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On Some Elementary Properties of Quadrilaterals

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Abstract. We study some elementary properties of convex quadrilaterals related to the idea of common harmonics of two pairs of points and detecting two similar parallelograms with diagonals parallel to those of the quadrilateral and such that the area of the quadrilateral is the mean geometric of their areas. We apply these results to the study of variation of area of quadrilaterals, whose angles remain fixed in measure.

1. The common harmonics

The common harmonics $\{X, Y\}$ of two pairs of collinear points $\{(A, B), (C, D)\}$ are points on their line, such that the resulting quadruples are simultaneously harmonic, i.e. their cross ratios ([4, p.70]) satisfy

$$\frac{XA}{XB} : \frac{YA}{YB} = \frac{XC}{XD} : \frac{YC}{YD} = -1.$$

A geometric view of the common harmonics results by considering them as the *limit points* of a non-intersecting pencil $\mathcal{P}$ of circles defined by the pairs $\{(A, B),$ $(C, D)\}$ ([5, p. 109]). In fact the two circles $\{\alpha, \beta\}$, respectively on diameters $\{AB, CD\}$, define a pencil, and if this pencil is non-intersecting, then it has two limit points $\{X, Y\}$, which are harmonic conjugate with respect to the diametral

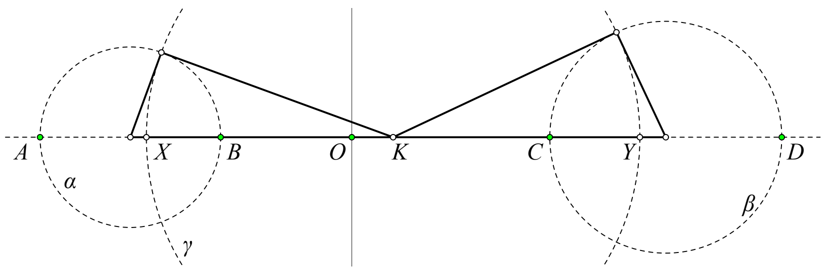

Figure 1. Common harmonics $\{X, Y\}$ of pairs $\{(A, B), (C, D)\}$

points $\{A, B\}$ of any circle of the pencil (see Figure 1). The common harmonics, as real points, exist precisely when the pencil is of non-intersecting type ([5, p. 118]). This case arrives when the intervals $\{(A, B), (C, D)\}$, as we say, *do not separate each other*, i.e. either are disjoint or one of them contains the other ([3, p. 284]). In the case the common harmonics exist, we can find them by intersecting their carrying line with a circle orthogonal to two circles of the aforementioned pencil. This, because the circles simultaneously orthogonal to those of the pencil

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$\mathcal{P}$ form another pencil $\mathcal{P}^{\perp}$ of intersecting type, all the circles of which pass through the points $\{X, Y\}$.

Lemma 1. *The common harmonics $\{X, Y\}$ of the pairs of collinear points $\{(A, B), (C, D)\}$, i.e. points for which their cross ratios satisfy $(ABXY) = (CDXY) = -1$, are given in coordinates by the following formula.*

$$x, y = \frac{1}{d+c-a-b}\left(cd - ab \pm \sqrt{(c-a)(c-b)(d-a)(d-b)}\right). \quad (1)$$

Proof. By the preceding remarks, in order to find quickly such an expression, it suffices to set two circles $\{\alpha, \beta\}$ on diameters $\{AB, CD\}$ and find the circle γ orthogonal to these two, with its center K on BC. Point K is the intersection of BC with the radical axis of the circles $\{\alpha, \beta\}$ and the intersection points of the circle γ with line BC define precisely the desired common harmonics $\{X, Y\}$ (see Figure 1). A short calculation, which I omit, shows that the corresponding coordinates are given by the stated formula. Here and in the sequel we use small letters $\{a, b, \dots\}$ for the line coordinates of points on a line represented with the corresponding capitals $\{A, B, \dots\}$. $\square$

