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Properties of a Pascal Points Circle in a Quadrilateral with Perpendicular Diagonals

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Abstract. The theory of a convex quadrilateral and a circle that forms Pascal points is a new topic in Euclidean geometry. The theory deals with the properties of the Pascal points on the sides of a convex quadrilateral and with the properties of circles that form Pascal points.

In the present paper, we shall continue developing the theory, and we shall define the concept of the "Pascal points circle".

We shall prove four theorems regarding the properties of the points of intersection of a Pascal points circle with a quadrilateral that has intersecting perpendicular diagonals.

1. Introduction: General concepts and Fundamental Theorem of the theory of a convex quadrilateral and a circle that forms Pascal points

First, we shall briefly survey the definitions of some essential concepts of the theory of a convex quadrilateral and a circle that forms Pascal points on its sides, and then we shall present this theory's Fundamental Theorem (see [1], [2], [3]). The theory considers the situation in which $ABCD$ is a convex quadrilateral for which there exists a circle ω that satisfies the following two requirements:

(i) Circle ω passes through point E, the point of intersection of the diagonals, and through point F, the point of intersection of the extensions of sides BC and AD.

(ii) Circle ω intersects sides BC and AD at interior points (points M and N, respectively, in Figure 1).

The Fundamental Theorem of the theory holds in this case.

The Fundamental Theorem.
Let there be: a convex quadrilateral; a circle that intersects a pair of opposite sides of the quadrilateral, that passes through the point of intersection of the extensions of these sides, and that passes through the point of intersection of the diagonals.
In addition, let there be four straight lines, each of which passes both through the point of intersection of the circle with a side of the quadrilateral and through the point of intersection of the circle with the extension of a diagonal.
Then there holds: the straight lines intersect at two points that are located on the other pair of opposite sides of the quadrilateral.
(In Figure 2, straight lines h and g intersect at point P on side AB, and straight lines i and j intersect at point Q on side CD).

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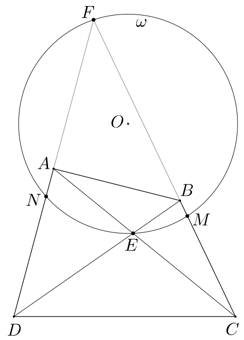

Figure 1

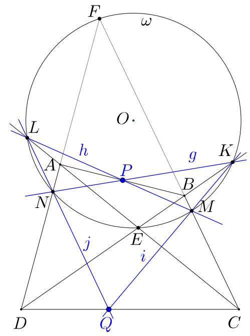

Figure 2

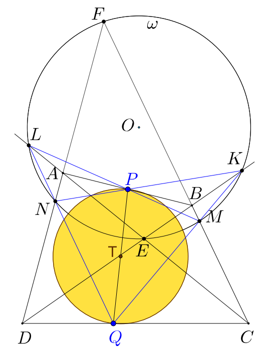

Figure 3

The Fundamental Theorem is proven using the general Pascal's Theorem (see [1]).

Definitions

Because the proof of the properties of the points of intersection P and Q is based on Pascal's Theorem,

(I) points P and Q are termed *Pascal points on sides AB and CD of the quadrilateral*;

(II) the circle that passes through points of intersection E and F and through two opposite sides is termed *a circle that forms Pascal points on the sides of the quadrilateral*.

We define a new concept: Pascal points circle.

(III) We shall call a circle whose diameter is segment PQ (see Figure 3) a *Pascal points circle*.

2. Properties of a quadrilateral with perpendicular intersecting diagonals, a circle that forms Pascal points, and a Pascal points circle.

Theorem 1.

Let $ABCD$ be a quadrilateral with perpendicular diagonals in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of the sides BC and AD; ω_{EF} is the circle whose diameter is segment EF. Then,

(a) *circle ω_{EF} forms Pascal points on sides AB and CD* (see Figure 4)*;*
there are an infinite number of circles that form Pascal points on sides AB and CD;

(b) *for every circle, ω, that intersects sides BC and AD at points M and N, respectively, and forms Pascal points P and Q on sides AB and CD, respectively, there holds:*

the point of intersection, T, of the tangents to circle ω at points M and N is the middle of segment PQ.

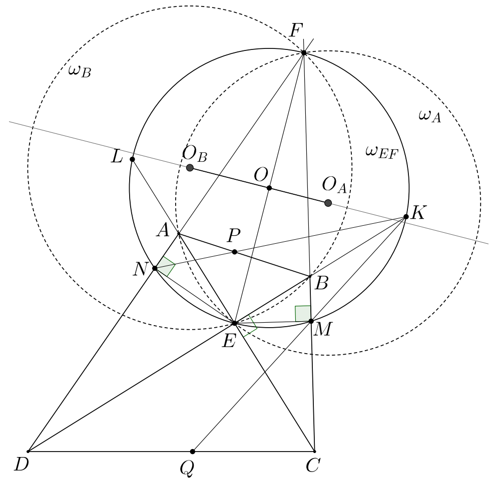

Figure 4.

Proof.
(a) Let us show that circle ω_{EF} intersects sides BC and AD at internal points.
In circle ω_{EF}, angle $\measuredangle EMF$ equals $90°$. Therefore, in right triangle $\triangle BCE$, segment EM is an altitude to hypotenuse BC, and hence it follows that the foot of altitude EM (point M in Figure 4) is an interior point of side BC. Similarly, we prove that point N (the base of the altitude to hypotenuse AD in right triangle $\triangle ADE$) is an internal point of side AD.
Based on the fundamental theorem, since circle ω_{EF} intersects sides BC and AD at internal points, this circle necessarily forms Pascal points on sides AB and CD. It is clear that if there is even one circle that passes through points E and F and also through internal points of sides BC and AD, then there must be an infinite number of such circles. Therefore, in our case, there are an infinite number of circles that pass through points E and F and through internal points of sides BC and AD. All these circles form Pascal points on sides AB and CD.

