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The First Sharp Gyrotriangle Inequality in Möbius Gyrovector Space $(\mathbb{D}, \oplus, \otimes)$

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Abstract. In this paper we present two different inequalities (with their reverse inequalities) written in a common type by using classical vector addition "$+$" and classical multiplication "$\cdot$" in Euclidean geometry and by using Möbius addition "$\oplus$" and Möbius scalar multiplication "$\otimes$" in Möbius gyrovector space $(\mathbb{D}, \oplus, \otimes)$. It is known that this Möbius gyrovector space form the algebraic setting for the Poincaré disc model of hyperbolic geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean geometry.

1. Introduction

There are many fundamental inequalities in mathematics and one of them is the famous "Triangle Inequality". In [7], Kato, Saito and Tamura presented the following "Sharp Triangle Inequality"in a Banach Space as follows:

Theorem 1. *For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X,*

$$\begin{aligned} &\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\| \\ &\leq \sum_{j=1}^{n} \|x_j\| \\ &\leq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \leq j \leq n} \|x_j\| \end{aligned}$$

holds.

Recently Mitani, Kato, Saito and Tamura [8] presented a new type Sharp Triangle Inequality as follows:

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Theorem 2. *For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X, $n \geq 2$ —small*

$$\begin{aligned}
&\|\sum_{j=1}^{n} x_j\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_k^*\| - \|x_{k+1}^*\|) \\
\leq &\sum_{j=1}^{n} \|x_j\| \\
\leq &\|\sum_{j=1}^{n} x_j\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|)
\end{aligned}$$

where $x_1^*, x_2^*, \cdots, x_n^*$ are the rearrangement of $x_1, x_2, \cdots, x_n$ satisfying $\|x_1^*\| \geq \|x_2^*\| \geq \cdots \geq \|x_n^*\|$ and $x_0^* = x_{n+1}^* = 0$.

2. A sharp triangle inequality in Banach space

Theorem 3. *For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X,*

$$\begin{aligned}
&\frac{\min_{1\leq j\leq n}\|x_j\|}{\max_{1\leq j\leq n}\|x_j\|}\left(\sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n} x_j\|\right) + \left\|\sum_{j=1}^{n} x_j\right\| \\
\leq &\sum_{j=1}^{n}\|x_j\| \qquad (1)\\
\leq &\frac{\max_{1\leq j\leq n}\|x_j\|}{\min_{1\leq j\leq n}\|x_j\|}\left(\sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n} x_j\|\right) + \left\|\sum_{j=1}^{n} x_j\right\|
\end{aligned}$$

holds.

Proof. Let us assume

$$\|x_r\| := \min\{\|x_1\|, \|x_2\|, \cdots, \|x_n\|\}$$

and

$$\|x_s\| := \max\{\|x_1\|, \|x_2\|, \cdots, \|x_n\|\}.$$

Clearly,

$$\frac{\|x_r\|}{\|x_s\|} \leq 1 \leq \frac{\|x_s\|}{\|x_r\|}$$

holds true. Using the generalized triangle inequality in X, we get

$$0 \leq \sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n} x_j\|.$$

Therefore, we immediately obtain

$$\begin{aligned} &\frac{\|x_r\|}{\|x_s\|}\left(\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|\right) + \left\|\sum_{j=1}^{n} x_j\right\| \\ &\leq \sum_{j=1}^{n}\|x_j\| \\ &\leq \frac{\|x_s\|}{\|x_r\|}\left(\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|\right) + \left\|\sum_{j=1}^{n} x_j\right\|. \end{aligned}$$

□

3. Gyrogroups and gyrovector spaces

The most general Möbius transformation of the complex open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

in the complex plane $\mathbb{C}$ is given by the polar decomposition [6]

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta} (z_0 \oplus z).$$

It induces the Möbius addition "$\oplus$" in the disc, allowing the Möbius transformation of the disc to be viewed as Möbius *left gyrotranslation*

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by rotation. Here $\theta \in \mathbb{R}$, $z_0 \in \mathbb{D}$ and Möbius substraction "$\ominus$" is defined by $a \ominus z = a \oplus (-z)$. Clearly $z \ominus z = 0$ and $\ominus z = -z$. The groupoid $(\mathbb{D}, \oplus)$ is not a group since it is neither commutative nor associative, but it has a group-like structure. The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration,

$$gyr : \mathbb{D} \times \mathbb{D} \to Aut(\mathbb{D}, \oplus)$$

given by the equation

$$gyr\,[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b} \tag{2}$$

where $Aut(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. Therefore, the *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyration in (2),

$$a \oplus b = gyr\,[a, b]\,(b \oplus a). \tag{3}$$

Coincidentally, the gyration $gyr[a, b]$ that repairs the breakdown of the commutative law of $\oplus$ in (3), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left* and *right gyroassociative laws*

$$\begin{aligned} a \oplus (b \oplus c) &= (a \oplus b) \oplus gyr\,[a, b]\, c \\ (a \oplus b) \oplus c &= a \oplus (b \oplus gyr\,[b, a]\, c) \end{aligned}$$

for all $a, b, c \in \mathbb{D}$.