2. The parallelogram associated to opposite vertices

It is well known, that the intersection points of an arbitrary line ε with four lines, passing through a common point O, define a cross ratio independent of the particular position of the line ε ([4, p.72], [2, p.251]). Thus, for every quadruple of lines, we can use such an arbitrary line ε, intersecting the quadruple at points $\{A, B, C, D\}$, and define the common harmonics $\{X, Y\}$ and the associated lines $\{OX, OY\}$, which we call the *common harmonics* of the line pairs $\{(OA, OB), (OC, OD)\}$. In particular, we may consider a convex quadrilateral $OBGC$ and from its vertex O draw parallels $\{OA, OD\}$ respectively to its sides $\{GC, GB\}$. Then define the common harmonics $\{OX, OY\}$ of the line pairs $\{(OA, OB), (OC, OD)\}$ (see Figure 1), and the parallelogram $OIGJ$ having its sides parallel to the lines$\{OX, OY\}$. In order to study this shape we use a coordinate system with origin at O and $x-$axis parallel to the diagonal BC of the

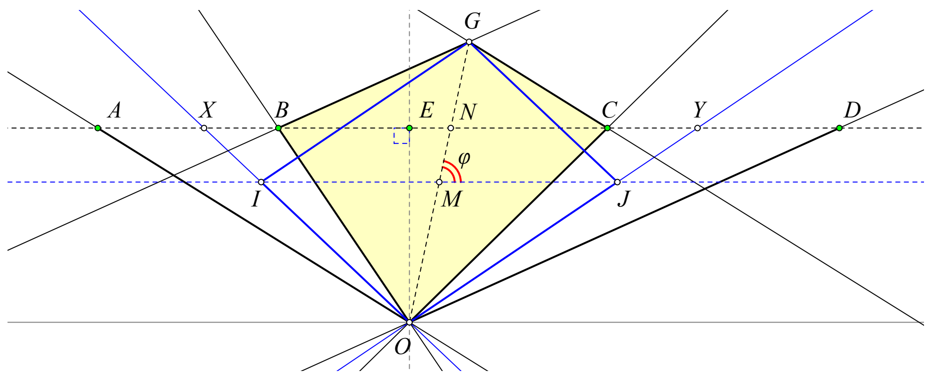

Figure 2. The parallelogram $OIGJ$

quadrilateral. In this system the $x-$coordinates of points $\{A, B, \dots\}$ are denoted by the corresponding small letters $\{a, b, \dots\}$, or occasionally by $\{A_x, B_y, \dots\}$ and the corresponding $y-$coordinates by $\{A_y, B_y, \dots\}$. A simple computation, which takes into account lemma-1 and which I omit, proves the first of the two next lemmata. In these E is the orthogonal projection of O on the diagonal BC.

Lemma 2. *The coordinates of G are*

$$G_x = \frac{cd - ba}{d - a}, \quad G_y = E_y \frac{c + d - a - b}{d - a},$$

the middle N of XY is on the diagonal OG and the diagonal IJ of the parallelogram $OJGI$ is parallel to the diagonal BC of $OBGC$.

Lemma 3. *The ratio of the areas of the quadrilaterals $\{OBGC, OIGJ\}$ depends only on the angles of the quadrilateral $OBGC$ and not on the length of its sides. This ratio is expressible through a cross ratio of the parallels to the four side-lines of the quadrilateral:*

$$\frac{|OIGJ|}{|OBGC|} = \sqrt{\frac{(c-a)(d-b)}{(c-b)(d-a)}} = \sqrt{(ABCD)}.$$

Proof. Using the angle ϕ between the diagonals and the formula for the area $|OBGC|$ $= \frac{1}{2}|OG||BC|\sin(\phi)$, for the two quadrilaterals (see Figure 2), which have in common the diagonal OG, we see that the stated ratio is equal to the ratio of the other diagonals IJ/BC. Using the similarity of triangles $\{IOJ, XOY\}$, we see that $IJ = XY\frac{G_y}{2E_y}$. Hence, using lemmata 1 and 2 we get:

$$\begin{aligned}
\frac{IJ}{BC} &= \frac{XY \cdot G_y}{2 \cdot BC \cdot E_y} = \frac{XY}{2 \cdot BC} \cdot \frac{c+d-a-b}{d-a} \\
&= \frac{\frac{2}{d+c-a-b}\sqrt{(c-a)(c-b)(d-a)(d-b)}}{2(c-b)} \cdot \frac{c+d-a-b}{d-a} \\
&= \frac{\sqrt{(c-a)(c-b)(d-a)(d-b)}}{(c-b)(d-a)} = \sqrt{\frac{(c-a)(d-b)}{(c-b)(d-a)}}.
\end{aligned}$$

