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A Cevian Locus and the Geometric Construction of a Special Elliptic Curve

Igor Minevich and Patrick Morton

Abstract. Given a triangle ABC, we determine the locus $\mathscr{L}$ of points P, for which the affine mapping $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a half-turn, where $T_P(ABC) = DEF$ is the cevian triangle of P, $T_{P'}(ABC)$ is the cevian triangle of the isotomic conjugate P' of P, and K is the complement map, with respect to ABC. This completes the determination of the points P for which the inconic $\mathcal{I} = \mathsf{M}(\tilde{\mathcal{C}}_O)$ of P, tangent to the sides of ABC at the points D, E, F, is congruent to the circumconic $\tilde{\mathcal{C}}_O$ of ABC whose center is $O = T_{P'}^{-1} \circ K(Q)$, where $Q = K(P')$. We show that the locus $\mathscr{L}$ is an elliptic curve minus six points, whose j-invariant is $j = \frac{2^4 11^3}{5^2}$, and use the cevian geometry of ABC and P to give a synthetic construction of this elliptic curve.

1. Introduction

In previous papers [7], [10], [12] we have studied several conics defined for an ordinary triangle ABC relative to a given point P, not on the sides of ABC or its anticomplementary triangle, including the inconic $\mathcal{I}$ and circumconic $\tilde{\mathcal{C}}_O$. These two conics are defined as follows. Let DEF be the cevian triangle of P with respect to ABC (i.e., the diagonal triangle of the quadrangle $ABCP$). Let K denote the complement map and ι the isotomic map for the triangle ABC, and set $P' = \iota(P)$ and $Q = K(P') = K(\iota(P))$. Furthermore, let T_P be the unique affine map taking ABC to DEF, and $T_{P'}$ the unique affine map taking ABC to the cevian triangle for P'.

The inconic $\mathcal{I}$ for P with respect to ABC is the unique conic which is tangent to the sides of ABC at the traces (diagonal points) D, E, F. (See [7], Theorem 3.9 for a proof that this conic exists.) If $\mathcal{N}_{P'}$ is the nine-point conic of the quadrangle $ABCP'$ relative to the line at infinity l_∞, then the circumconic $\tilde{\mathcal{C}}_O$ is defined to be $\tilde{\mathcal{C}}_O = T_{P'}^{-1}(\mathcal{N}_{P'})$. (See [10], Theorems 2.2 and 2.4; and [2], p. 84.) These two conics are generalizations of the classical incircle and circumcircle of a triangle, and coincide with these circles when the point $P = Ge$ is the Gergonne point of the triangle. In that case, the point $Q = I$ is the incenter. In general, the point Q is the center of the inconic $\mathcal{I}$. The center O of the circumconic $\tilde{\mathcal{C}}_O$ is given by the affine formula

$$O = T_{P'}^{-1} \circ K(Q),$$

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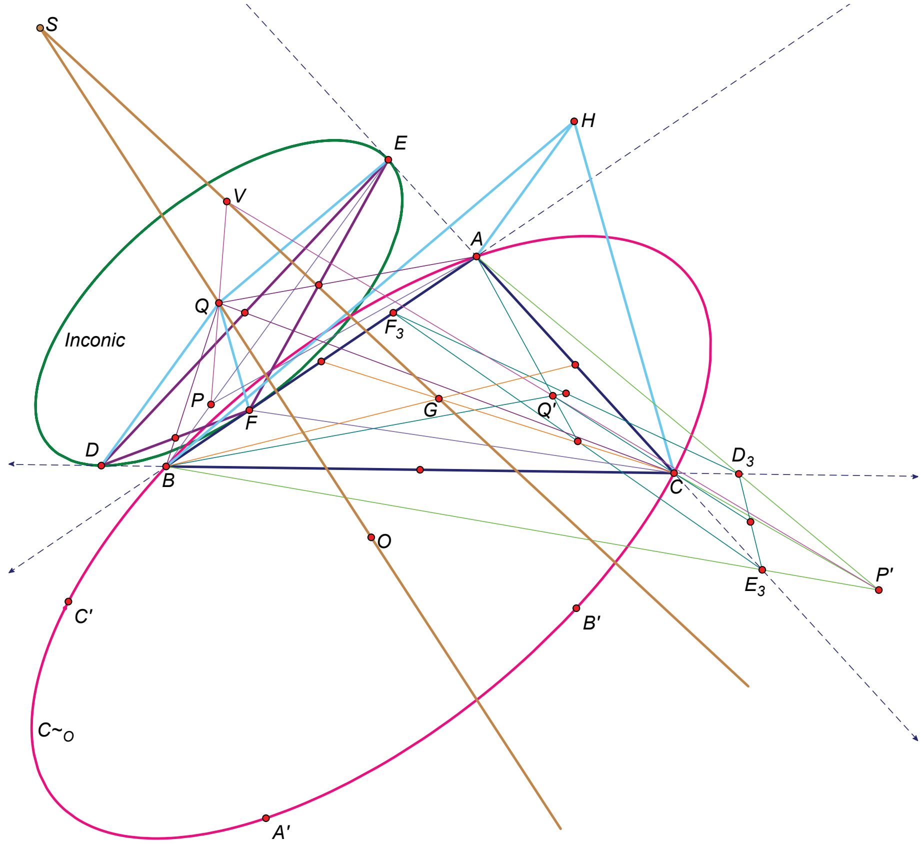

Figure 1. The conics $\tilde{\mathcal{C}}_O$ (strawberry) and $\mathcal{I}$ (green).

since the center of the conic $\mathcal{N}_{P'}$ turns out to be $K(Q)$.

We also showed in [10], Theorem 3.4, that the affine map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a homothety or translation which maps the circumconic $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$. If G is the centroid of ABC and $Q' = K(P)$, then the center of the map M is the point

$$S = OQ \cdot GV = OQ \cdot O'Q', \text{ where } V = PQ \cdot P'Q',$$

and $O' = T_P^{-1} \circ K(Q')$ is the generalized circumcenter for the point P'.

In [12] we showed that for a fixed triangle ABC the locus of points P, for which the map M is a translation, is an elliptic curve minus 6 points, and that this elliptic curve has infinitely many points defined over $\mathbb{Q}(\sqrt{2})$. Thus, there are infinitely many points P for which the conics $\mathcal{I}$ and $\tilde{\mathcal{C}}_O$ are congruent to each other. In this note we determine the remaining points P for which these two conics are congruent by determining (synthetically) the locus of points P for which the map M is a half-turn. We show, for example, that M being a half-turn is equivalent to the point P lying on the circumconic $\tilde{\mathcal{C}}_{O'}$, where $O' = T_P^{-1} \circ K(Q')$ is as above for the point

P', and this is also equivalent to the point $P' = \iota(P)$ lying on the circumconic $\tilde{\mathcal{C}}_O$. (By contrast, we showed in [12] that M is a translation if and only if the point P lies on $\tilde{\mathcal{C}}_O$.) This is interesting, since if $P = Ge$ is the Gergonne point of ABC, then $P' = Na$ is the Nagel point, which always lies *inside* the circumcircle $\tilde{\mathcal{C}}_O$. Given triangle ABC, the locus of all such points P turns out to be another elliptic curve (minus 6 points; see Theorem 9). As in [12], this elliptic curve can be constructed synthetically using a locus of affine maps defined for points on certain open arcs of a conic. In [12] the latter conic was a hyperbola, while here the conic needed to construct the elliptic curve is a circle. (See Figure 4.)