(b) Let us employ the following property that holds true for a convex quadrilateral (whose diagonals are not necessarily perpendicular) and a circle, ω, that forms Pascal points P and Q on sides AB and CD.
We denote: M and N are the intersection points of circle ω with sides BC and AD, respectively, and K and L are the intersection points of circle ω with the extensions of diagonals BD and AC, respectively (see Figure 5).
It thus holds that the four points P, Q, T, and R (P and Q are the two Pascal points, T is the point of intersection of the tangents to the circle at points M and

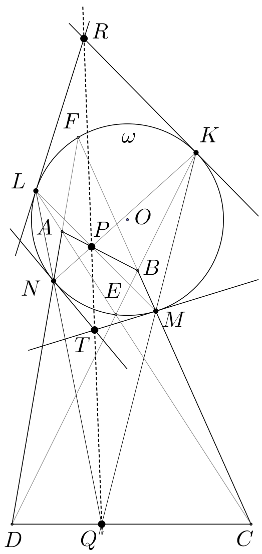

Figure 5

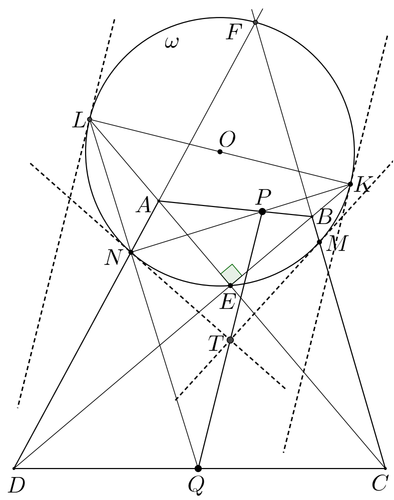

Figure 6

N, and R is the point of intersection of the tangents to the circle at points K and L) constitute a harmonic quadruple, in other words, there holds: $\frac{QT}{TP} = \frac{QR}{RP}$ (see [3, Theorem 1].)

In our case, the quadrilateral has perpendicular diagonals.
Therefore, $\measuredangle KEL = 90°$ and segment KL is a diameter of ω. Therefore, the tangents to circle ω at points K and L are parallel to each other (see Figure 6). In this case, their point of intersection, R, is at infinity, and ratio $\frac{QR}{RP}$ equals 1. Hence it also holds that $\frac{QT}{TP} = 1$, or $QT = TP$. In other words, point T is the middle of segment PQ. □

Theorem 2.
Let $ABCD$ be a quadrilateral with perpendicular diagonals in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD; ω is a circle that passes through points E and F and intersects sides BC and AD at points M and N, respectively; P and Q are Pascal points formed using ω on sides AB and CD, respectively; σ_{PQ} is a circle whose diameter is segment PQ (a Pascal points circle); T is the center of circle σ_{PQ} (see Figure 7).
Then,
(a) *Sides BC and AD each have at least one common point with circle σ_{PQ}. In other words:*

(1) *In the case that the center, O, of circle ω does not belong to straight lines BF and AF, circle σ_{PQ} intersects sides BC and AD at two points each. Two of these four points of intersection are N and M; the other two points are denoted as V and W* (see Figure 7).

(2) *When center O lies on straight line BF, circle σ_{PQ} is tangent to side BC at point M. In this case, point V coincides with M.*

(3) *When center O lies on straight line AF, circle σ_{PQ} is tangent to side AD at point N. In this case, point W coincides with N.*

(b) *Points V, T, and W lie on the same straight line. This property holds even in cases when point V coincides with point M or point W coincides with point N.*
(c) *Circles ω and σ_{PQ} are perpendicular to each other.*

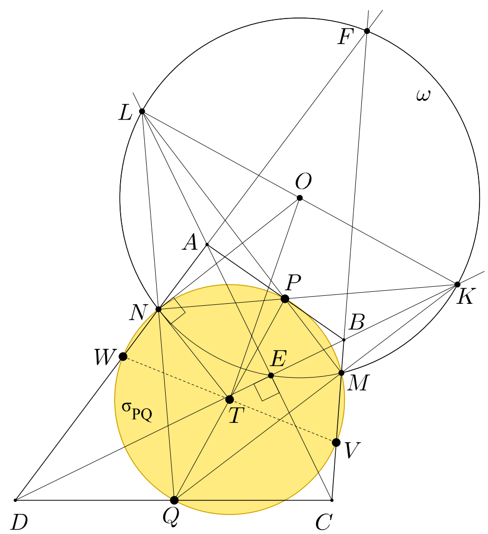

Figure 7.

Proof.
(a) We shall use the method of complex numbers in plane geometry. (The principles of the method and a system of formulas that we use in the proofs appear, for example, in [6, pp. 154-181]; some isolated formulas may be found in [4], [5]).
Let us choose a system of coordinates such that circle ω is the unit circle (O is the origin, and the radius is $OE = 1$). In this system, the equation of circle ω is $z \cdot \overline{z} = 1$, where z is the complex coordinate of some point Z that belongs to circle ω, and $\overline{z}$ is the conjugate of z.
We denote the complex coordinates of points K, L, M and N by k, l, m and n, respectively. These points are located on unit circle ω, therefore there holds: $\overline{k} = \frac{1}{k}$, $\overline{l} = \frac{1}{l}$, $\overline{m} = \frac{1}{m}$, and $\overline{n} = \frac{1}{n}$.
Point P is the point of intersection of straight lines KN and LM. Let us express the complex coordinate of P (and its conjugate) using the coordinates of points K,

L, M, and N. We shall use the following formula:
Let $A(a)$, $B(b)$, $C(c)$, and $D(d)$ be four points on the unit circle, and let $S(s)$ be the point of intersection of straight lines AB and CD. Then the coordinate s and its conjugate $\overline{s}$ satisfy:

$$\overline{s} = \frac{a+b-c-d}{ab-cd} \quad \textit{and} \quad s = \frac{bcd+acd-abd-abc}{cd-ab} \tag{I}$$

In our case, segment KL is a diameter of circle ω. Therefore, $k=-l$, and the expressions for p and $\overline{p}$ are:

$$\overline{p} = \frac{n+k-m-l}{nk-ml} = \frac{2l+m-n}{l\,(m+n)} \quad \textit{and} \quad p = \frac{2mn+nl-ml}{m+n}.$$

Now, let us find complex coordinate t of point T (which is the point of intersection of the tangents to the unit circle at points M and N). We use the following formula:
Let $S(s)$ be the point of intersection of the tangents to the unit circle at points $A(a)$ and $B(b)$, which are located on the circle. Then coordinate s and its conjugate $\overline{s}$ satisfy:

$$s = \frac{2ab}{a+b} \quad \textit{and} \quad \overline{s} = \frac{2}{a+b}. \tag{II}$$