The last equality results from the definition of the expression of the cross ratio through the coordinates $(ABCD) = \frac{a-c}{b-c} : \frac{a-d}{b-d}$. This quantity expresses also the cross ratio of the ordered quadruple of lines (OA, OB, OC, OD), which remains fixed if the angles of the quadrilateral do not change. $\square$

Starting with the opposite to O, vertex G of the quadrilateral, and doing the same work as before, leads to a parallelogram, which is identical to the previous one. The reason for this is apparent in Figure 3-I. The two configurations on either side of the diagonal BC of the quadrilateral are similar. The relations between triangles, more explicitly, being $\{OAB \sim GCA,\ OBC \sim GA'B',\ OCB \sim GB'B\}$. From the similarity follows that $\{(OX, GX'), (OY, GY')\}$ are pairs of parallel lines (see Figure 3-II), hence they coincide with the side-lines of the parallelogram $OIGJ$ of lemma 3. The following theorem recapitulates the results so far.

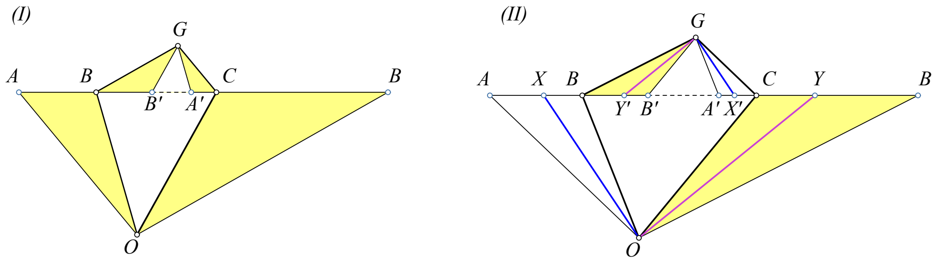

Figure 3. The same parallelogram relative to opposite vertices

Theorem 4. *Each pair of opposite vertices of a convex quadrilateral defines a parallelogram whose side-lines are the common harmonics of pairs of lines parallel to its sides and passing through the vertices of the pair. The diagonal joining the opposite vertices is also diagonal of the parallelogram. The other diagonals of the two quadrilaterals are parallel. In addition the ratio of the areas of the two quadrilaterals depends only on the angles of the convex quadrilateral.*

3. The two parallelograms

Repeating the construction of the previous section for the other pair of opposite points, we obtain a second parallelogram, one diagonal of which coincides now with the diagonal BC of the quadrangle of reference $OBGC$. Drawing the parallel ε' to the diagonal $\varepsilon = BC$ and doing the construction of the common

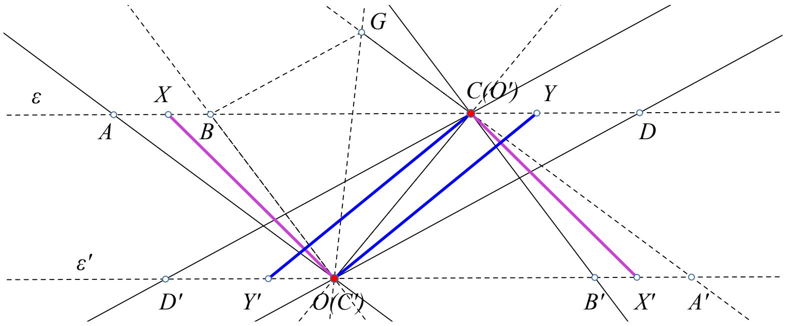

Figure 4. The common harmonics of adjacent vertices are pairwise parallel

harmonics this time on C (see Figure 4), we see that the parallels to the sides of the quadrilateral $OBGC$ from C form a quadruple of lines respectively parallel to those of the quadruple of lines through O. It follows that the common harmonics $\{CX', CY'\}$ of the pairs of lines $\{(CA', CB'), (CO, CD')\}$ are respectively parallel to the common harmonics $\{OX, OY\}$ constructed in the previous section. This leads to the following theorem.

