triangles with sides in arithmetic progression, find all factorizations of $12m^2$ that result in integer values of β and δ.

For example, for $m = 1$, the possible factorizations are $(12)(1)$, $(6)(2)$, and $(4)(3)$. Only the second factorization gives integer values: $\beta = 4$ and $\delta = 2$. The resulting triangle has sides $a = 6$, $b = 8$, and $c = 10$.

The results for values of m from 1 through 4 are shown in Table 3.

m	δ	α	β	γ	a	b	c
1	2	2	4	6	6	8	10
2	11	2	13	24	15	26	37
2	4	4	8	12	12	16	20
2	1	6	7	8	13	14	15
3	2	26	28	30	54	56	58
3	6	6	12	18	18	24	30
4	47	2	49	96	51	98	145
4	22	4	26	48	30	52	74
4	13	6	19	32	25	38	51
4	8	8	16	24	24	32	40
4	2	24	26	28	50	52	54

Table 3.

A second interesting parameter associated with a triangle is the ratio of the radii of its circumcircle and incircle. MacLeod [8] has shown that, for triangles with sides a, b, and c, this ratio is given by:

$$N = \frac{2abc}{(a+b-c)(a+c-b)(b+c-a)}. \tag{9}$$

He points out that triangles for which this ratio is an integer are "relatively rare" and finds some of them. In fact, it will be shown that, with the exception of the equilateral triangle, no right triangles, isosceles triangles, or triangles whose sides are in arithmetic progression have an integer ratio.

First consider right triangles. Only primitive triangles need be considered. For such triangles, the sides can be represented by $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$, where m and n are relatively prime and of opposite parity. Then, from equation (9):

$$N = \frac{4mn(m^2-n^2)(m^2+n^2)}{(2n(m-n))(2n(m+n))(2m(m-n))}. \tag{10}$$

Simplification yields:

$$2n(m+n)N = m^2 + n^2. \tag{11}$$

This implies that n must divide m. The only possibility is $n = 1$. But then the left side of (11) would be even and the right side odd. Therefore there are no right triangles with integer N.

Next consider isosceles triangles. Letting $b = c$ in equation (9) gives $N = \frac{2b^2}{a(2b-a)}$. This may be written as a quadratic equation in a: $Na^2 - 2Nab + 2b^2 = 0$. This equation has solutions $a = \frac{b(N \pm I)}{N}$ where $I^2 = (N-1)^2 - 1$. Thus $N = 2$ and $I = 0$, and the only solution is $a = b = c$, the equilateral triangle.

Lastly consider triangles with sides in arithmetic progression. Letting $a = b + d$ and $c = b - d$ in equation (9) gives $N = \frac{2b(b^2-d^2)}{b^2-4d^2}$ or $\frac{b^2}{d^2} = \frac{2(2N-1)}{N-2}$. Thus $2(2N-1)(N-2) = J^2$ for some integer J. This may be rewritten as $(4N-5)^2 - (2J)^2 = (4N-5+2J)(4N-5-2J) = 9$. One factorization yields $J = 2$

and the non-integer $N = \frac{5}{2}$. The only other possible factorization yields $J = 0$ and $N = 2$, once again the equilateral triangle.

Conclusion

Insight is gained from the geometric approach to the solution of the problem of $A = mP$. Equation (5), in terms of the general angle, θ, leads immediately to equation (3) for right triangles and to equation (7) for the general case.

When a triangle is described in terms of the segments into which its sides are divided by an inscribed circle, equation (7) permits determination of all integer sided triangles for which the area is an integer multiple of the perimeter. The angle θ, opposite to the greatest side c, is shown to have a relatively small number of allowed values.

The ratio of the radii of circumcircle to incircle is considered.

The special cases of right triangles, isosceles triangles, and triangles with sides in arithmetic progression are solved generally for the two parameters.

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