In our case, we obtain for coordinate t and its conjugate $\bar{t}$ the following:

$$t = \frac{2mn}{m+n} \quad \textit{and} \quad \bar{t} = \frac{2}{m+n}.$$

Point T is the center of circle σ_{PQ}. Therefore, the equation of circle σ_{PQ} is

$$(z-t)\left(\overline{z}-\overline{t}\right) = r^2_{\sigma_{PQ}},$$

where $r_{\sigma_{PQ}}$ is the radius of the circle, and z is the complex coordinate of some point Z that belongs to the circle.
Let us find the square of the radius of circle σ_{PQ}. Point P lies on the circle, therefore the following equality holds: $(p-t)\left(\overline{p}-\overline{t}\right) = r^2_{\sigma_{PQ}}$.
Let us substitute the expressions for p and $\overline{p}$ in the left-hand side of the equality. We obtain:

$$\left(\frac{2mn+nl-ml}{m+n} - \frac{2mn}{m+n}\right)\left(\frac{2l+m-n}{l\,(m+n)} - \frac{2}{m+n}\right)$$
$$= \frac{nl-ml}{m+n} \cdot \frac{m-n}{l\,(m+n)} = -\left(\frac{m-n}{m+n}\right)^2.$$

In other words, there holds $r^2_{\sigma_{PQ}} = -\left(\frac{m-n}{m+n}\right)^2$. Therefore, the equation for circle σ_{PQ} is

$$\left(z - \frac{2mn}{m+n}\right)\left(\overline{z} - \frac{2}{m+n}\right) = -\left(\frac{m-n}{m+n}\right)^2. \tag{1}$$

Now let us find the equations of straight lines BC and AD and, subsequently, their points of intersection with circle σ_{PQ}.

We use the following formula of a straight line passing through two points $A(a)$ and $B(b)$ belonging to the unit circle:

$$z + ab\overline{z} = a + b. \tag{III}$$

In accordance with this formula, the equation of straight line BC (which passes through points $F(f)$ and $M(m)$ that belong to the unit circle) shall be $z + fm\overline{z} = f + m$. Hence:

$$\overline{z} = -\frac{1}{fm}z + \frac{f+m}{fm}. \tag{2}$$

We substitute the expression for $\overline{z}$ from (2) into (1) and obtain:

$$\begin{aligned}
&\left(z - \frac{2mn}{m+n}\right)\left(-\frac{1}{fm}z + \frac{f+m}{fm} - \frac{2}{m+n}\right) + \left(\frac{m-n}{m+n}\right)^2 = 0,\\
&-\frac{1}{fm}z^2 + \left(\frac{f+m}{fm} - \frac{2}{m+n} + \frac{2mn}{fm\,(m+n)}\right)z\\
&-\frac{2mn\,(f+m)}{fm\,(m+n)} + \frac{4mn}{(m+n)^2} + \frac{(m-n)^2}{(m+n)^2} = 0.
\end{aligned}$$

This leads to the following quadratic equation:

$$(m+n)\,z^2 - \left(3mn - fm + fn + m^2\right)z + fmn + 2m^2n - fm^2 = 0. \tag{3}$$

The solutions of this equation are:

$$\begin{aligned}
z_{1,2} &= \frac{3mn - fm + fn + m^2 \pm \sqrt{\begin{array}{c}\left(3mn - fm + fn + m^2\right)^2\\ -4\,(m+n)\left(fmn + 2m^2n - fm^2\right)\end{array}}}{2\,(m+n)}\\
&= \frac{3mn - fm + fn + m^2 \pm \sqrt{(m-n)^2\,(m+f)^2}}{2\,(m+n)}.
\end{aligned}$$

Equation (3) is a quadratic equation with complex coefficients.
It follows that if expression $(m-n)^2\,(m+f)^2$ does not equal 0, then (3) will have two solutions. In the present case it necessarily holds that $m \neq -f$, and hence it follows that points F and M are not the ends of the diameter of circle ω. This means that the center, O, of the circle does not belong to straight line MF (the line BF). In this case the two solutions of the equation are:

$$z_1 = \frac{3mn - fm + fn + m^2 + (m-n)\,(m+f)}{2\,(m+n)} = \frac{2mn + 2m^2}{2\,(m+n)} = m,$$

and

$$z_2 = \frac{3mn - fm + fn + m^2 - (m-n)\,(m+f)}{2\,(m+n)} = \frac{2mn - fm + fn}{m+n}.$$

It is clear that the first solution is the complex coordinate of point M, and the second solution is the coordinate of another point that belongs to straight line BC (denoted by V).

In other words: $v=\dfrac{2mn-fm+fn}{m+n}$.
Similarly, one can prove that in the case where the center, O, of circle ω does not lie on straight line AF, circle σ_{PQ} will intersect straight line AD at two points: 1) at point N, and 2) at some other point (designated as W) whose complex coordinate can be expressed as $w=\dfrac{2mn-fn+fm}{m+n}$.
If $(m-n)^2(m+f)^2=0$ holds, then Equation (3) has a single solution. For two different points M and N located on unit circle ω, there holds $m\neq n$, therefore necessarily there holds $m=-f$. In other words, points F and M are the ends of a diameter of circle ω, and therefore center O of the circle belongs to straight line BF.
In this case, the only solution of the equation is:

$$z=\frac{3mn-fm+fn+m^2}{2(m+n)}=\frac{3mn+m^2-mn+m^2}{2(m+n)}=\frac{2mn+2m^2}{2(m+n)}=m.$$

In other words, in this case, line BF is tangent to circle σ_{PQ} at point M.
Similarly, we can prove that when the center, O, of ω belongs to line AF, line AF will be tangent to circle σ_{PQ} at point N.