Theorem 5. *The two parallelograms of pairs of opposite vertices of a general convex quadrilateral are homothetic w.r. to a point Q of the Newton line of the*

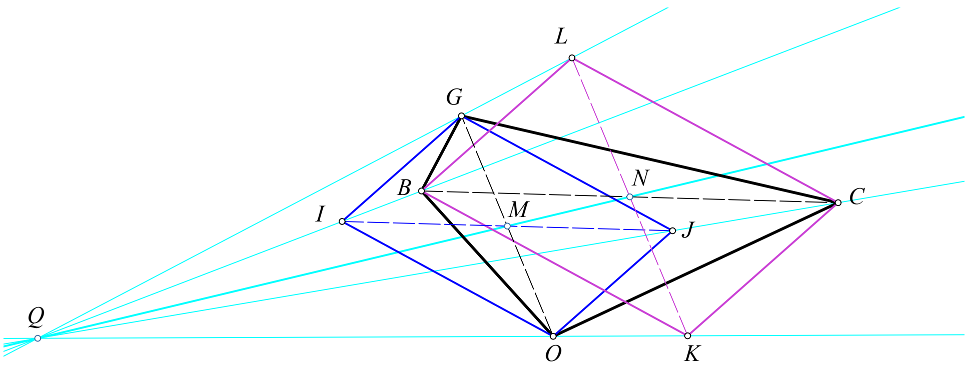

Figure 5. The two homothetic paralellolograms of a convex quadrilateral

quadrilateral. In addition, the area v of the quadrilateral is the geometric mean $v^2 = v_1 \cdot v_2$ of the areas of the two parallelograms.

Proof. By the preceding remarks, the two parallelograms have parallel sides. Besides, since their diagonals are parallel to the diagonals of the quadrilateral, they form the same angle. Thus, the parallelograms are similar and consequently also homothetic. Since their centers are the middles $\{M, N\}$ of the diagonals of the quadrilateral of reference, their homothety center lies on line MN, which is the Newton line of the quadrilateral (see Figure 5). For the last claim use lemma 3, by which the ratio v/v_1 is expressible through a cross ratio. Using the configuration of figure-4, we easily see that v/v_2 involves the inverse cross ratio of the previous one, so that the product $(v/v_1)(v/v_2) = 1$. □

Remark. Using the ratio of lemma 3, we can express the ratio QM/QN of the homothety of the two parallelograms and through it also find arithmetically the location of Q on the Newton line (see Figure 5). In fact, $QM/QN = GO/LK = \sqrt{v_1/v_2} = \sqrt{(v_1/v)(v/v2)} = v_1/v$, the last quotient being expressible through a cross ratio as in the aforementioned lemma.

In the sequel we shall call these two the *associated parallelograms* of the convex quadrilateral.

4. Rhombi, rectangles and squares

For special quadrilaterals the corresponding associated parallelograms obtain characteristic shapes. The simplest case is the one of *orthodiagonal quadrilaterals*, i.e. quadrilaterals, whose the diagonals are orthogonal. Since the diagonals of the parallelograms are parallel to the diagonals of the quadrilateral of reference, we obtain the following characteristic property (see Figure 6).

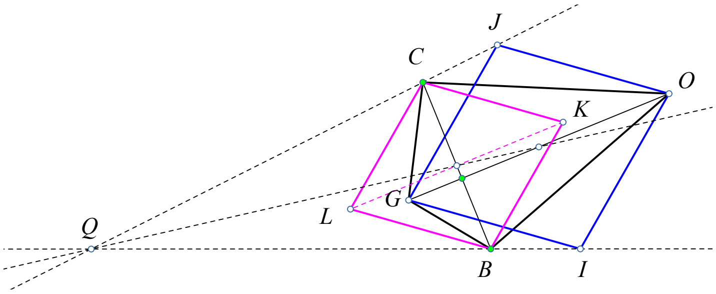

Figure 6. Rhombi as associated parallelograms of orthodiagonal quadrilaterals

Corollary 6. *For orthodiagonal quadrilaterals the associated paralleograms are rhombi.*