We adhere to the notation of [7]: $D_0E_0F_0$ is the medial triangle of ABC, with D_0 on BC, E_0 on CA, F_0 on AB (and the same for further points D_i, E_i, F_i); DEF is the cevian triangle associated to P; $D_2E_2F_2$ the cevian triangle for $Q = K \circ \iota(P)$; $D_3E_3F_3$ the cevian triangle for $P' = \iota(P)$; and G the centroid of ABC. As above, T_P and $T_{P'}$ are the unique affine maps taking triangle ABC to DEF and $D_3E_3F_3$, respectively. See [7] and [9] for the properties of these maps. Also, the generalized orthocenter for P with respect to ABC is the point $H = K^{-1}(O)$, which is also the intersection of the lines through the vertices A, B, C which are parallel, respectively, to the lines QD, QE, QF. Finally, the point Z is defined to be the center of the cevian conic $\mathcal{C}_P = ABCPQ$. (See [9].)

We also refer to the papers [7], [9], [10], and [11] as I, II, III, IV, respectively. See [1], [2], [3] for results and definitions in triangle geometry and projective geometry.

2. The locus of P for which M is a half-turn.

In this section we determine necessary and sufficient conditions for the map M to be a half-turn. We start with the following lemma.

Lemma 1. (a) *If the point P (not on a side of ABC or $K^{-1}(ABC)$) lies on the Steiner circumellipse $\iota(l_\infty)$ of ABC, then the map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a homothety with ratio $k = 4$, and is therefore not a half-turn.*

(b) *If the point P lies on a median of triangle ABC, but does not lie on the Steiner circumellipse $\iota(l_\infty)$ of ABC, then $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is not a half-turn.*

Proof. To prove (a), we use the result of I, Theorem 3.14, according to which P lies on $\iota(l_\infty)$ if and only if the maps T_P and $T_{P'}$ satisfy $T_P T_{P'} = K^{-1}$. If this condition holds, then because the map M is symmetric in P and P' (see III, Proposition 3.12b and IV, Lemma 5.2),

$$\mathsf{M} = T_{P'}K^{-1}T_P = T_{P'}T_PT_{P'}T_P = (T_{P'}T_P)^2 = (T_P^{-1}K^{-1}T_P)^2 = T_P^{-1}K^{-2}T_P.$$

The similarity ratio of the dilatation K^{-1} is -2, so the similarity ratio of K^{-2} is 4, which proves part (a).

For (b), suppose P lies on the median AG (G the centroid of ABC) and the map M is a half-turn. Then, since P' also lies on AG, we have that $D = D_0 = D_3$, so that

$$\mathsf{M}(A) = T_PK^{-1}T_{P'}(A) = T_PK^{-1}(D_3) = T_PK^{-1}(D_0) = T_P(A) = D = D_0$$

and the center S of M is the midpoint of AD_0. In particular, $\mathsf{M}(B)$ and $\mathsf{M}(C)$ are the reflections in S of B and C on the line $\ell = K^{-1}(BC)$. We claim that the line ℓ is tangent to the circumconic $\tilde{\mathcal{C}}_O$ at the point A. This is because the affine reflection ρ through the line $AG = AP$ in the direction of the line BC takes the triangle ABC to itself, and maps P to P, so it also takes the circumconic $\tilde{\mathcal{C}}_O$ to itself. Hence the tangent to $\tilde{\mathcal{C}}_O$ at A maps to itself, which implies that it must be parallel to BC (since the only other ordinary fixed line is AG, which lies on the center O of $\tilde{\mathcal{C}}_O$ and cannot be a tangent at an ordinary point). But ℓ is the unique line through A parallel to BC, so ℓ must be the tangent. (Also see III, Corollary 3.5.) It follows that the points $\mathsf{M}(B)$ and $\mathsf{M}(C)$, neither of which is A, must be exterior points of the conic $\tilde{\mathcal{C}}_O$. On the other hand, we claim that $D = D_0$ is an *interior* point of $\tilde{\mathcal{C}}_O$. This is because the segment BC, parallel to the tangent ℓ at A, is a chord of $\tilde{\mathcal{C}}_O$, and B and C lie on the same branch of $\tilde{\mathcal{C}}_O$, if the latter is a hyperbola (any tangent to a hyperbola separates the two branches). It follows that the segments $D\mathsf{M}(B)$ and $D\mathsf{M}(C)$ join the interior point D to exterior points, and so must each contain a point on $\tilde{\mathcal{C}}_O$. However, M is a half-turn mapping the circumconic $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$, so that $\mathsf{M}(\mathcal{I}) = \tilde{\mathcal{C}}_O$. Hence, $\tilde{\mathcal{C}}_O$ must be inscribed in the triangle $\mathsf{M}(ABC) = D\mathsf{M}(B)\mathsf{M}(C)$, meaning that $\tilde{\mathcal{C}}_O$ touches all three extended sides of the triangle. But by what we just showed the intersections of $\tilde{\mathcal{C}}_O$ with the sides of $D\mathsf{M}(B)\mathsf{M}(C)$ lie on the segments joining the vertices. Hence, the point D lies on the two tangents $b = D\mathsf{M}(B)$ and $c = D\mathsf{M}(C)$ to $\tilde{\mathcal{C}}_O$, implying that D is an *exterior* point of $\tilde{\mathcal{C}}_O$. This contradiction proves the lemma. $\square$

Proposition 2. *If the points P and P' are ordinary and do not lie on the sides or medians of triangles ABC and $K^{-1}(ABC)$, and H does not coincide with a vertex of ABC, the following are equivalent:*

(1) $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ *is a half-turn;*
(2) P *is on* $\tilde{\mathcal{C}}_{O'}$;
(3) P' *is on* $\tilde{\mathcal{C}}_O$;
(4) $T_P(P) = O'$;
(5) $T_{P'}(P') = O$;
(6) O' *lies on* $\mathcal{N}_P$;
(7) O *lies on* $\mathcal{N}_{P'}$.

Proof. First note that (4) $\iff$ (5): this follows on applying the affine reflection η from Part II to (4) (η is the harmonic homology with axis GZ and center $PP' \cdot l_\infty$), and using that

$$\eta(P) = P', \;\; \eta(O) = O', \;\text{ and }\; \eta \circ T_P = T_{P'} \circ \eta.$$

By III, Proposition 3.12 and IV, Lemma 5.2, M commutes with η, since η commutes with K and

$$\mathsf{M} \circ K^{-1} = (T_P \circ K^{-1}) \circ (T_{P'} \circ K^{-1}) = (T_{P'} \circ K^{-1}) \circ (T_P \circ K^{-1}).$$

Hence, $T_{P'} \circ K^{-1} \circ T_P = \eta \mathsf{M} \eta = \mathsf{M}$. This shows that the locus of points P for which M is a half-turn is invariant under $P \to P'$.

We now show that (1) is equivalent to (4) and (5). Namely, if M is a half-turn, then since $\mathsf{M}(O') = Q'$, we have to have $\mathsf{M}(Q') = O'$. But

$$\mathsf{M}(Q') = T_P K^{-1} T_{P'}(Q') = T_P K^{-1}(Q') = T_P(P),$$

and so $O' = T_P(P)$. Conversely, if $O' = T_P(P)$, then $\mathsf{M}(Q') = O'$, so $\mathsf{M}^2(O') = O'$, which implies that M must be a homothety with similarity ratio $k = \pm 1$, since M^2 fixes the point $O' \neq S$. However, k can't be $+1$, since in that case M is the identity and $O' = Q'$, impossible by the argument of III, Theorem 3.9. Therefore, $k = -1$, so M is a half-turn. (Note that $O' \neq S$, since otherwise $O = \eta(O') = \eta(S) = S$; but the points O, O', Q, Q' are distinct, by the proof of III, Theorem 3.9, as long as P does not lie on a median of ABC or on $\iota(l_\infty)$.) This shows that (1) $\iff$ (4) $\iff$ (5).