(b) Let us prove that points V, T, and W lie on the same straight line (see Figure 7).
We shall use the following formula, which gives the relation between the coordinates of any three collinear points $A(a)$, $B(b)$, and $C(c)$:

$$a\left(\overline{b}-\overline{c}\right)+b\left(\overline{c}-\overline{a}\right)+c\left(\overline{a}-\overline{b}\right)=0. \tag{IV}$$

According to this formula, points V, T, and W are collinear provided the following equality holds:

$$v\left(\overline{t}-\overline{w}\right)+t\left(\overline{w}-\overline{v}\right)+w\left(\overline{v}-\overline{t}\right)=0. \tag{4}$$

Let us first calculate the conjugates of coordinates v and w:

$$\overline{v}=\frac{\overline{2mn-fm+fn}}{m+n}=\frac{\dfrac{2}{mn}-\dfrac{1}{fm}+\dfrac{1}{fn}}{\dfrac{1}{m}+\dfrac{1}{n}}=\frac{2f-n+m}{f(m+n)}$$

and similarly

$$\overline{w}=\frac{2f-m+n}{f(m+n)}.$$

We substitute the expressions for t, $\overline{t}$, v, $\overline{v}$, w, and $\overline{w}$ into (4), to obtain:

$$\begin{aligned}&\frac{2mn-fm+fn}{m+n}\left(\frac{2}{m+n}-\frac{2f-m+n}{f(m+n)}\right)\\&+\frac{2mn}{m+n}\left(\frac{2f-m+n}{f(m+n)}-\frac{2f-n+m}{f(m+n)}\right)\\&+\frac{2mn-fn+fm}{m+n}\left(\frac{2f-n+m}{f(m+n)}-\frac{2}{m+n}\right)=0.\end{aligned}$$

After simplifying the left-hand side, we have:

$$\frac{2mn - fm + fn}{m+n} \cdot \frac{m-n}{f\,(m+n)} + \frac{2mn}{m+n} \cdot \frac{2n-2m}{f\,(m+n)}$$
$$+ \frac{2mn - fn + fm}{m+n} \cdot \frac{m-n}{f\,(m+n)} = 0,$$
$$\frac{m-n}{f\,(m+n)^2} \cdot \underbrace{(2mn - fm + fn - 4mn + 2mn - fn + fm)}_{=0} = 0.$$

We have thus obtained $0 = 0$.
In other words, (IV) is satisfied, and therefore points V, T, and W must be on the same straight line, and segment VW is a diameter of circle σ_{PQ}.

(c) In (a), we proved that circles ω and σ_{PQ} intersect at points M and N, and therefore $r_\omega = OM$ and $r_{\sigma_{PQ}} = TM$. Let us find the distance, OT, between the centers of circles ω and σ_{PQ}:

$$OT^2 = (t-0)\left(\bar{t}-\overline{0}\right) = \left(\frac{2mn}{m+n} - 0\right)\left(\frac{2}{m+n} - 0\right) = \frac{4mn}{(m+n)^2}.$$

Now we calculate the sum $r_\omega^2 + r_{\sigma_{PQ}}^2$:

$$r_\omega^2 + r_{\sigma_{PQ}}^2 = 1 - \left(\frac{m-n}{m+n}\right)^2 = \frac{(m+n)^2 - (m-n)^2}{(m+n)^2} = \frac{4mn}{(m+n)^2}.$$

Therefore, the equation $r_\omega^2 + r_{\sigma_{PQ}}^2 = OT^2$ holds , and, specifically, $OM^2 + TM^2 = OT^2$ holds.
It thus follows that angle $\measuredangle OMT$ is a right angle, and therefore line OM is tangent to circle σ_{PQ}, and line TM is tangent to ω.
We obtained that the tangents to circles ω and σ_{PQ} at the point of their intersection, M, are perpendicular to each other.
Therefore the circles are perpendicular to each other. □

Conclusions from Theorem 2.
(1) We obtained that the two segments PQ and VW are diameters of circle σ_{PQ}. Therefore their lengths are equal, and they bisect each other (at point T). It follows that quadrilateral $PVQW$ is a rectangle (see Figure 8).

Note: Rectangle $PVQW$ (in which two opposite vertices are Pascal points) is usually different from the rectangle inscribed in quadrilateral $ABCD$ in such a manner that its sides are parallel to diagonals AC and BD, which are perpendicular to each other. (In Figure 8 rectangle $PXYZ$ is inscribed in the quadrilateral and its sides are parallel to the diagonals of the quadrilateral.)

(2) For any quadrilateral, $ABCD$, with perpendicular diagonals and any circle, ω_i, that forms a pair of Pascal points P_i and Q_i on sides AB and CD, one can define a rectangle that is inscribed in quadrilateral $ABCD$ as follows:
We construct a Pascal points circle $\sigma_{P_iQ_i}$ that intersects sides BC and AD at points V_i and W_i (in addition to points N_i and M_i). Points P_i, V_i, Q_i, and W_i define a rectangle inscribed in quadrilateral $ABCD$.

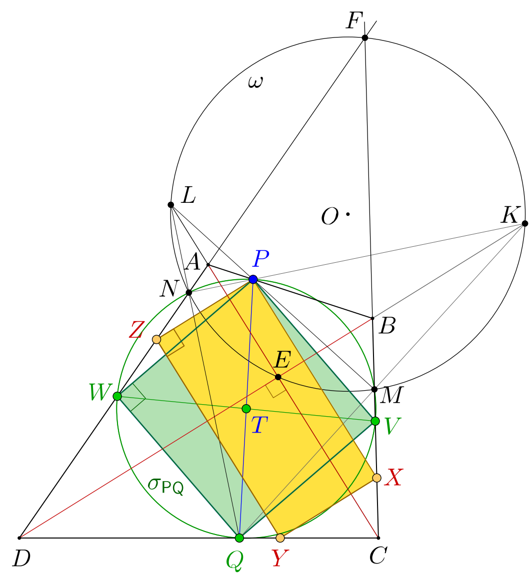

Figure 8.

(3) Let ω be a circle that passes through points E and F, intersects sides BC and AD at points M and N, respectively, and forms Pascal points P and Q on sides AB and CD. T is the point of intersection of the tangents to circle ω at points M and N.
In this case, the circle whose center is at point T and whose radius is segment TM is the Pascal points circle σ_{PQ}.

Explanation: In Theorem 1 we proved that the tangents to circle ω at points M and N intersect in the middle of segment PQ (at point T). In Theorem 2 we proved that Pascal points circle σ_{PQ} passes through points M and N.