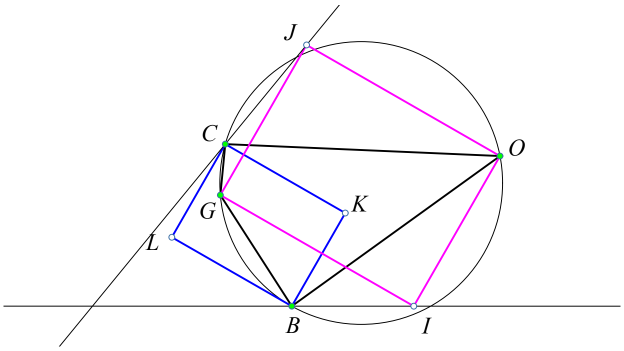

Figure 7. Rectangles characteristic for inscriptible quadrilaterals

Theorem 7. *For inscriptible in a circle quadrilaterals the associated parallelograms are rectangles.*

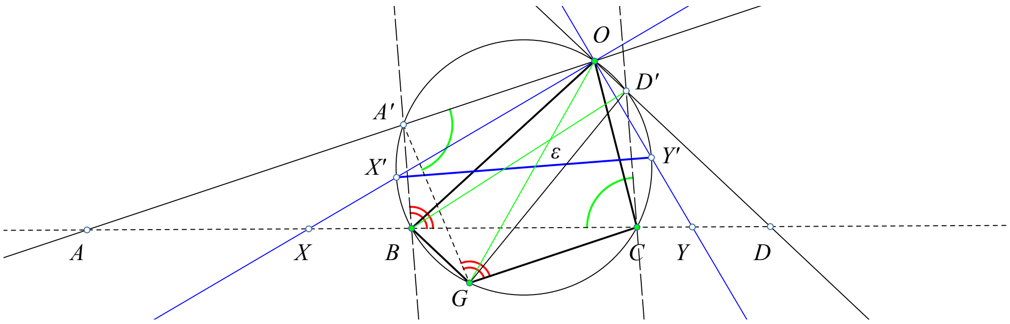

Figure 8. The orthogonal common harmonics $\{OX, OY\}$

Proof. For the proof repeat the construction of the common harmonics $\{OX, OY\}$ of the parallel to the sides through the vertex O of the quadrilateral $OBGC$, as in section 2. It suffices to show that $\{OX, OY\}$ are orthogonal. Let the paralells $\{OA, OD\}$ to sides $\{GC, GB\}$ intersect the circumcircles at points $\{A', D'\}$. We notice first that lines $\{BA', CD'\}$ are parallel (see Figure 8). This follows from a simple angle chasing argument: $\phi = \widehat{A'BC} = \widehat{A'GC}$ and $\psi = \widehat{BCD'} = \widehat{OA'G}$. Later because the two chords $\{OG, BD'\}$ are equal, since $BGD'O$ is an isosceles trapezium. From the isosceles trapezium $GA'OC$ we have also $\phi+\psi = \pi$, thereby proving the stated parallelity. Consider then the isosceles trapezium $BA'D'C$ and draw the diameter ε orthogonal to its parallel sides, intersecting the circle at points $\{X', Y'\}$. Later points are the middles of the arcs $\{\widehat{A'B}, \widehat{D'C}\}$, hence the lines $\{OX', OY'\}$ are orthogonal and they are also bisectors of the respective angles $\{\widehat{COD}, \widehat{AOB}\}$. Thus, these orthogonal lines define also the common harmonics of the pairs $\{(OA, OB), (OC, OD)\}$, thereby proving the claim. $\square$

Combining the last corollary and theorem, we obtain also the following characterization.

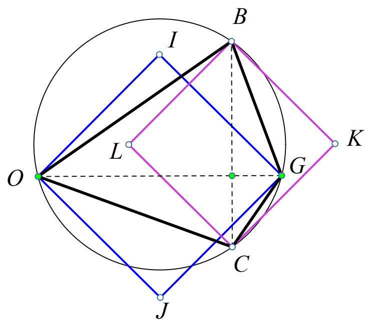

Figure 9. The case of squares

Corollary 8. *The associated parallelograms of orthodiagonal quadrilaterals, which are also inscriptible in a circle, are squares.*

5. The hyperbola arc

Next application of the associated parallelograms gives a generalization of the well known property of hyperbolas, according to which the parallelograms $OBGC$ with sides parallel to the asymptotes, where O is the center of the hyperbola and the opposite vertex G is a point of the hyperbola, have constant area ([7, p.192]). A more general problem would be to ask for the geometric locus of points G, which projected along the directions $\{\alpha, \delta\}$ on two intersecting at point O lines $\{\beta, \gamma\}$ form quadrilaterals $OBGC$ of constant area (see Figure 10). Next theorem handles this case in a slightly different formulation.