Definition 1. *A groupoid* $(\mathbb{G}, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms*
(G1) For each $a \in G$*, there is an element* $0 \in G$ *such that* $0 \oplus a = a$*.*
(G2) For each $a \in G$*, there is an element* $b \in G$ *such that* $b \oplus a = 0$*.*
(G3) For all $a, b \in G$*, there exists a unique element* $gyr[a, b]c \in G$ *such that*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$$

(G4) For all $a, b \in G$*,* $gyr\,[a, b] \in Aut(\mathbb{G}, \oplus)$ *where* $Aut(\mathbb{G}, \oplus)$ *is automorphism group.*
(G5) For all $a, b \in G$*,* $gyr\,[a, b] = gyr\,[a \oplus b, b]$*.*

Definition 2. *A gyrogroup* $(\mathbb{G}, \oplus)$ *is gyrocommutative if its binary operation obeys the* gyrocommutative law
(G6) $a \oplus b = gyr\,[a, b]\,(b \oplus a)$ *for all* $a, b \in G$*.*

Clearly, with these properties, one can now readily check that the Möbius complex disc groupoid $(\mathbb{D}, \oplus)$ is a gyrocommutative gyrogroup. For more details, we refer [1]-[4].

Now define the secondary binary operation $\boxplus$ in $\mathbb{G}$ by

$$a \boxplus b = a \oplus gyr\,[a, \ominus b]\,b.$$

The primary and secondary operations of $\mathbb{G}$ are collectively called the dual operations of gyrogroups. Let a, b be two elements of a gyrogroup $(\mathbb{G}, \oplus)$. Then the unique solution of the equation

$$a \oplus x = b$$

for the unknown x is

$$x = \ominus a \oplus b$$

and the unique solution of the equation

$$x \oplus a = b$$

for the unknown x is

$$\begin{aligned} x &= b \boxminus a \\ &= b \boxplus (\ominus a). \end{aligned}$$

For further details see [3].

Identifying complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way:

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2.$$

Then the equations

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \operatorname{Re}(\overline{u}v) \\ \|\mathbf{u}\| &= |u|\,. \end{aligned} \tag{4}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathbb{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc $\mathbb{R}^2_1 = \left\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < 1\right\}$ of $\mathbb{R}^2$. In fact we get from that (4)

$$\begin{aligned} u \oplus v &= \frac{u+v}{1+\overline{u}v} \\ &= \frac{(1+u\overline{v})(u+v)}{(1+\overline{u}v)(1+u\overline{v})} \\ &= \frac{\left(1+\overline{u}v+u\overline{v}+|v|^2\right)u+\left(1-|u|^2\right)v}{1+\overline{u}v+u\overline{v}+|u|^2|v|^2} \qquad (5) \\ &= \frac{\left(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2\right)\mathbf{u}+\left(1-\|\mathbf{u}\|^2\right)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\ &= \mathbf{u}\oplus\mathbf{v} \end{aligned}$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_1$.

Let $\mathbb{V}$ be any inner-product space and

$$\mathbb{V}_s = \{v \in \mathbb{V} : \|v\| < s\}$$

be the open ball of $\mathbb{V}$ with radius $s > 0$. Möbius addition in $\mathbb{V}_s$ is motivated by (5). It is given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{\left(1+\left(2/s^2\right)\mathbf{u}\cdot\mathbf{v}+\left(1/s^2\right)\|\mathbf{v}\|^2\right)\mathbf{u}+\left(1-\left(1/s^2\right)\|\mathbf{u}\|^2\right)\mathbf{v}}{1+(2/s^2)\,\mathbf{u}\cdot\mathbf{v}+(1/s^4)\,\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \qquad (6)$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$. Without loss of generality, we may assume that $s = 1$ in (6). However we prefer to keep s as a free positive parameter in order to exhibit the results that in the limit as $s \to \infty$, the ball $\mathbb{V}_s$ expands the whole of its real inner product space $\mathbb{V}$, and Möbius addition $\oplus$ reduces to vector addition $+$ in $\mathbb{V}$, i.e.,