Furthermore, if (3) holds, then P' is on $\tilde{\mathcal{C}}_O$, so the latter conic lies on the vertices of quadrangle $ABCP'$ (since $\tilde{\mathcal{C}}_O$ is a circumconic), so the center O must lie on $\mathcal{N}_{P'}$, by definition of the 9-point conic. (See Part III, paragraph before Prop. 2.4.) Thus (3) implies (7). Also, (7) implies (3), because O being on $\mathcal{N}_{P'}$ implies O is the center of a conic on $ABCP'$. If O is not the midpoint of a side of ABC (which holds if and only if H is not a vertex), there is a unique such conic, namely $\tilde{\mathcal{C}}_O$. Hence P' lies on this conic. This shows that (3) $\iff$ (7). Similarly, (2) $\iff$ (6).

Now suppose that (3) holds. Then (7) holds, so $P' \in \tilde{\mathcal{C}}_O$. But $P' \in \mathcal{C}_P$, so P' is the fourth intersection of the circumconics $\mathcal{C}_P$ and $\tilde{\mathcal{C}}_O$, and therefore coincides with the point $\tilde{Z} = R_O K^{-1}(Z)$, by III, Theorem 3.14; here R_O is the half-turn about O and Z is the center of $\mathcal{C}_P$. Now, in the proof of III, Theorem 3.14 we showed that $T_{P'}(\tilde{Z}) = T_{P'}(P')$ lies on OP'. But $T_{P'}(P')$ lies on OQ (III, Proposition 3.12), so this forces $T_{P'}(P') = O$, i.e. (5), provided we can show that the line OP' is distinct from OQ.

However, if $OP' = OQ$, then $OP' = P'Q = QG$ so O, Q, G are collinear. Then $K^{-1}(O) = H$ is also on this line, so Q, H, P' are all on this line. We claim that these three points, Q, H, P', must all be distinct by our hypothesis on P. If $P' = Q = K(P')$ then $P' = G = P$, which can't hold because P is not on a median of ABC. If $Q = H$, then using the map $\lambda = T_{P'} \circ T_P^{-1}$ and III, Theorem 2.7 gives $Q = \lambda(H) = \lambda(Q) = P'$, so $P' = G$. Finally, if $P' = H$, then taking complements gives that $Q = O = T_{P'}^{-1}(K(Q))$ so $T_{P'}(Q) = K(Q)$, implying (by I, Theorem 3.7) that $P' = K(Q) = K^2(P')$ and therefore $P' = G = P$ once again. Therefore, the three distinct points Q, H, P', which all lie on the conic $\mathcal{C}_P$, are collinear, which is impossible. This shows that $OP' \cap OQ = O$, so $T_{P'}(P') = O$ and (3) $\Rightarrow$ (5) $\Rightarrow$ (1).

For the rest, it suffices to show that (1) $\Rightarrow$ (3). This is because of the symmetry of M in P and P': for example, (3) $\Rightarrow$ (1) $\Rightarrow$ (2) (switching P and P') and conversely, so (2) and (3) are equivalent, as are (6) and (7), and everything is equivalent to (1). Now assume (1). We will prove (7). Since (1) implies (5), we know $T_{P'}(P') = O$, so $T_{P'}(OP') = K(Q)O$ by the formula for O. But by III, Corollary 3.13(b) we know $K^{-1}(Z)$ lies on OP', so $T_{P'} \circ K^{-1}(Z) = Z$ (III, Prop. 3.10) lies on $K(Q)O$.

But Z also lies on $QN = K(P'O)$, which is parallel to OP'. This easily implies Z is an ordinary point and $QZ \parallel OP'$; for this, note $Q \neq Z$ since Z is the center of the conic $\mathcal{C}_P$, while Q is an ordinary point lying *on* $\mathcal{C}_P$. Also, $Z \neq K(Q)$, since $T_P \circ K^{-1}(Z) = Z$, while $T_P \circ K^{-1}(K(Q)) = Q$. Therefore, using the fact that the lines OP' and OQ are distinct, it follows that $QK(Q)Z$ and $P'K(Q)O$ are similar triangles. But $K(Q)$ is the midpoint of segment $P'Q$, so these triangles must be congruent. Hence, $K(Q)$ is also the midpoint of segment OZ, and O is the reflection of Z in the point $K(Q)$. Since $K(Q)$ is the center of $\mathcal{N}_{P'}$ and Z lies on $\mathcal{N}_{P'}$, this means O also lies on $\mathcal{N}_{P'}$, which is (7). $\square$

Corollary 3. *With the hypotheses of Proposition 2, the map* M *is a half-turn if and only if* $K^{-1}(S) = Z$*, which holds if and only if* $QZP'O$ *is a parallelogram.*

Proof. If M is a half-turn, the above argument shows that segments $QZ \cong OP'$; hence, $QZP'O$ is a parallelogram, and $P'Z \parallel QO$. Since $K(P') = Q$, this implies that line $P'Z$ is the same as the line $K^{-1}(QO) = P'H$ and Z is the midpoint of $P'H$, since the map K^{-1} doubles lengths of segments. But S is the midpoint of QO, so $K^{-1}(S) = Z$ is the midpoint of $P'H$. Conversely, if $K^{-1}(S) = Z$ then Z lies on $K^{-1}(OQ) = P'H$ (since S lies on OQ) and because $QZ \parallel P'O$, $QZP'O$ is a parallelogram with center $K(Q)$, the midpoint of $P'Q$. Hence, O lies on $\mathcal{N}_{P'}$ and M is a half-turn. $\square$

Remark. The condition of Corollary 3 is a necessary condition for M to be a half-turn, without the hypothesis that H not be a vertex. This follows from the last paragraph in the proof of Proposition 2, since that hypothesis is not used to prove that (1) $\Rightarrow$ (7). Lemma 1 then shows that $K^{-1}(S) = Z$ is a necessary condition for M to be a half-turn, without any extra hypotheses.

In the following proposition we will make use of the polarity induced by the conic $\mathcal{C}_P = ABCPQ = PQP'Q'H$.

Proposition 4. *If the hypotheses of Proposition 2 hold and the map* M *is a half-turn, then:*

(1) *The lines* $O'P$ *and* OP' *are tangents to the conic* $\mathcal{C}_P$ *at* P *and* P'*.*
(2) *The point* $V = PQ \cdot P'Q'$ *is the midpoint of segment* OO' *and line* $OO' = K^{-1}(PP')$*.*
(3) *On the line* GV*, the signed ratios* $\frac{GS}{SV} = \frac{5}{3}$ *and* $\frac{ZG}{GV} = \frac{5}{4}$*.*

Proof. (See Figure 2.) (1) As in the proof of Proposition 2 we have $P' = \tilde{Z} = R_O \circ K^{-1}(Z)$, so $K^{-1}(Z)$ lies on OP'; by symmetry, it also lies on $O'P$. We will show that pole of PP' is $K^{-1}(Z)$. Then (1) follows, since $K^{-1}(Z)$ is conjugate to both P and P', and so lies on the polars of P and P', which are the tangents to $\mathcal{C}_P$ at P and P'. Hence, $K^{-1}(Z)P = O'P$ and $K^{-1}(Z)P' = OP'$ are tangents to $\mathcal{C}_P$.