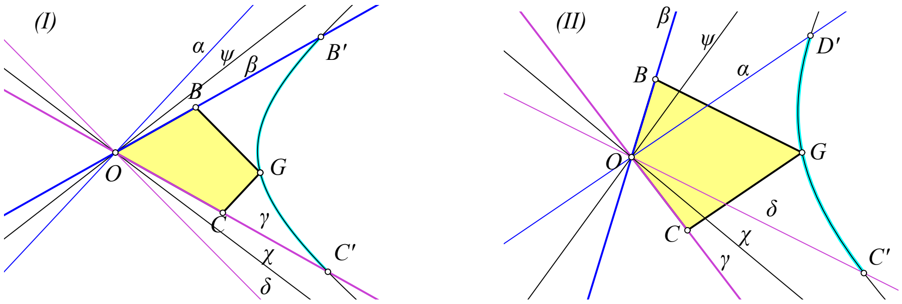

Figure 10. Locus of G for quadrilateral $OBGC$ of constant area

Theorem 9. *If the oriented convex quadrilateral $OBGC$ has sides pointing in fixed directions and its vertex O, as well as, its area is fixed, then the vertex G, opposite to the fixed one, lies on an arc of a hyperbola. The hyperbola is the one passing through G and having for asymptotes the common harmonics $\{\chi, \psi\}$ of the line pairs $\{(\alpha, \beta), (\gamma, \delta)\}$, where $\{\beta = OB, \gamma = OC\}$ and $\{\alpha, \delta\}$ are the parallels to $\{GC, GB\}$ from O.*

Proof. The proof follows directly from lemma-3, by drawing from G parallels to $\{\chi, \psi\}$ and defining the associated parallelogram $OIGJ$, as in figure-2. Since the area of the parallelogram is constant for G varying on the hyperbola and the area ratio $|OBGC|/|OIGJ|$ is constant, the area of $OIGJ$ remains also constant. The convexity requirement restricts the points G on an arc of the hyperbola determined by the lines $\{\beta, \gamma\}$, in the case the angle $\omega = \widehat{BOC}$ is acute (see Figure 10-I), and determined by the lines $\{\alpha, \delta\}$ in the case ω is obtuse (see Figure 10-II). □

6. Affine invariance

As is well known, affine transformations preserve the parallelity of lines ([1, p.191]). Since they are particular kinds of projective transformations, they preserve also the cross ratio of pencils of lines ([6, p.77]). This implies the *affine invariance* of the ratio of the areas of the two associated parallelograms of a convex quadrilateral, i.e. the invariance of this number for quadrilaterals, which map to each other by affine transformations. I take here the opportunity to make a short deviation into the way we can find a representative of a class of affinely equivalent convex quadrilaterals. The procedure is very simple and is illustrated by Figure 11. In this we fix a square $I_0J_0K_0L_0$ and for every other convex quadrilateral $q = ABCD$ we form its circumscribed parallelogram $IJKL$, with sides parallel to its diagonals. The correspondence of vertices $\{f_q(I) = I_0, f_q(J) = J_0, f_q(K) = K_0, f_q(L) = L_0\}$, extends to an affine transformation of the whole plane into itself, which we may denote by the same symbol f_q, and which maps q into the, inscribed in the square, quadrilateral $q' = f_q(q) = A'B'C'D'$. The general properties of affine transformations imply that the diagonals $\{A'C', B'D'\}$ are parallel to the sides of the square, hence equal, and the ratios $\{r_1, r_2\}$ on the diagonals, defined by their intersection