$$\lim_{s\longrightarrow\infty}\mathbf{u}\oplus\mathbf{v} = \mathbf{u}+\mathbf{v}$$

and

$$\lim_{s\to\infty}\mathbb{V}_s = \mathbb{V}.$$

Möbius scalar multiplication "$\otimes$" is given by the equation

$$r\otimes\mathbf{v} = s\tanh\left(r\tanh^{-1}\left(\|\mathbf{v}\|/s\right)\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \qquad (7)$$

where $r \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$, $\mathbf{v} \neq 0$ and $r \otimes 0 = 0$.

Definition 3 (Real inner product gyrovector spaces)**.** *A real inner product space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:*

(1) G is a subset of a real inner product space $\mathbb{V}$ called the carrier of G, $G \subset \mathbb{V}$,

from which it inherits its inner product, "$\cdot$" and norm "$\|\cdot\|$" which are invariant under gyroautomorphisms, that is,

$$gyr[u,v]a \cdot gyr[u,v]b = a \cdot b$$

for all points $a, b, u, v \in G$.
(2) G admits a scalar multiplication $\otimes$, possessing the following properties. For all real numbers r, r_1, r_2 and all points $a \in G$.
(III.1) $1 \otimes a = a$
(III.2) $(r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a$
(III.3) $(r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a)$
(III.4) $\frac{|r|\otimes a}{\|r\otimes a\|} = \frac{a}{\|a\|}$
(III.5) $gyr[u,v](r \otimes a) = r \otimes gyr[u,v]a$
(III.6) $gyr[r_1 \otimes v, r_2 \otimes v] = I$
(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one dimensional vectors

$$\|G\| = \{\pm\|a\| : a \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$, $a, b \in G$.
(III.7) $\|r \otimes a\| = |r| \otimes \|a\|$
(III.8) $\|a \oplus b\| \leq \|a\| \oplus \|b\|$.

Clearly, Möbius scalar multiplication possesses the properties above. For the proof of (III.8) in the complex unit disc $\mathbb{D}$ we refer [5].

Theorem 4. *A Möbius gyrogroup $(\mathbb{V}_s, \oplus)$ with Möbius scalar multiplication $\otimes$ in (7) forms a gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, see* [1].

Definition 4. *The gyrodistance function in $(\mathbb{V}_s, \oplus, \otimes)$ is given by*

$$d(\mathbf{A}, \mathbf{B}) = \|\mathbf{B} \ominus \mathbf{A}\|$$

for $\mathbf{A}, \mathbf{B} \in \mathbb{V}_s$, see [1].

The gyrodistance function in hyperbolic geometry gives rise to a gyrotriangle inequality which involves a gyroaddition $\oplus$. In contrast, the familiar hyperbolic distance function in the literature is designed so as to give rise to a triangle inequality which involves the addition $+$. The connection between the gyrodistance function and the standard hyperbolic distance function is described in [3].

4. A sharp gyrotriangle inequality and its reverse in Möbius gyrovector space $(\mathbb{D}, \oplus, \otimes)$

Lemma 5. *For all nonzero elements of $x_1, x_2, \cdots, x_n$ in $(\mathbb{D}, \oplus, \otimes)$*

$$\left|\bigoplus_{j=1}^{n} x_j\right| \leq \bigoplus_{j=1}^{n} |x_j| \tag{8}$$

holds, where $\bigoplus\limits_{j=1}^{n} |x_j| = |x_1| \oplus |x_2| \oplus \cdots \oplus |x_n|$, $|x_j| = |x_j \ominus 0|$ and $\oplus_{j=1}^{n} x_j = (\cdots((x_1 \oplus x_2) \oplus x_3) \oplus \cdots \oplus x_{n-1}) \oplus x_n$.

Proof. Firstly, the set

$$|\mathbb{D}| = \{\pm|a| : a \in \mathbb{D}\} \subset (-1, 1)$$

is a commutative group with the operation $\oplus$. Therefore, by the associativity property $\oplus$, we get

$$\left|\bigoplus_{j=1}^{n} u_j\right| = \left|\left(\bigoplus_{j=1}^{n-1} u_j\right) \oplus u_n\right| = \cdots = \left|u_1 \oplus \left(\bigoplus_{j=2}^{n} u_j\right)\right|$$

where $u_j \in [0,\ 1)$ for $1 \leq j \leq n$. For $x \in [0, 1)$, define the function