We do this by showing that $T_{P'}(P)$ and $T_P(P')$ are conjugate to $K^{-1}(Z)$. This implies that the polar of $K^{-1}(Z)$ is the join of $T_{P'}(P)$ and $T_P(P')$, which is PP' by II, Corollary 2.2(c). By symmetry it suffices to consider $T_{P'}(P)$. We use the

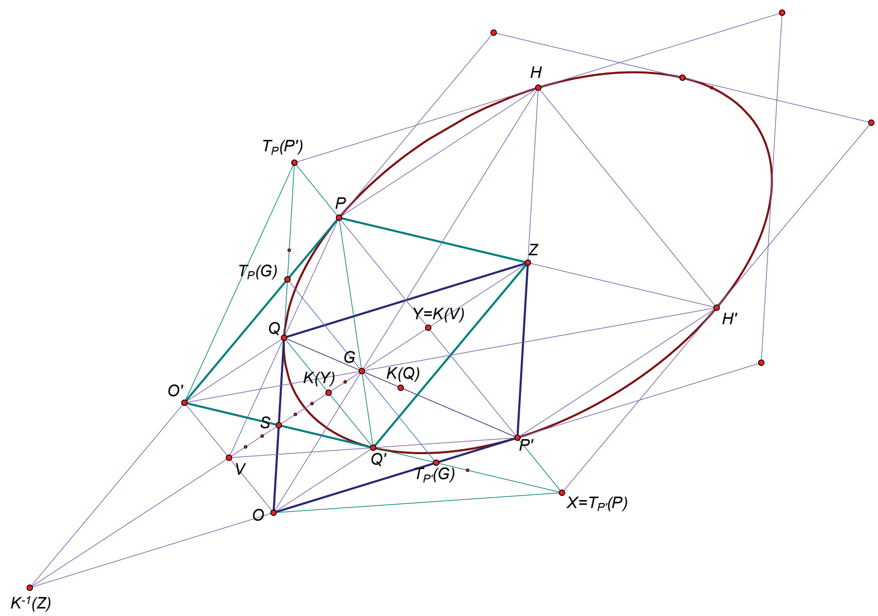

Figure 2. The parallelograms $QZP'O$ and $Q'ZPO'$ when M is a half-turn.

fact from Part IV, Prop. 3.10, that $O'Q'$ is tangent to $\mathcal{C}_P$ at Q'. Applying the map λ gives that $\lambda(O'Q')$ is tangent to $\lambda(\mathcal{C}_P) = \mathcal{C}_P$ at $\lambda(Q') = H'$, by II, Theorem 3.2 and III, Theorem 2.7. Using $T_P(P) = O'$ from Proposition 2 we know that

$$\lambda(O') = T_{P'} \circ T_P^{-1}(O') = T_{P'}(P),$$

so $T_{P'}(P)$ lies on the tangent to $\mathcal{C}_P$ at H' and is conjugate to H'. Also, P, Q', $K^{-1}(P)$ are collinear points, so applying the map $T_{P'}$ gives that $T_{P'}(P)$ is collinear with $T_{P'}(Q') = Q'$ and

$$T_{P'} \circ K^{-1}(P) = T_{P'} \circ K^{-1} \circ T_P(Q') = \mathsf{M}(Q') = O'.$$

Therefore, $T_{P'}(P)$ lies on the tangent $O'Q'$ and so is conjugate to the point Q'. Thus, the polar of $T_{P'}(P)$ is $Q'H'$, which lies on $K^{-1}(Z)$ since $Z, K(Q'), O' = K(H')$ are collinear, using the fact from Corollary 3 that $Q'ZPO'$ is a parallelogram and $K(Q')$ is the midpoint of the diagonal $Q'P$. This shows that $K^{-1}(Z)$ is conjugate to $T_{P'}(P)$, as desired. Note that this also shows that $T_{P'}(P) \neq T_P(P')$, since the polar of $T_P(P')$ is QH, and QH cannot be the same line as $Q'H'$, since the four points Q, H, Q', H' all lie on the conic $\mathcal{C}_P$.

(2) From the fact that OP' and OQ (IV, Prop. 3.10) are tangent to $\mathcal{C}_P$, it follows that $P'Q$ is the polar of O with respect to $\mathcal{C}_P$, and likewise, PQ' is the polar of O'. Hence, the pole of OO' is $P'Q \cdot PQ' = G$. On the other hand, the polar of G is the line VV_∞, where V_∞ is the infinite point on the line PP', by II, Proposition 2.3(a). Therefore, V lies on OO'. Since GV is the fixed line of the affine reflection η, V

must be the midpoint of segment OO'. Then V lies on the parallel lines $K^{-1}(PP')$ (II, Prop. 2.3(e)) and OO', so $K^{-1}(PP') = OO'$.

(3) From (2) we have $PP' = K(OO') = K^2(HH') = NN'$, where $N = K(O)$ and $N' = K(O')$ are the centers of the nine-point conics $\mathcal{N}_H$ and $\mathcal{N}_{H'}$. Thus, the line PP' is halfway between the parallel lines OO' and HH'. Taking complements, QQ' is halfway between $NN' = PP'$ and OO'. Also, the center S of the map M is located halfway between the lines QQ' and OO', since $\mathsf{M}(OO') = QQ'$. Let $X = T_{P'}(P) = O'Q' \cdot PP' = O'S \cdot PP'$ and $Y = K(V) = GV \cdot PP'$. Then triangles $VO'S$ and YXS are similar, with similarity ratio $1/3$, because S is the midpoint of segment $O'Q'$ and Q' is the midpoint of segment $O'X$, so that

$$|SX| = |SQ'| + |Q'X| = |SO'| + 2|SO'| = 3|SO'|.$$

Furthermore, V on OO' implies that $\mathsf{M}(V)$ is on QQ', so that $\mathsf{M}(V) = QQ' \cdot GV$ is the midpoint of $VY = VK(V)$ and therefore coincides with $K(Y)$. Then from $|SK(Y)| = |S\mathsf{M}(V)| = |SV|$, $|K(Y)G| = \frac{1}{3}|K(Y)Y|$, and $|K(Y)Y| = |VK(Y)| = |V\mathsf{M}(V)| = 2|SV|$ we find that

$$\begin{aligned}\frac{|GS|}{|SV|} &= \frac{|SK(Y)| + |K(Y)G|}{|SV|} = \frac{|SV| + |K(Y)Y|/3}{|SV|} \\ &= \frac{|SV| + 2|SV|/3}{|SV|} = \frac{5}{3}.\end{aligned}$$

Since S lies between G and V, this proves $\frac{GS}{SV} = \frac{5}{3}$. Now $\frac{ZG}{GV} = \frac{2GS}{8SV/3} = \frac{3}{4}\frac{GS}{SV} = \frac{5}{4}$. $\square$

Remark. The conditions of Proposition 4 are also sufficient for M to be a half-turn. We leave the verification of this for parts (1) and (2) to the reader. We will verify this for condition (3) in the next section.

3. Constructing the elliptic curve locus.

Now suppose a parallelogram $QZP'O$ is given, and with it: $K(Q)$ as the midpoint of QP'; G as the point for which the signed distance QG satisfies $QG = \frac{1}{3}QP'$; and $S = K(Z)$ (Corollary 3). Then Proposition 4 shows that the point V is determined by G and S. This determines, in turn, the points P and Q' uniquely, since P is the reflection in Q of the point V (see II, p. 26) and $Q' = K(P)$. Further, O' is also determined as the reflection of Q' in S, or as the reflection of O in V. Therefore, the parallelogram determines P, Q, P', Q' and $H = K^{-1}(O)$, and hence the conic $\mathcal{C}_P$ on these 5 points (by III, Theorem 2.8). Thus, any triangle ABC for which M is a half-turn with the given parallelogram $QZP'O$ must be inscribed in the conic $\mathcal{C}_P$. Furthermore, the affine maps T_P and $T_{P'}$ are also determined, since

$$T_P(PQQ') = O'QP \quad \text{and} \quad T_{P'}(P'QQ') = OP'Q'. \tag{1}$$

Defining the maps T_P and $T_{P'}$ by (1), we will show that $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a half-turn about S.