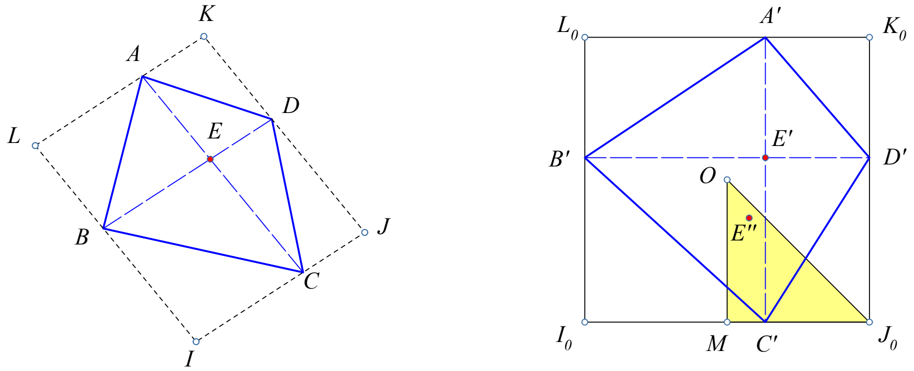

Figure 11. $A'B'C'D'$ representative of affine equivalents to $ABCD$

point, are preserved, i.e.

$$r_1 = \frac{EA}{EC} = \frac{E'A'}{E'C'}, \qquad r_2 = \frac{EB}{ED} = \frac{E'B'}{E'D'}.$$

Affine equivalent quadrilaterals map to equal corresponding inscribed in the square quadrilaterals $A'B'C'D'$. Later are characterized by the two ratios $\{r_1, r_2\}$. Using the symmetries of the square, for given $\{r_1, r_2\}$, we can in general construct 8 different but euclidean isometric inscribed in $I_0J_0K_0L_0$ quadrilaterals. This implies that the *moduli space*, i.e. the set of unique representatives for each class of affine equivalent quadrilaterals, can be set into bijective correspondence with the region defined by the triangle OMJ_0 shown in Figure 11 and representing the $1/8$ of the square. The points of this area determine a unique ordered pair of ratios (r_1, r_2) and a corresponding representative for the class of affinely equivalent quadrilaterals characterized by these two ratios. For example, point E'' in the figure represents $q = ABCD$ and results from E' by rotating it by $90°$ about the center O of the square. The boundary lines of the triangle have a special meaning. The point O represents the class of all parallelograms. The points of the segment OM represent the class of all quadrilaterals, whose one only of the diagonals is bisected by their intersection point. The points of the segment OJ_0 represent classes of quadrilaterals with diagonals divided by their intersection point into equal ratios, i.e. equilateral trapezia. Figure 12 shows the associated parallelograms of the affine representative $A'B'C'D'$ of $ABCD$. As noticed in section 4, they are rhombi, which in this case have one diagonal equal to the side of the square. The ratio of their areas, which is also equal to the area ratio of the original parallelograms, is equal to the ratio of their non-equal diagonals.

References

[1] H. S. M. Coxeter, *Introduction to Geometry*, John Wiley and Sons Inc., New York, 1961.

[2] H. Dörrie, 100 *Great Problems of Elementary Mathematics*, Dover Publications, Inc. New York, 1965.

[3] C. V. Durell, *A Course of Plane Geometry II*, MacMillan and Co. London, 1910.

[4] H. Eves, *A Survey of Geometry*, Allyn and Bacon, Inc., Boston, 1963.

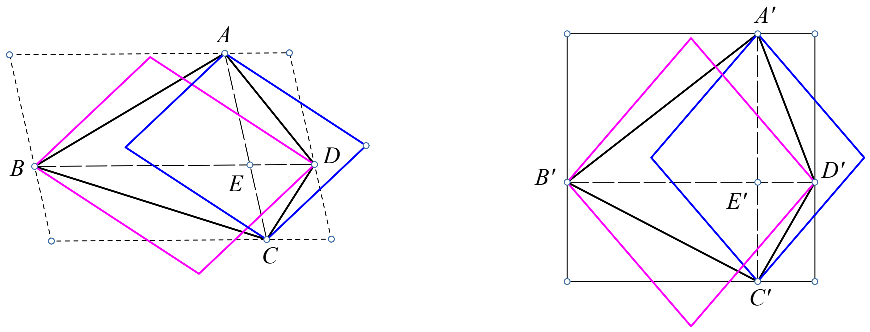

Figure 12. Affine representatives of associated parallelograms are rhombi

[5] D. Pedoe, *A Course of Geometry*, Dover, New York, 1990.
[6] J. Richter-Gebert, *Perspectives on Projective Geometry*, Springer, New York, 2011.
[7] G. Salmon, *A Treatise on Conic Sections*, Longmans, Green and Co., London, 1917.

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