Theorem 3.
Let $ABCD$ be a quadrilateral with perpendicular diagonals in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD; ω_{EF} is a circle whose diameter is segment EF; Circle ω_{EF} intersects sides BC and AD at points M_0 and N_0, respectively, and forms Pascal points P_0 and Q_0 on sides AB and CD; $\sigma_{P_0Q_0}$ is the Pascal points circle *of points P_0 and Q_0. Then:*
(a) *Circle $\sigma_{P_0Q_0}$ intersects the sides of quadrilateral $ABCD$ at 8 points, as follows:*
It intersects side AB at points P_0 and M_1, side BC at M_0 and V_0, side CD at points Q_0 and N_1, side AD at N_0 and W_0. (In Figure 9, one can observe the fours points of intersection mentioned in Theorem 2, N_0, M_0, V_0, and W_0, and also two additional points of intersection, M_1 and N_1).
(b) *Chords V_0N_0, W_0M_0, Q_0M_1, and P_0N_1 of the circle intersect at point E.*

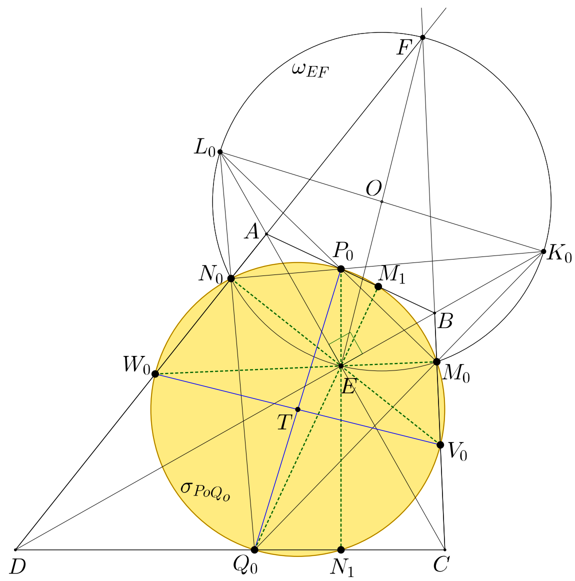

Figure 9.

Proof.
The center, O, of circle ω_{EF} does not belong to straight lines FB and FA. Therefore, from Theorem 2, circle $\sigma_{P_0Q_0}$ intersects each of the sides BC and AD at two points (at points M_0 and V_0, and at points N_0 and W_0, respectively). Therefore, it remains to be proven that $\sigma_{P_0Q_0}$ intersects each of the other two sides at two points.
We will first prove one additional property that holds for the points of intersection of circle $\sigma_{P_0Q_0}$ with sides BC and AD. We will show that chords V_0N_0 and W_0M_0 both pass through point E.
We choose a system of coordinates such that circle ω_{EF} is the unit circle (O is the origin and the radius, OE, equals 1).
From formula (IV) in the proof of Theorem 2, points $N_0\,(n)$, $E\,(e)$ and $V_0\,(v)$ are collinear provided the following equality holds:

$$n\left(\overline{e}-\overline{v}\right)+e\left(\overline{v}-\overline{n}\right)+v\left(\overline{n}-\overline{e}\right)=0.$$

For e, $\overline{e}$, v, and $\overline{v}$ there holds: $e=-f$ (because segment EF is the diameter of the unit circle), $\overline{e}=-\dfrac{1}{f}$, $v=\dfrac{2mn-fm+fn}{m+n}$, and $\overline{v}=\dfrac{2f-n+m}{f\,(m+n)}$ (see the proof of Theorem 2).
We substitute these expressions in the left-hand side of the formula above, and

obtain:

$$\begin{aligned}
& n\left(\overline{e}-\overline{v}\right)+e\left(\overline{v}-\overline{n}\right)+v\left(\overline{n}-\overline{e}\right)\\
&= n\left(-\frac{1}{f}-\frac{2f-n+m}{f\left(m+n\right)}\right)-f\left(\frac{2f-n+m}{f\left(m+n\right)}-\frac{1}{n}\right)+\frac{2mn-fm+fn}{m+n}\left(\frac{1}{n}+\frac{1}{f}\right)\\
&= n\cdot\frac{-2m-2f}{f\left(m+n\right)}-f\cdot\frac{fn-n^2+mn-fm}{fn\left(m+n\right)}+\frac{\left(2mn-fm+fn\right)\left(f+n\right)}{fn\left(m+n\right)}\\
&= \frac{0}{fn\left(m+n\right)}\\
&= 0.
\end{aligned}$$

In other words, the equality holds and therefore the points N_0, E, and V_0 are collinear (see Figure 9).
Similarly, we also prove that points M_0, E, and W_0 are collinear.
To find the remaining two points of intersection, we follow the following path:
The first stage is to find two points that can be candidates for the intersection of circle $\sigma_{P_0Q_0}$ with sides AB and CD. The second stage is to prove that these two points are really the points of intersection of circle $\sigma_{P_0Q_0}$ with sides AB and CD. It is reasonable to assume that the property satisfied for the four points of intersection of circle $\sigma_{P_0Q_0}$ with sides BC and AD shall also hold for the four points of intersection of circle $\sigma_{P_0Q_0}$ with sides AB and CD. Therefore, at the first stage we shall choose points M_1 and N_1 to be our candidates, which are the intersection points of line AB with line Q_0E and line CD with line P_0E, respectively.
At the second stage, we shall prove that the points M_1 and N_1 belong to circle $\sigma_{P_0Q_0}$.
Using formula (IV) we obtain the equation of straight line AB.
The formula holds for three collinear points $A(a)$, $B(b)$, and $C(c)$. If we replace the coordinate of point C by the coordinate of some point $Z(z)$ that belongs to straight line AB, we obtain the equation of AB:

$$a\left(\overline{b}-\overline{z}\right)+b\left(\overline{z}-\overline{a}\right)+z\left(\overline{a}-\overline{b}\right)=0.$$

This can be put in the form:

$$\overline{z}=\frac{\overline{a}-\overline{b}}{a-b}z+\frac{a\overline{b}-\overline{a}b}{a-b}. \tag{V}$$

Let us express the coordinates of $A(a)$ and $B(b)$ (and their conjugates) using the coordinates of points F, E, K, L, M, and N, which lie on the unit circle. We shall use the formulas (I) from the proof of Theorem 2.
In our case, segments K_0L_0 and EF are diameters of circle ω_{EF}. Therefore, there holds that $k=-l$ and $e=-f$. The following expressions are therefore obtained:

$$a=\frac{2nl+fl-fn}{n+l},\qquad \overline{a}=\frac{2f+n-l}{f\left(n+l\right)},$$
$$b=\frac{2ml+fl+fm}{l-m},\qquad \overline{b}=\frac{2f+m+l}{f\left(m-l\right)}.$$