Lemma 5. *Given collinear and distinct ordinary points G, V, Z and an ordinary point P not on GZ, if $Q' = K(P)$ and P' is the reflection of V in the point Q', and Q is the midpoint of PV, then:*

(a) *there is a unique conic $\mathcal{C}$ with center Z which lies on P, P', Q, and Q';*

(b) *with respect to any triangle ABC with vertices on $\mathcal{C}$ whose centroid is G, and whose vertices do not coincide with any of the points P, P', Q or Q', the point P' is the isotomic conjugate of P, and $\mathcal{C}$ coincides with the conic $\mathcal{C}_P$ for ABC.*

Proof. For (a), first note that PQ', $P'Q'$, and PQ do not lie on Z, since $PQ' \cdot GZ = G$ and $P'Q' \cdot GZ = PQ \cdot GZ = V$, which are distinct from Z by assumption. Suppose that Z is not on PP'. The point G, being $2/3$ of the way from P to the midpoint Q' of VP', is the centroid of triangle PVP'. Hence, $K(P') = Q$ and $VG = GZ$ is a median of triangle PVP', implying that GZ intersects PP' at the midpoint of segment PP'. Also, $QQ' \parallel PP'$, so $V_\infty = PP' \cdot QQ'$ is on the line at infinity.

Now let $\mathcal{C}$ be the conic with center Z, lying on the points P, Q, P'. This exists and is unique, since Z does not lie on $PQ, P'Q$ or PP'. With respect to this conic, Z is conjugate to V_∞, and so is the midpoint $GZ \cdot PP'$, since P and P' lie on $\mathcal{C}$. Since Z is not on PP', GZ is the polar of V_∞. Now, V_∞ lies on QQ', so the point $Q_m = GZ \cdot QQ'$, which is the midpoint of segment QQ', is conjugate to V_∞. Let Q^* be the second intersection of QQ' with C. Note that $Q^* \neq Q$; otherwise V_∞ would be conjugate to Q, so Q would lie on its polar GZ, implying that P also lies on GZ, which is contrary to assumption. Hence, V_∞ is conjugate to the midpoint of QQ^*, which must be Q_m. This implies that $Q^* = Q'$, so Q' lies on $\mathcal{C}$.

Now suppose Z is on PP'. Then Z is not on QQ', since $QQ' \parallel PP'$, so there is a unique conic $\mathcal{C}$ through P, Q, Q' with center Z. As above, the pole of GZ is V_∞, and switching the point pairs P, P' and Q, Q' in the argument above gives that P' lies on $\mathcal{C}$.

For (b), the triangle ABC determines the conic $\mathcal{C} = \mathcal{C}_P = ABCPQ'$, since P and Q' cannot lie on any of the sides of ABC and P does not lie on a median of ABC (see the proof of II, Theorem 2.1). We know that this conic has center Z, since ABC is inscribed in $\mathcal{C}$. Furthermore, the pole of GZ with respect to $\mathcal{C}$ is $V_\infty = l_\infty \cdot PP'$, as above. But the isotomic conjugate P^* of P with respect to ABC is the unique point $P^* \neq P$ on PV_∞ lying on the conic $\mathcal{C}_P$ (see II, p. 26), so that means $P^* = P'$. □

Lemma 6. *With the assumptions of Lemma 5, suppose the signed ratio $\frac{ZG}{GV} = \frac{5}{4}$. Then the tangent to the conic $\mathcal{C}$ at Q is $K(P'Z) = QK(Z)$.*

Proof. (See Figure 3.) As in the proof of Proposition 4, in triangle $PP'V$, the midpoint of PP' is the complement $Y = K(V)$ of V, and so the midpoint of QQ' is the complement $K(Y)$ of Y. Let $U = K^{-1}(V)$. Then $GV = 2GY = 4GK(Y)$ and $ZG = 5GK(Y)$, so $ZY = ZG - GY = 3GK(Y)$. In addition, $UG = 8GK(Y)$ so $UZ = UG - ZG = 3GK(Y) = ZY$; hence, Z is the midpoint of YU. Let H' be the reflection of P in Z. Then triangles $H'UZ$ and PYZ are

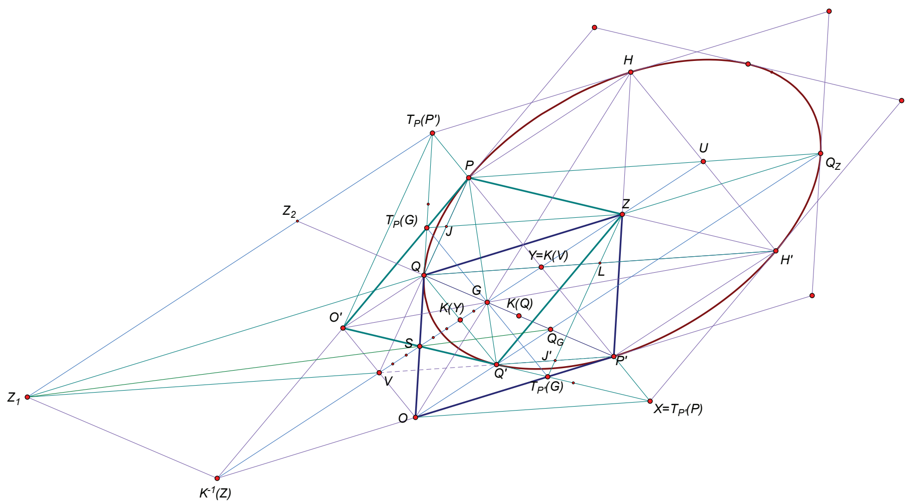

Figure 3. Proof of Lemma 3.2.

congruent so $H'U \parallel PY = P'Y$. Also, triangles PYZ and $PP'H'$ are similar (SAS), so $P'H' \parallel YZ = UY$. This implies $P'H'UY$ is a parallelogram and $P'H' = UY = VY$. Hence $P'H'YV$ is also a parallelogram and $H'Y \parallel P'V$. Since triangles QYP and $VP'P$ are similar (with similarity ratio $1/2$), $QY \parallel P'V$, so Y lies on $H'Q$.

Let Q_G and Q_Z be the reflections of Q in G and Z, respectively. Then $QQ_GQ_Z \sim QGZ$, so the line Q_GQ_Z is parallel to GZ and lies halfway between P' and GZ. Hence, Q' lies on Q_GQ_Z. Also, U lies on PQ_Z, since P, U, Q_Z are the reflections of H', Y, Q in Z, and we proved H', Y, Q are collinear in the previous paragraph. Moreover, $PQ_Z \parallel H'Q \parallel P'V$, since $P'H'YV$ is a parallelogram.