$$f_c(x) = x \oplus c = \frac{x + c}{1 + xc}$$

where c is a constant in $[0, 1)$. Since

$$f_c'(x) = \frac{1 - c^2}{(1 + xc)^2} > 0$$

holds true, we get that f_c must be a monotonically increasing function. Therefore,

$$\begin{aligned} |x_1 \oplus x_2 \oplus x_3| &= |(x_1 \oplus x_2) \oplus x_3| \\ &\leq |x_1 \oplus x_2| \oplus |x_3| \\ &\leq |x_1| \oplus |x_2| \oplus |x_3| \end{aligned}$$

by (III.8) and by the increasing property of $f_{|x_3|}$. Following this way, one can easily get

$$\begin{aligned} \left|\bigoplus_{j=1}^{n} x_j\right| &= \left|\left(\bigoplus_{j=1}^{n-1} x_j\right) \oplus x_n\right| \\ &\leq \left|\bigoplus_{j=1}^{n-1} x_j\right| \oplus |x_n| \\ &\leq |x_1| \oplus |x_2| \oplus \cdots \oplus |x_{n-1}| \oplus |x_n| \\ &= \bigoplus_{j=1}^{n} |x_j| \end{aligned}$$

by (III.8) and by the increasing properties of $f_{|x_n|}, f_{|x_{n-1}|}, \ldots, f_{|x_3|}$.

□

Theorem 6. *For all nonzero elements of* $x_1, x_2, \cdots, x_n$ *in* $(\mathbb{D}, \oplus, \otimes)$

$$\begin{aligned}
&\left(\frac{\min_{1\le j\le n}|x_j|}{\max_{1\le j\le n}|x_j|} \otimes \left(\bigoplus_{j=1}^{n}|x_j| \ominus \left|\bigoplus_{j=1}^{n} x_j\right|\right)\right) \oplus \left|\bigoplus_{j=1}^{n} x_j\right| \\
\le\ & \bigoplus_{j=1}^{n}|x_j| \\
\le\ & \left(\frac{\min_{1\le j\le n}|x_j|}{\max_{1\le j\le n}|x_j|} \otimes \left(\bigoplus_{j=1}^{n}|x_j| \ominus \left|\bigoplus_{j=1}^{n} x_j\right|\right)\right) \oplus \left|\bigoplus_{j=1}^{n} x_j\right|
\end{aligned}$$

holds.

Proof. Firstly, we have

$$\begin{aligned}
0 &\le \bigoplus_{j=1}^{n}|x_j| \boxminus \left|\bigoplus_{j=1}^{n} x_j\right| \\
&= \bigoplus_{j=1}^{n}|x_j| \ominus \left|\bigoplus_{j=1}^{n} x_j\right|
\end{aligned}$$

by (8). Let us assume

$$|x_r| := \min\{|x_1|, |x_2|, \cdots, |x_n|\}$$

and

$$|x_s| := \max\{|x_1|, |x_2|, \cdots, |x_n|\}.$$

Clearly we get

$$\frac{|x_r|}{|x_s|} \le 1 \le \frac{|x_s|}{|x_r|}.$$

Now define a function g_u on $\mathbb{R}^+$ by

$$\begin{aligned}
g_u(x) &= x \otimes u \\
&= \tanh(x \tanh^{-1} u)
\end{aligned}$$

where u is a constant in $[0, 1)$. Since

$$g_u'(x) = \frac{\tanh^{-1} u}{\cosh^2(x \tanh^{-1} u)} > 0,$$

we see that g_u is a monotonically increasing function on $\mathbb{R}^+$. Since $\frac{|x_r|}{|x_s|} \le 1 \le \frac{|x_s|}{|x_r|}$ holds, we get

$$g_w\left(\frac{|x_r|}{|x_s|}\right) \le g_w(1) \le g_w\left(\frac{|x_s|}{|x_r|}\right)$$

where $w := \bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \in [0,1)$. Hence we obtain

$$\begin{aligned} &\frac{|x_r|}{|x_s|} \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \\ &\leq 1 \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \\ &\leq \frac{|x_s|}{|x_r|} \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \end{aligned}$$

and this yields

$$\begin{aligned} &\left(\frac{|x_r|}{|x_s|} \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \right) \oplus \left| \bigoplus_{j=1}^{n} x_j \right| \\ \leq\ & \bigoplus_{j=1}^{n} |x_j| \\ \leq\ & \left(\frac{|x_s|}{|x_r|} \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \right) \oplus \left| \bigoplus_{j=1}^{n} x_j \right| . \end{aligned}$$

□

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