We substitute these expressions into (V) to obtain:

$$\overline{z} = \frac{\dfrac{2f+n-l}{f(n+l)} - \dfrac{2f+m+l}{f(m-l)}}{\dfrac{2nl+fl-fn}{n+l} - \dfrac{2ml+fl+fm}{l-m}} z + \frac{\dfrac{2nl+fl-fn}{n+l} \cdot \dfrac{2f+m+l}{f(m-l)} - \dfrac{2f+n-l}{f(n+l)} \cdot \dfrac{2mk+fl+fm}{l-m}}{\dfrac{2nl+fl-fn}{n+l} - \dfrac{2ml+fl+fm}{l-m}}.$$

After simplifying, we obtain:

$$\overline{z} = \frac{fm-fn-2fl-ml-nl}{fl(fm+fn+2mn+ml-nl)} z + \frac{2fnl+2fml+2f^2l+2mnl-f^2n+f^2m+nl^2-ml^2}{fl(fm+fn+2mn+ml-nl)}. \qquad (5)$$

Similarly, we can obtain the equation of line QE: we replace the letters a and b in (V) with the letters e and q to obtain: $\overline{z} = \dfrac{\overline{q}-\overline{e}}{q-e} z + \dfrac{q\overline{e}-\overline{q}e}{q-e}$.

In our case there holds: $e = -f$ and $\overline{e} = -\dfrac{1}{f}$.

Point Q is the point of intersection of straight lines KM and LN. In addition, in our case there holds that $k = -l$. Therefore, from the formulas (I) for q and $\overline{q}$, we obtain the following expressions: $q = \dfrac{2mn+ml-nl}{m+n}$ and $\overline{q} = \dfrac{2l+n-m}{l(m+n)}$.

We substitute these expressions in the equation of straight line QE, and obtain:

$$\overline{z} = \frac{\dfrac{2l+n-m}{l(m+n)} + \dfrac{1}{f}}{\dfrac{2mn+ml-nl}{m+n} + f} z + \frac{\dfrac{2mn+ml-nl}{m+n} \cdot \left(-\dfrac{1}{f}\right) + \dfrac{2l+n-m}{l(m+n)} \cdot f}{\dfrac{2mn+ml-nl}{m+n} + f},$$

and after simplifying:

$$\overline{z} = \frac{2fl+fn-fm+ml+nl}{fl(2mn+ml-nl+fm+fn)} z + \frac{2f^2l+f^2n-f^2m-2mnl-ml^2+nl^2}{fl(2mn+ml-nl+fm+fn)}. \qquad (6)$$

By equating the right-hand sides of (5) and (6), we obtain an expression for complex coordinate z_{M_1} of the intersection point of straight lines AB and QE:

$$z_{M_1} \underset{\text{we denote}}{=} m_1 = \frac{f^2n-f^2m-2lmn-fml-fnl}{fm-fn-2fl-ml-nl}.$$

The expression for the conjugate of m_1 is: $\overline{m}_1 = \dfrac{ml-nl-2f^2-fm-fn}{f(nl-ml-2mn-fm-fn)}$.

We now prove that point M_1 belongs to circle $\sigma_{P_0Q_0}$.

From formula (1) in the proof of Theorem 2, the equation of circle $\sigma_{P_0Q_0}$ is:

$$\left(z-\frac{2mn}{m+n}\right)\left(\overline{z}-\frac{2}{m+n}\right)=-\left(\frac{m-n}{m+n}\right)^2.$$

Let us substitute the expressions for m_1 and $\overline{m}_1$ in the equation of the circle. We obtain:

$$\left(\frac{f^2n-f^2m-2lmn-fml-fnl}{fm-fn-2fl-ml-nl}-\frac{2mn}{m+n}\right)$$
$$\times\left(\frac{ml-nl-2f^2-fm-fn}{f\left(nl-ml-2mn-fm-fn\right)}-\frac{2}{m+n}\right)$$
$$=-\left(\frac{m-n}{m+n}\right)^2$$

Let us check if this equality is a true statement.

Observe the left-hand side of the equality. After adding fractions and collecting similar terms, we obtain:

$$\frac{fn^2-fm^2-lm^2-ln^2+2mnl-2m^2n-2mn^2}{\left(fm-fn-2fl-ml-nl\right)\left(m+n\right)}$$
$$\times\frac{lm^2-ln^2-fm^2-fn^2+2fmn-2fnl+2fml}{\left(nl-ml-2mn-fm-fn\right)\left(m+n\right)}.$$

After factoring the expressions in the numerators, we obtain:

$$\frac{\left(n-m\right)\left(fm+fn-nl+ml+2mn\right)}{\left(fm-fn-2fl-ml-nl\right)\left(m+n\right)}\cdot\frac{\left(m-n\right)\left(ml+nl-fm+fn+2fl\right)}{\left(nl-ml-2mn-fm-fn\right)\left(m+n\right)}$$
$$=\frac{-\left(m-n\right)^2\left(-fm-fn+nl-ml-2mn\right)\left(-ml-nl+fm-fn-2fl\right)}{\left(fm-fn-2fl-ml-nl\right)\left(nl-ml-2mn-fm-fn\right)\left(m+n\right)^2}$$
$$=-\left(\frac{m-n}{m+n}\right)^2.$$

We have obtained an identical expression on both sides of the equality. Therefore the last equality is a true statement, and therefore point M_1 belongs to circle $\sigma_{P_0Q_0}$. Since point M_1 belongs to line AB, it follows that circle $\sigma_{P_0Q_0}$ intersects straight line AB at point M_1.

Similarly, we can prove that circle $\sigma_{P_0Q_0}$ intersects straight line CD at point N_1.

In summary, we have shown that the four chords V_0N_0, W_0M_0, Q_0M_1, and P_0N_1 of circle $\sigma_{P_0Q_0}$ pass through point E, which is, therefore, their point of intersection. □

Conclusions from Theorems 1-3.

In proving Theorems 1-3 we considered a quadrilateral whose two opposite sides, BC and AD, are not parallel, and we did not require any additional conditions concerning the remaining opposite sides.