Let Z_1 be the reflection of Q_Z in Q. Then $Z_1QV \cong Q_ZQP$ (SAS), so $Z_1V \parallel PQ_Z \parallel P'V$, hence Z_1 lies on $P'V$. Now let J' be the midpoint of $P'Q'$ and L the midpoint of QH'. Then ZL is a midline in triangle PQH', so $ZL \parallel PQ = QV$. Since $P'Q' \parallel H'Q = QL$, and ZL lies in the conjugate direction to QH', it follows that Z, L, J' are collinear. Hence, $Z_1QV \sim Z_1ZJ'$, which implies, since $Z_1Q = 2QZ$, that $Z_1V = 2VJ' = 2 \cdot (\frac{3}{4}VP') = \frac{3}{2}VP'$. Let $Z^* = K^{-1}(Z)$. Then $Z^*G = 2 \cdot GZ = \frac{5}{2} \cdot VG$, by hypothesis, so $Z^*V = \frac{3}{2} \cdot VG$. Hence, triangles Z_1Z^*V and $P'GV$ are similar (SAS) and $Z_1Z^* \parallel P'G$. Let $S = Z_1Q_G \cdot GZ$. Since $Z_1Z^* \parallel Q_GG$, we have similar triangles Z_1Z^*S and Q_GGS. Moreover, $Z_1ZS \sim Z_1Q_ZQ_G$, since $SZ = GZ \parallel Q_GQ_Z$, with $Z_1Z = 3ZQ_Z$, so $Z_1S = 3SQ_G$. It follows that $Z^*S = 3SG$ and therefore $Z^*G = 4SG = 2GZ$. This implies $S = K(Z)$.

Let Z_2 be the intersection of $P'G$ with the line through Z_1 parallel to GZ. Also, let Z_∞ be the point at infinity on GZ. Then we have the following chain of perspectivities:

$$GVZK(Y) \overset{Q}{\overline{\overline{\wedge}}} P'VZ_1Q' \overset{Z_\infty}{\overline{\overline{\wedge}}} P'GZ_2Q_G \overset{Z_1}{\overline{\overline{\wedge}}} VGZ_\infty S.$$

The resulting projectivity on the line GZ is precisely the involution of conjugate points on GZ with respect to $\mathcal{C}$, because G and V are conjugate points (they are vertices of the diagonal triangle GVV_∞ of the inscribed quadrangle $PP'QQ'$) and the polar of Z is the line at infinity, which intersects GZ in Z_∞. This gives that $K(Y)$ is conjugate to S. But S is also conjugate to V_∞, since S lies on its polar GV. This implies that the polar of S is $K(Y)V_\infty = QQ'$. Thus, the tangent to $\mathcal{C}$ at Q is $QS = K(P'Z)$. □

Proposition 7. *Under the assumptions of Lemmas 5 and 6, for any triangle ABC with vertices on $\mathcal{C}$ whose centroid is G, and whose vertices do not coincide with any of the points P, P', Q or Q', the map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a half-turn.*

Proof. By Lemmas 5 and 6, the tangent at Q to $\mathcal{C}_P$ goes through $K(Z)$. But the tangent at Q is OQ (IV, Prop. 3.10), hence the generalized insimilicenter S for ABC is $S = OQ \cdot GZ = K(Z)$, where Z is the center of $\mathcal{C}_P = \mathcal{C}$. Now the proposition follows from Corollary 3. □

Theorem 8. *Let G, V, Z be any distinct, collinear, and ordinary points with signed ratio $\frac{ZG}{GV} = \frac{5}{4}$, and P an ordinary point not on GZ. Define $Q' = K(P)$ (complement taken with respect to G) and let P' be the reflection of V in Q' and Q the midpoint of PV. Finally let $\mathcal{C}$ be the conic guaranteed by Lemma 5(a). For any point A on the arc $\mathscr{A} = PQQ'P'$ of $\mathcal{C}$ distinct from these four points, there is a unique pair of points $\{B, C\}$ on $\mathcal{C}$ (and then on the same arc), such that ABC has centroid G. For each such triangle, the map M is a half-turn, and this map is independent of A. Conversely, if ABC is inscribed in the conic $\mathcal{C}$ with centroid G, then $A \in \mathscr{A} - \{P, Q, Q', P'\}$.*

Proof. We start by showing that the hypotheses of the theorem can be satisfied for suitable points G, V, Z, P, for which the conic $\mathcal{C}$ is a circle. Start with a circle $\mathcal{C}$ with center Z. Pick points P', Q on $\mathcal{C}$ and O so that $QZP'O$ is a square. Let S be the midpoint of OQ and $G = SZ \cdot QP'$. Reflect P' and Q in GZ to obtain the points P and Q' on $\mathcal{C}$. Let Y on GZ be the midpoint of PP'. Also, let $V = PQ \cdot P'Q'$ on GZ. Since $G = QP' \cdot SZ$ is on the bisector of $\angle SQZ$ and $\frac{ZQ}{QS} = \frac{2}{1}$, we have $\frac{ZG}{GS} = \frac{2}{1}$, so $K(Z) = S$. This implies that $K(ZP') = SK(P')$ is parallel to ZP' and half the length, so $K(P') = Q$. In the same way, $K(P) = Q'$. Then in triangle PVQ' the segment VG bisects the angle at V, so $\frac{PV}{VQ'} = \frac{PG}{GQ'} = \frac{P'G}{GQ} = \frac{2}{1}$. Thus, $VP' = VP = 2 \cdot VQ'$. It follows that Q' is the midpoint of VP', Q is the midpoint of VP, and G is the centroid of VPP'. Hence, $Y = K(V)$. This shows that the hypotheses of Lemma 5 hold, so $\mathcal{C}$ is the conic of that lemma. By the same argument as in the proof of Proposition 4(2), using the fact that OQ and OP' are tangent to $\mathcal{C}$ at Q and P', respectively, and that GVV_∞ is a self-polar triangle with respect to $\mathcal{C}$, it follows that $K^{-1}(PP') = OO'$ and V is the midpoint of OO'.

Now, letting M be the half-turn about S, the same argument as in the proof of Proposition 4(3) gives that $\frac{ZG}{GV} = \frac{5}{4}$.

If G, V, Z, P are any points satisfying the hypotheses, then there is an affine map taking triangle VPP' to the corresponding triangle constructed in the previous paragraph, so that G (the centroid of VPP') and Z go to the similarly named points and the trapezoid $PP'Q'Q$ is mapped to the corresponding trapezoid for the circle. Then the image of the new conic $\mathcal{C}$ is the circle of the previous paragraph, so $\mathcal{C}$ must be an ellipse.

Given A on the arc $\mathscr{A} = PQQ'P'$ of $\mathcal{C}$, define $D_0 = K(A)$. Now P' and $K(P') = Q$ are on $\mathcal{C}$, as are P and $K(P) = Q'$. We claim that P and P' are the only two points R on $\mathcal{C}$ for which $K(R)$ is also on $\mathcal{C}$. This is because $K(\mathcal{C})$ is a conic with center $K(Z) = S$, meeting $\mathcal{C}$ at Q, Q', and lying on the point $K(Q)$. Note that the map K fixes all points on l_∞, so $K(\mathcal{C})$ induces the same involution on l_∞ that $\mathcal{C}$ does. It follows that there are exactly two points in $K(\mathcal{C}) \cap \mathcal{C}$. Since the point Q is on the given arc $\mathscr{A}$ and $K(Q)$, as the midpoint of segment QP', is interior to $\mathcal{C}$, it follows that the same is true for the point $D_0 = K(A)$, for any A on $\mathscr{A}$, while D_0 lies outside of $\mathcal{C}$ when A is on $\mathcal{C} - \mathscr{A}$. Now consider the reflection $\mathcal{C}'$ of the conic $\mathcal{C}$ in the point D_0. When D_0 lies inside $\mathcal{C}$, it also lies inside $\mathcal{C}'$, and therefore the two conics $\mathcal{C}, \mathcal{C}'$ overlap. Since reflection in D_0 fixes all the points on l_∞, the conic $\mathcal{C}'$ induces the same involution on l_∞ that $\mathcal{C}$ does. Therefore, they have exactly two points in common. Labeling these points as B and C, it is clear that D_0 is the midpoint of segment BC, and from this and $K(A) = D_0$ it follows that G is the centroid of ABC. On the other hand, if A lies outside of $\mathscr{A}$, then D_0 lies outside of $\mathcal{C}$, and in this case, $\mathcal{C}'$ does not intersect $\mathcal{C}$ (so there can be no triangle inscribed in $\mathcal{C}$ with centroid G). Applying the same argument to the points B and C instead of A shows that B and C are also on the arc $\mathscr{A}$. The next to last assertion follows from Proposition 7 and the comments preceding Lemma 5. $\square$

We can use the proof of Theorem 8 to give a construction of the locus $\mathscr{L}$ of points P, for a given triangle ABC, for which the map M is a half-turn. To do this, start with the construction of the points $Q_1 Z_1 P_1' O_1$ on circle $\mathcal{C}$, as in the first paragraph of the proof. Pick a point A_1 on the arc $\mathscr{A} = P_1 Q_1 Q_1' P_1'$, and determine the unique pair of points B_1, C_1 on $\mathscr{A}$ so that the centroid of $A_1 B_1 C_1$ is the point $G_1 = Z_1 S_1 \cdot Q_1 P_1'$, with S_1 the midpoint of segment $Q_1 O_1$. Then determine the unique affine map A for which $\mathsf{A}(A_1 B_1 C_1) = ABC$. The points $P = \mathsf{A}(P_1)$ and $P' = \mathsf{A}(P_1')$ describe the locus $\mathscr{L}$, as A runs over all affine maps with $A_1 \in \mathscr{A}$. This locus is shown in Figure 4, and turns out to be an elliptic curve $\mathcal{E}$ minus 6 points, as we show below. For the pictured triangle $A_1 B_1 C_1$ and its half-turn M_1, the map $\mathsf{M} = \mathsf{A} \circ \mathsf{M}_1 \circ \mathsf{A}^{-1}$ is a half-turn about the point $S = \mathsf{A}(S_1)$. Note that $\mathcal{E}$ is tangent to the sides of the anticomplementary triangle $K^{-1}(ABC)$ of ABC at the vertices.

An equation for the curve $\mathcal{E}$ can be found using barycentric coordinates. It can be shown (see [13]) that homogeneous barycentric coordinates of the points S and

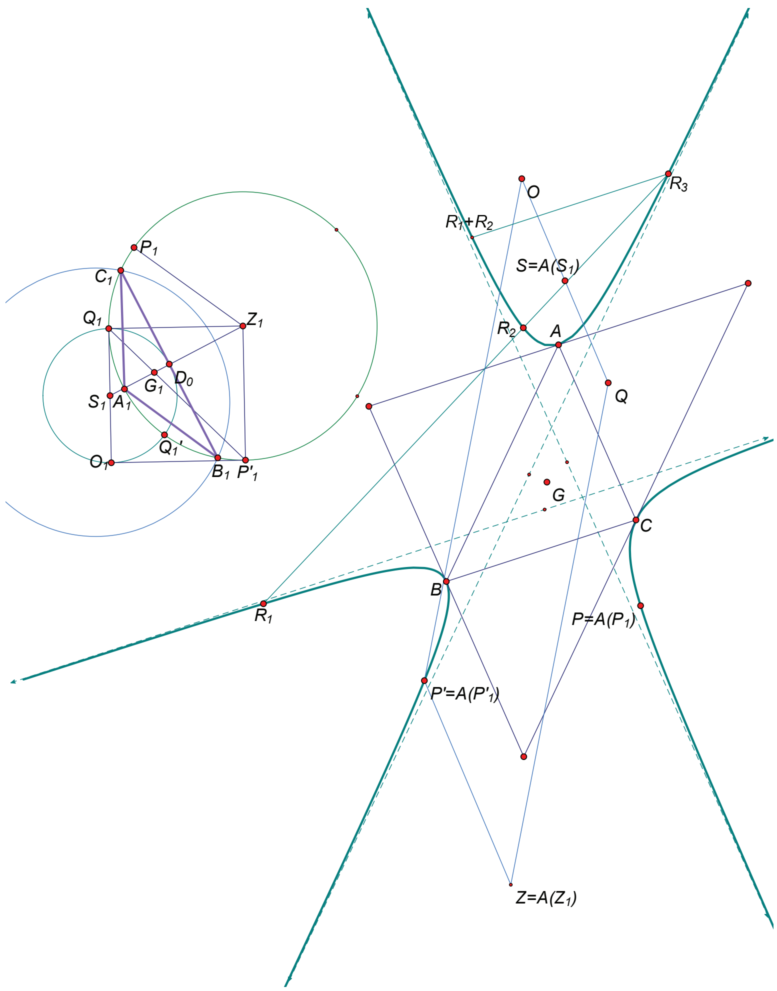

Figure 4. Elliptic curve locus of P with M a half-turn.

Z are

$$S = (x(y+z)^2, y(x+z)^2, z(x+y)^2), \quad Z = (x(y-z)^2, y(z-x)^2, z(x-y)^2),$$

where $P = (x, y, z)$. Using the remark after Corollary 3, we compute that the points $P = (x, y, z)$, for which M is a half-turn, satisfy $S = K(Z)$, so the coordinates of P satisfy the equation

$$\mathcal{E}: \quad x^2(y+z) + y^2(x+z) + z^2(x+y) - 2xyz = 0.$$

Note that $P \in \mathcal{E} \Rightarrow P' \in \mathcal{E}$. Setting $z = 1 - x - y$, where (x, y, z) are absolute barycentric coordinates, we get the affine equation for $\mathcal{E}$:

$$(5x - 1)y^2 + (5x - 1)(x - 1)y - x^2 + x = 0. \tag{2}$$

This is the case $a = -5$ of the geometric normal form

$$(ax + 1)y^2 + (ax + 1)(x - 1)y + x^2 - x = 0$$

which we consider in [13]. Rational points on $\mathcal{E}$ are $(x, y, z) = (1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, which correspond to the vertices A, B, C. The points on $l_\infty \cap \mathcal{E}$ are $(x, y, z) = (0, 1, -1), (1, 0, -1)$, and $(1, -1, 0)$, which are the infinite points on the sides of ABC. No other points on the sides or medians of ABC or $K^{-1}(ABC)$ lie on the curve $\mathcal{E}$. Using (2) we can check directly that the curve $\mathcal{E}$ is tangent to $K^{-1}(ABC)$ at the points A, B, C and that it has no singular points. It follows from this that $\mathcal{E}$ is an elliptic curve, whose points form an abelian group under the addition operation given by the chord-tangent construction. (See [5], p. 67, or [14].) In Figure 4 the sum of the points R_1 and R_2 on $\mathcal{E}$ is the point $R_1 + R_2$, taking the point $A_\infty = BC \cdot l_\infty = (0, 1, -1)$ as the base point (identity for the addition operation on the curve). With the base point A_∞, the point A has order 2, while $B_\infty = AC \cdot l_\infty$ and $C_\infty = AB \cdot l_\infty$ have order 3, and the points B, C have order 6.

Note that if $P \in \iota(l_\infty)$ is a point on the Steiner circumellipse lying on $\mathcal{E}$, then $P' \in \mathcal{E} \cap l_\infty$, so that P' is one of the points $A_\infty = (0, 1, -1), B_\infty = (1, 0, -1), C_\infty = (1, -1, 0)$, whose isotomic conjugates are A, B, C. Other than the vertices of ABC, no points on the Steiner circumellipse lie on $\mathcal{E}$. Furthermore, the Steiner circumellipse is inscribed in the triangle $K^{-1}(ABC)$, while $\mathcal{E}$ is tangent to the sides of this triangle at A, B, C. By Proposition 7 and Theorem 8, any point P for which M is a half-turn has the property that the points $Q = K(P')$ and $Q' = K(P)$ are exterior to triangle ABC. It follows that P and P' are exterior to triangle $K^{-1}(ABC)$. Hence, *all* the points of $\mathcal{E} - \{A, B, C\}$ are exterior to triangle $K^{-1}(ABC)$, as pictured in Figure 4.