In the case that sides AB and CD also intersect (we denote the point of their intersection by G), there will be circles that pass through points E and G and form Pascal points on sides BC and AD. In this case, for these circles Theorems similar to Theorems 1-3 shall hold (the proofs of these theorems are similar to the proofs

of Theorems 1-3).
According to these theorems, the circle whose diameter is segment EG (we denote it by ψ_{EG}) satisfies the following properties:

(a) The circle ψ_{EG} forms Pascal points on sides BC and AD (for now we denote these points by $\overline{P}$ and $\overline{Q}$, respectively).

(b) The circle ψ_{EG} and the circle whose diameter is segment $\overline{P}\,\overline{Q}$ are perpendicular to each other, and they intersect at the points at which circle ψ_{EG} intersects sides AB and CD (for now we denote these points by $\overline{M}$ and $\overline{N}$, respectively).

(c) The circle whose diameter is segment $\overline{P}\,\overline{Q}$ intersects sides AB and CD at points $\overline{V}$ and $\overline{W}$ (in addition to points $\overline{M}$ and $\overline{N}$). The four points $\overline{P}$, $\overline{V}$, $\overline{Q}$, and $\overline{W}$ define a rectangle inscribed in quadrilateral $ABCD$.

Theorem 4.
Let $ABCD$ be a quadrilateral with perpendicular diagonals in which E is the point of intersection of the diagonals, F is the point of intersection of the extensions of sides BC and AD, and G is the point of intersection of the extensions of the sides AB and CD; ω_{EF} is a circle whose diameter is segment EF which forms Pascal points P_0 and Q_0 on sides AB and CD, respectively; $\sigma_{P_0Q_0}$ is the Pascal points circle *of points P_0 and Q_0, which intersects the sides of quadrilateral $ABCD$ at the following eight points: P_0, Q_0, M_0, N_0, V_0, W_0, M_1, and N_1* (see Theorem 3)*; ψ_{EG} is the circle whose diameter is segment EG. Then:*
(a) *The circle ψ_{EG} intersects sides AB and CD at points M_1 and N_1, respectively.*
(b) *Circles ψ_{EG} and $\sigma_{P_0Q_0}$ are perpendicular to each other.*
(c) *Points V_0 and W_0 are the Pascal points formed by circle ψ_{EG} on sides BC and AD, respectively.*
(d) *The angle between diameters EF and EG of circles ω_{EF} and ψ_{EG} is equal to one of the two angles between diameters P_0Q_0 and V_0W_0 of circle $\sigma_{P_0Q_0}$* (in Figure 10, there holds: $\measuredangle FEG = \measuredangle V_0EQ_0$).

Proof.
(a) In circle $\sigma_{P_0Q_0}$, inscribed angle $\measuredangle P_0M_1Q_0$ rests on diameter P_0Q_0. It therefore holds that $\measuredangle P_0M_1Q_0 = 90°$, and therefore also $\measuredangle EM_1G = 90°$. Hence it follows that point M_1 belongs to the circle whose diameter is EG (circle ψ_{EG}).
Similarly, $\measuredangle P_0N_1Q_0 = 90°$. Therefore $\measuredangle EN_1G = 90°$ and therefore $N_1 \in \psi_{EG}$.

(b) Inscribed angles $\measuredangle P_0N_1M_1$ and $\measuredangle P_0Q_0M_1$ rest on the same arc, $\widehat{P_0M_1}$, in circle $\sigma_{P_0Q_0}$ (see Figure 11). Therefore $\measuredangle P_0Q_0M_1 = \measuredangle P_0N_1M_1$.
In addition, for angle $\measuredangle TQ_0M_1$ (which is another name for the angle $P_0Q_0M_1$) there holds: $\measuredangle TQ_0M_1 = \measuredangle TM_1Q_0$ (because they are the base angles of isosceles triangle TQ_0M_1). Therefore:

$$\measuredangle TM_1Q_0 = \measuredangle P_0N_1M_1. \tag{7}$$

Similarly, in circle ψ_{EG} there holds that $\measuredangle EN_1M_1 = \measuredangle EGM_1$, and also $\measuredangle O_1GM_1 = \measuredangle O_1M_1G$. Therefore:

$$\measuredangle O_1M_1G = \measuredangle EN_1M_1. \tag{8}$$

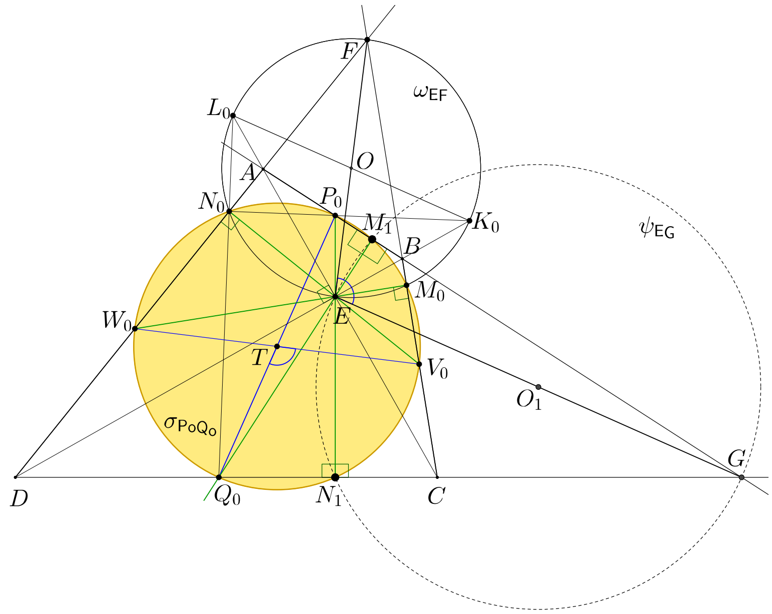

Figure 10.

Since angles $\measuredangle P_0N_1M_1$ and $\measuredangle EN_1M_1$, which appear in the right-hand side of equalities (7) and (8), are the same angle, therefore $\measuredangle TM_1Q_0 = \measuredangle O_1M_1G$.
Now, consider angle $\measuredangle TM_1O_1$:

$$\begin{aligned}\measuredangle TM_1O_1 &= \measuredangle TM_1Q_0 + \measuredangle Q_0M_1O_1 \\ &= \measuredangle TM_1Q_0 + (\measuredangle Q_0M_1G - \measuredangle O_1M_1G) \\ &= \measuredangle TM_1Q_0 + 90^\circ - \measuredangle O_1M_1G \\ &= 90^\circ.\end{aligned}$$

We obtained that $\measuredangle TM_1O_1 = 90^\circ$, and therefore M_1T is tangent to circle ψ_{EG}, and M_1O_1 is tangent to circle $\sigma_{P_0Q_0}$. Hence it follows that circles $\sigma_{P_0Q_0}$ and ψ_{EG} are perpendicular to each other.