We now check that the hypotheses of Proposition 2 hold for all the points in $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$. By the results of [12], the points for which the generalized orthocenter H is a vertex are contained in the union of three conics, $\overline{\mathcal{C}}_A \cup \overline{\mathcal{C}}_B \cup \overline{\mathcal{C}}_C$, which lie inside the Steiner circumellipse. By what we said above, none of the points in $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$ can lie on any of these conics, so H is never a vertex for these points. Hence, the hypotheses of Proposition 2 are satisfied for any P in $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$ and Corollary 3 implies that the map M for the point P is a half-turn. Thus, we have the following result.

Theorem 9. *The locus of points P, not lying on the sides of triangles ABC or $K^{-1}(ABC)$, for which $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a half-turn, coincides with the set of points whose barycentric coordinates lie in $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$.*

Every point P on $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$ is a point for which $QZP'O$ is a parallelogram. This yields an affine map A, for which $\mathsf{A}(Q_1 Z_1 P_1' O_1) = QZP'O$, and implies by the proof of Theorem 8 that $\mathsf{A}^{-1}(ABC) = A_1 B_1 C_1$ is a triangle with centroid G_1, inscribed on the arc $\mathscr{A}$. Lemma 5 shows that P_1 is the isotomic conjugate of P_1' with respect to $A_1B_1C_1$; hence $P = \mathsf{A}(P_1)$. This shows that every point on $\mathcal{E}$ except the vertices and points at infinity is $P = \mathsf{A}(P_1)$ for some affine mapping A in the "locus" of maps with $A_1 \in \mathscr{A}$.

Now, each point A_1 on $\mathscr{A}$ yields two points on $\mathcal{E}$, since A maps both points P_1 and P_1' to points on $\mathcal{E}$. Alternatively, with a given triangle $A_1B_1C_1$, there are affine maps $\mathsf{A}, \widetilde{\mathsf{A}}$ for which $\mathsf{A}(A_1B_1C_1) = ABC$ and $\widetilde{\mathsf{A}}(A_1C_1B_1) = ABC$. We claim first that $\mathsf{A}(P_1) = -\widetilde{\mathsf{A}}(P_1)$, i.e., that $P = \mathsf{A}(P_1)$ and $\tilde{P} = \widetilde{\mathsf{A}}(P_1)$ are negatives on the curve $\mathcal{E}$ with respect to the addition on the curve. This is equivalent to the fact that the line $P\tilde{P}$ through these two points is parallel to BC. This is obvious from the fact that $\widetilde{\mathsf{A}} = \rho \circ \mathsf{A}$, where, as in the proof of Lemma 1, ρ is the affine reflection in the direction of the line BC, fixing the points on the median $AD_0 = AG$.

We claim now that the points on $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$ are in $1-1$ correspondence with the collection of point-map pairs (A_1, A) and $(A_1, \widetilde{\mathsf{A}})$ for $A_1 \in \mathscr{A} - \{P_1, Q_1, P_1', Q_1'\}$. Suppose that (A_1, A_1) and (A_2, A_2) map to the same point P on $\mathcal{E}$, for $A_1, A_2 \in \mathscr{A} - \{P_1, Q_1, P_1', Q_1'\}$. Then $\mathsf{A}_1(A_1B_1C_1) = ABC = \mathsf{A}_2(A_2B_2C_2)$ or $\mathsf{A}_1(A_1B_1C_1) = ABC = \mathsf{A}_2(A_2C_2B_2)$, since the labeling of the points B_i, C_i can be switched; and for these maps, $\mathsf{A}_1(P_1) = P = \mathsf{A}_2(P_1)$. Then $\mathsf{A}_1^{-1}\mathsf{A}_2(A_2B_2C_2) = A_1B_1C_1$ or $A_1C_1B_1$ and $\mathsf{A}_1^{-1}\mathsf{A}_2(P_1) = P_1$. But then the map $\mathsf{A}_1^{-1}\mathsf{A}_2$ also fixes the points Q_1, P_1', Q_1', since G_1 is the centroid for both triangles. Hence, $\mathsf{A}_1^{-1}\mathsf{A}_2$ is the identity and $\mathsf{A}_1 = \mathsf{A}_2$, so that $A_1B_1C_1 = \mathsf{A}_1^{-1}(ABC) = A_2B_2C_2$ or $A_2C_2B_2$. Therefore, $(A_2, \mathsf{A}_2) = (A_1, \mathsf{A}_1)$.

Thus, we have proved the following.

Theorem 10. *Given a circle $\mathcal{C}$ with center Z_1, points Q_1 and P_1' on $\mathcal{C}$, and point O_1 for which $Q_1Z_1P_1'O_1$ is a square, then with the points G_1, P_1, S_1 as in Figure 4, the set of points*

$$\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$$

on the elliptic curve $\mathcal{E}$ coincides with the set of points $\mathsf{A}(P_1)$, where $A_1 \in \mathscr{A} = P_1Q_1Q_1'P_1'$ is a point on the arc $\mathscr{A}$ distinct from the points in $\{P_1, Q_1, Q_1', P_1'\}$, B_1, C_1 are the unique points on $\mathscr{A}$ for which $A_1B_1C_1$ has centroid G_1, and A is an affine map for which $\mathsf{A}(A_1B_1C_1) = ABC$ or $\mathsf{A}(A_1C_1B_1) = ABC$.

Note finally that the discriminant of (2) with respect to y is $D = (x-1)(5x-1)(5x^2-2x+1)$, so (2) is birationally equivalent to the curve

$$Y^2 = (X-1)(5X-1)(5X^2-2X+1).$$

Putting $X = \frac{u}{u-4}, Y = \frac{8v}{(u-4)^2}$ shows that this curve is, in turn, birationally equivalent (over $\mathbb{Q}$) to

$$v^2 = (u+1)(u^2+4), \tag{3}$$

which has j invariant $j = \frac{2^4 11^3}{5^2}$. This curve is curve (20A1) in Cremona's tables [4], and has the torsion subgroup $T = \{O, (-1,0), (0, \pm 2), (4, \pm 10)\}$ of order 6 and rank $r = 0$ over $\mathbb{Q}$. The curve $\mathcal{E}$ has infinitely many real points defined over quadratic extensions of $\mathbb{Q}$, including, for example,

$$P = (-4 + \sqrt{19}, -1, 3), \quad \left(\frac{9 + \sqrt{89}}{2}, -2, 1 \right).$$

This shows that there are infinitely many points for which the map M is a half-turn. There are even infinitely many such points defined over the field $\mathbb{Q}(\sqrt{6})$, since the points $(u, v) = (2, 2\sqrt{6})$ and $(u, v) = (\frac{2}{3}, \frac{10\sqrt{6}}{9})$ have infinite order on (3).

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Igor Minevich: Dept. of Mathematics, Rose-Hulman Institute of Technology, 5500 Wabash Avenue CM129, Terra Haute, Indiana, USA 47803-3920
E-mail address: `minevich@rose-hulman.edu`

Patrick Morton: Dept. of Mathematical Sciences, Indiana University - Purdue University at Indianapolis (IUPUI), 402 N. Blackford St., Indianapolis, Indiana, USA 46202
E-mail address: `pmorton@math.iupui.edu`