(c) Circles $\sigma_{P_0Q_0}$ and ψ_{EG} intersect at an additional point: N_1. Therefore the tangent to circle ψ_{EG} at point N_1 also passes through the center, T, of circle $\sigma_{P_0Q_0}$.
We obtained that the tangents to circle ψ_{EG} at points M_1 and N_1 intersect at point T. Points M_1 and N_1 are the points of intersection of circle ψ_{EG} with sides AB and CD. Therefore, from Conclusion 3 of Theorem 2, we have that the circle whose center is point T and whose radius is segment TM_1 is the *Pascal points circle* of the points formed by circle ψ_{EG} on sides AB and CD.
On the other hand, in Theorem 3, we have proven that Pascal points circle $\sigma_{P_0Q_0}$ passes through points M_1 and N_1, and its center is at point T.
Therefore the *Pascal points circle* of the points formed by circle ψ_{EG} is circle $\sigma_{P_0Q_0}$.

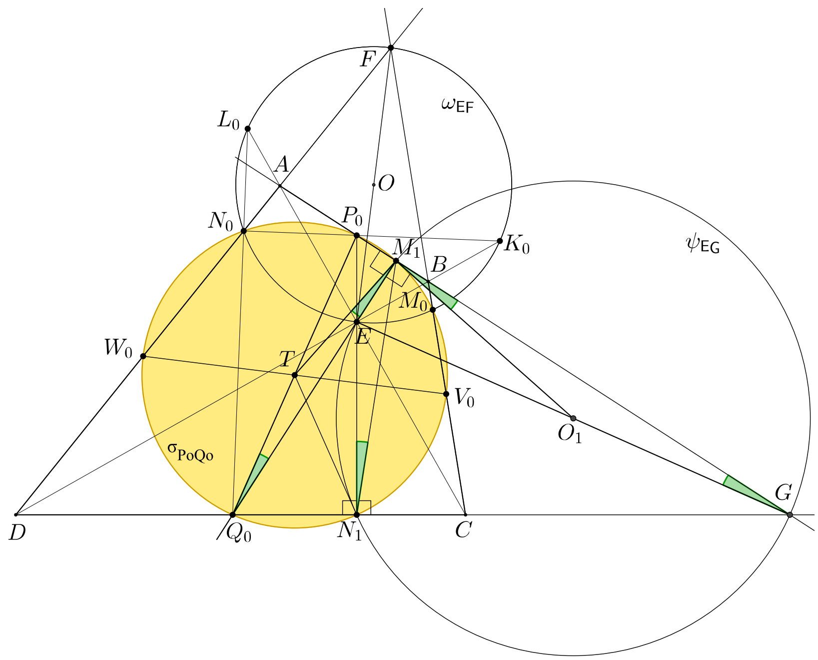

Figure 11.

In Theorem 3 we saw that circle $\sigma_{P_0Q_0}$ intersects side BC at points M_0 and V_0 and also intersects side AD at points N_0 and W_0.
Of the four chords that connect a point on side BC with a point on side AD (M_0W_0, N_0V_0, M_0N_0, and V_0W_0), only V_0W_0 passes through T, the center of the circle $\sigma_{P_0Q_0}$. In other words, only V_0W_0 is a diameter of circle $\sigma_{P_0Q_0}$.
Therefore points V_0 and W_0 are Pascal points formed by circle ψ_{EG} on sides BC and AD.

(d) Let us prove that straight line FE is perpendicular to diameter V_0W_0.
From Theorem 3, we have that segments W_0M_0 and V_0N_0 pass through point E.
In circle $\sigma_{P_0Q_0}$, angles $\measuredangle W_0M_0V_0$ and $\measuredangle V_0N_0W_0$ are inscribed angles resting on diameter V_0W_0 (see Figure 10). Therefore, they are right angles.
We obtained that segments W_0M_0 and V_0N_0 in triangle FV_0W_0 are altitudes to sides FV_0 and FW_0, respectively, and that E is their point of intersection. It follows that straight line FE contains the third altitude (the altitude to side W_0V_0) of triangle FV_0W_0, and therefore $EF \perp V_0W_0$.
Similarly, one can prove that $EG \perp P_0Q_0$.
In summary, segments EF and EG are perpendicular to diameters V_0W_0 and P_0Q_0, respectively, of circle $\sigma_{P_0Q_0}$, and therefore angle $\measuredangle FEG$ is equal to one of the angles between diameters V_0W_0 and P_0Q_0. □

Conclusion from Theorems 2-4.
In a quadrilateral, $ABCD$, in which diagonals are perpendicular and intersect at

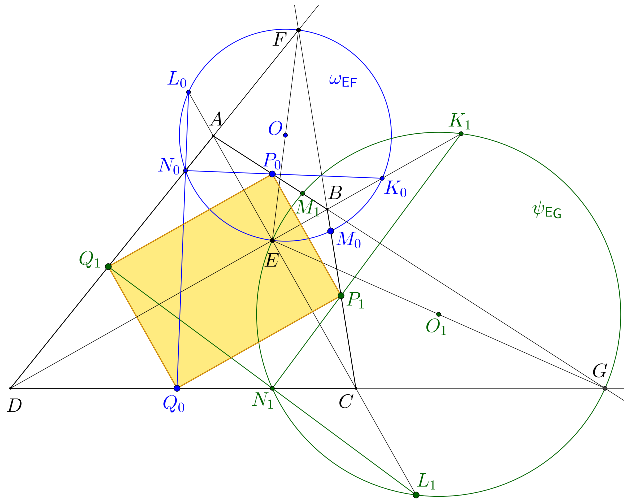

Figure 12.

point E, and the extensions of the opposite sides intersect at points F and G, there holds: the Pascal points formed by circles ω_{EF} and ψ_{EG} are the vertices of a rectangle inscribed in the quadrilateral (see Figure 12).

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